1.1 Introduction

The traces of optimization can be found in the development of calculus. As far back as 1629, Pierre de Fermat showed that the necessary condition for an extremum (minima or maxima) for a real-valued function of one variable is that the derivative must be zero. A class of extremum problem, that has been the favourite of giants like Jean Bernoulli, Leonhard Euler, Andrien Legendre and Carl Gustav Jacobi since the end of the eighteenth century and the beginning of nineteenth century, is known as the calculus of variation. The calculus of variation is an infinite-dimensional problem of unconstrained optimization in which the functional to be minimized is defined by an integral. In the second half of the nineteenth century, Karl Weierstrass posed the crucial question of existence of a solution of minima and maxima and answered it for a fairly general situation. In 1939, Nobel Laureate Leonid Vitalevich Kantorovich formulated many problems of Economics in the form of optimization problem for linear functionals known as the mathematical method of production of planning and optimization and made extensive study of such problems.
Chapter 1. Preliminaries

Variational inequalities have its origin in the calculus of variations associated with the minimization of finite dimensional functionals. The theory of variational inequality was jointly established by a French and an Italian named Guido Stampacchia and Jacques-Louis Lions in the early nineteen sixties. They use variational inequality as a tool for solving the problems arising in nonlinear partial differential equations and mechanics. Variational inequality plays an important part as a mathematical model in the study of many real life problems, in particular equilibrium problems. It provides us a tool for formulating and qualitatively analyzing the equilibrium problems in terms of existence and uniqueness of solutions, stability, sensitivity analysis, and algorithms for computational purposes.

The terminology of variational inclusions was introduced and studied by Hassouni and Moudafi [40] in 1994. They proposed a perturbed iterative algorithm for their problem, which was considered by many researchers around the globe, later on. Variational inclusion problems are the most widely and comprehensively studied real world mathematical problems and have a large variety of applications in the area of optimization, image processing, economics, network problems, transportation equilibrium and applied sciences, see [12, 14, 20, 21, 22, 28]. The very well known idea for solving inclusion problems by means of fixed point technique is due to Browder [15].

In different sections of this chapter, we present some definitions and results which are essential for presentation of results in the subsequent chapters.

1.2 Basic definitions and results

In this section, we present some basic definitions and results of functional analysis which will be used in the subsequent chapters. Throughout this thesis, we take $\mathcal{E}$ to be a real ordered Banach space, $\mathcal{H}$ to be a real ordered Hilbert space endowed with a norm $\| \cdot \|$, and an inner product $\langle \cdot, \cdot \rangle$, $\theta$ as zero element in $\mathcal{E}$ and $\mathcal{H}$, unless otherwise specified. $d$ is the metric induced by the norm $\| \cdot \|$, $2^\mathcal{E}$ (respectively, $\text{CB}(\mathcal{E})$) is the family of all nonempty (respectively, closed and bounded) subsets of $\mathcal{E}$, and $\mathcal{D}(\cdot, \cdot)$ is the Hausdorff metric on $\text{CB}(\mathcal{E})$ defined by

$$\mathcal{D}(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$ and $d(A, y) = \inf_{x \in A} d(x, y)$. 

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Definition 1.2.1. Let $\mathcal{C}(\neq \emptyset)$ be a closed, convex subset of $\mathcal{E}$. Then $\mathcal{C}$ is said to be a cone if

(i) for $x \in \mathcal{C}$ and $\lambda > 0$, $\lambda x \in \mathcal{C}$;

(ii) if $x$ and $-x \in \mathcal{C}$, then $x = \theta$.

Definition 1.2.2 ([29]). $\mathcal{C}$ is called a normal cone iff there exists a constant $\lambda_{\mathcal{C}} > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_{\mathcal{C}}\|y\|$, where $\lambda_{\mathcal{C}}$ is called the normal constant of $\mathcal{C}$.

Definition 1.2.3 ([29]). For arbitrary elements $x, y \in \mathcal{E}$, $x \leq y$ iff $x - y \in \mathcal{C}$, then the relation “$\leq$” is a partial ordered relation in $\mathcal{E}$. The real Banach space $\mathcal{E}$ endowed with the ordered relation “$\leq$” defined by $\mathcal{C}$ is called ordered real Banach space.

Definition 1.2.4. A relation “$\leq$” defined as $x \leq y$ iff $x - y \in \mathcal{C}$ for $x, y \in \mathcal{H}$ is known as partial order relation expounded by $\mathcal{C}$ in $\mathcal{H}$, then $(\mathcal{H}, “\leq”) \text{ is called a real ordered Hilbert space.}$

Definition 1.2.5 ([72]). For arbitrary elements $x, y \in \mathcal{E}$, $\text{lub}\{x,y\}$ and $\text{glb}\{x,y\}$ mean least upper bound and greatest upper bound of the set $\{x,y\}$. Suppose $\text{lub}\{x,y\}$ and $\text{glb}\{x,y\}$ exist, some binary relations are defined as follows:

(i) $x \lor y = \text{lub}\{x,y\}$;

(ii) $x \land y = \text{glb}\{x,y\}$;

(iii) $x \oplus y = (x - y) \lor (y - x)$;

(iv) $x \otimes y = (x - y) \land (y - x)$.

The operations $\lor, \land, \oplus \text{ and } \otimes$ are called OR, AND, XOR and XNOR operations, respectively.

Definition 1.2.6 ([72]). For arbitrary elements $x, y \in \mathcal{E}$, if $x \leq y \text{ (or } y \leq x \text{) holds, then } x \text{ and } y \text{ are called comparable to each other and is denoted by } x \asymp y.$

Proposition 1.2.7 ([29]). For any positive integer $n$, if $x \asymp y_n \text{ and } y_n \to y^* \text{ (} n \to \infty \text{), then } x \asymp y^*.$
**Definition 1.2.8** ([53]). Let $A, B : \mathcal{E} \to \mathcal{E}$ be two mappings. Then

(i) $A$ is said to be comparison, if for each $x, y \in \mathcal{E}$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$, and $y \propto A(y)$.

(ii) $A$ and $B$ is said to be comparison with each other, if for each $x \in \mathcal{E}$, $A(x) \propto B(x)$, (denoted by $A \propto B$).

Obviously, if $A$ is comparison, then $A \propto I$, where $I$ is an identity mapping on $\mathcal{E}$.

**Definition 1.2.9** ([2]). A map $A : \mathcal{E} \to \mathcal{E}$ is called $\beta$-ordered compression map, if it is a comparison mapping and

$$A(x) \oplus A(y) \leq \beta (x \oplus y), \text{ for } 0 < \beta < 1.$$ 

**Lemma 1.2.10** ([29]). If $x$ and $y$ both are comparable to each other then lub\{x, y\} and glb\{x, y\} exist, $x - y \propto y - x$, and $\theta \leq (x - y) \lor (y - x)$.

**Lemma 1.2.11** ([29, 55]). Let $\mathcal{C}$ be a normal cone with normal constant $\lambda_{\mathcal{C}}$ in $\mathcal{E}$, then each $x, y \in \mathcal{E}$, the hereunder connections are following:

(i) $\|\theta \oplus \theta\| = \|\theta\| = \theta$;

(ii) $\|x \land y\| \leq \|x\| \land \|y\| \leq \|x\| + \|y\|$;

(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_{\mathcal{C}} \|x \oplus y\|$;

(iv) if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.

(v) $\lim_{x \to x_0} \|A(x) - A(x_0)\| = 0$ iff $\lim_{x \to x_0} A(x) \oplus A(x_0) = 0$.

**Definition 1.2.12** ([49]). Let $A : \mathcal{E} \to \mathcal{E}$ to be a single-valued map. Then

(i) $A$ is called a $\gamma$-order non-extended mapping if there exists a constant $\gamma > 0$ such that

$$\gamma (x \oplus y) \leq A(x) \oplus A(y) \text{ for all } x, y \in \mathcal{E};$$

(ii) $A$ is called strongly comparison map if it is a comparison mapping and $A(x) \propto A(y)$ iff $x \propto y$, for all $x, y \in \mathcal{E}$. 


1.2. Basic definitions and results

Lemma 1.2.13 ([46, 49, 50, 51, 53, 72]). Let “≤” be a partial order relation defined by the cone $C$ with a normal constant $\lambda_C$ in $E$ in the Definition 1.2.3. Then hereinafter connections hold:

(i) $x \oplus y = y \oplus x = -(x \odot y) = -(y \odot x), x \oplus x = \theta$;

(ii) $\theta \leq x \oplus \theta$;

(iii) if $x \propto y$, then $0 \leq x \oplus y$;

(iv) if $\lambda$ is real, then $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;

(v) if $x, y$ and $w$ can be comparative to each other, then $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$;

(vi) if $x \propto s, t$ and $y \propto s, t$, then $(x + y) \oplus (s + t) \leq (x \oplus s + y \oplus t) \land (x \oplus t + y \oplus s)$ provided $(x + y) \lor (s + t)$ exist;

(vii) if $x, y, r, w$ can be compared with each other, then $(x \land y) \oplus (r \land w) \leq ((x \oplus r) \lor (y \oplus w)) \land ((x \oplus w) \lor (y \oplus r))$;

(viii) if $x \leq y$ and $s \leq t$, then $x + s \leq y + t$;

(ix) if $x \propto \theta$, then $-x \oplus \theta \leq x \leq x \oplus \theta$;

(x) if $x \propto y$, then $(x \oplus \theta) \oplus (y \oplus \theta) \leq (x \oplus y) \oplus \theta = x \oplus y$;

(xi) $(x \oplus \theta) - (y \oplus \theta) \leq (x - y) \oplus \theta$;

(xii) if $\theta \leq x$, $x \neq \theta$, and $\alpha > 0$, then $\theta \leq \alpha x$ and $\alpha x \neq \theta$, for all $x, y, r, s, t, w \in E$ and $\alpha, \lambda \in \mathbb{R}$;

(xiii) if $x \propto \theta$, then $\alpha x \oplus \beta x = |\alpha - \beta|x = (\alpha \oplus \beta)x$.

Definition 1.2.14 ([53]). A map $A : E \times E \to E$ is called $(\alpha_1, \alpha_2)$-restricted-accretive map, if it is a comparison mapping and $\exists$ constants $0 \leq \alpha_1, \alpha_2 \leq 1$ such that

$$(A(x, \cdot) + I(x)) \oplus (A(y, \cdot) + I(y)) \leq \alpha_1(A(x, \cdot) \oplus A(y, \cdot)) + \alpha_2(x \oplus y),$$

for all $x, y \in E$,

where $I$ is the identity map on $E$. 

Theorem 1.2.15 (Nadler’s Theorem [58]). Let \((X,d)\) be a complete metric space. If \(F : X \rightarrow CB(X)\) is a set-valued contraction mapping, then \(F\) has a fixed point.

Remark 1.2.16 ([51]). Let \(H : \mathcal{H} \rightarrow \mathcal{H}\) be a single-valued mapping and \(M : \mathcal{H} \rightarrow CB(\mathcal{H})\) be a set-valued mapping. Then the following relations hold:

(i) Every \(\lambda\)-ordered monotone mapping is a \(\lambda\)-weak ordered different comparison mapping.

(ii) If \(H = I\) (identity mapping), then a \(\gamma_1\)-ordered rectangular mapping is an ordered rectangular mapping.

Lemma 1.2.17 ([72]). For any members \(x, y, w \in \mathcal{E}\), the following relations hold:

(i) if \(x \leq y\), then \(x \lor y = y\), \(x \land y = x\);

(ii) \((x + w) \lor (y + w)\) exists and \((x + w) \lor (y + w) = (x \lor y) + w\);

(iii) \((x \land y) = (x + y) - (x \lor y)\);

(iv) for \(\lambda \geq 0\), one can have \(\lambda(x \lor y) = \lambda x \lor \lambda y\);

(v) for \(\lambda \leq 0\), one can have \(\lambda(x \land y) = \lambda x \lor \lambda y\);

(vi) the converse parts of (v) and (vi) hold if \(x \neq y\);

(vii) either \(x \lor y\) or \(x \land y\) exists, then \(\mathcal{E}\) is a lattice;

(viii) \((x + w) \land (y + w)\) exists and \((x + w) \land (y + w) = (x \land y) + w\);

(ix) \((x \land y) = -(x \lor -y)\);

(x) \((-x) \land (x) \leq \theta \leq (-x) \lor x\).

Definition 1.2.18 ([53]). Let \(\mathcal{E}\) be a real ordered Banach space, \(\Sigma\) be a non-void open subset of \(\mathcal{E}\) in which parameter \(\rho\) takes values and \(A, B : \mathcal{E} \times \Sigma \rightarrow \mathcal{E}\) be two parametric maps. Then

(i) \(A\) is said to be a comparison with respect to the argument \(\rho\), if for any \(\rho \in \Sigma\), each \(x(\rho), y(\rho) \in \mathcal{E}\), and \(x(\rho) \propto y(\rho)\) then \(A(x(\rho), \rho) \propto A(y(\rho), \rho)\), \(x(\rho) \propto A(x(\rho), \rho)\), and \(y(\rho) \propto A(y(\rho), \rho)\) hold.
(ii) \(A\) and \(B\) are termed as comparative to each other with respect to the argument \(\rho\), if for each \(x(\rho) \in E\), \(A(x(\rho), \rho) \propto B(x(\rho), \rho)\) (denoted by \(A \propto B\)) holds.

**Definition 1.2.19** ([53]). Let \(E\) be a real ordered Banach space, \(C\) be a normal cone with normal constant \(N\) in \(E\), \(\Sigma\) be a non-void open subset of \(E\) in which parameter \(\rho\) takes values, and \(A : E \times \Sigma \to E\) be a parametric map. Then \(A\) is pronounced as \(\beta\)-ordered compression with respect to the second argument \(\rho\), if \(A\) is comparative with respect to argument \(\rho\), and there exists a constant \(0 < \beta < 1\) such that for any \(\rho \in \Sigma\),

\[
A(x(\rho), \rho) \oplus A(y(\rho), \rho) \leq \beta(x(\rho) \oplus y(\rho))
\]

holds.

**Definition 1.2.20** ([53]). Let \(E\) be a real ordered Banach space, \(C\) be a normal cone with normal constant \(N\) in \(E\), \(\Sigma\) be a non-void open subset of \(E\) in which parameter \(\rho\) takes values, \(A, B : E \times \Sigma \to E\) be two parametric maps, and \(I\) be an identity map on \(E \times E\).

(i) Map \(A\) is said to be restricted-accretive with respect to the argument \(\rho\), if \(A\) is comparative, and there exist two constants \(0 < \alpha_1, \alpha_2 \leq 1\) such that for arbitrary \(x(\rho), y(\rho) \in E\) and \(\rho \in \Sigma\),

\[
(A(x(\rho), \rho) + I(x(\rho), \rho)) \oplus (A(y(\rho), \rho) + I(y(\rho), \rho)) \leq \alpha_1(A(x(\rho), \rho) \oplus A(y(\rho), \rho)) \\
+ \alpha_2(x(\rho) \oplus y(\rho))
\]

holds.

(ii) Map \(A : E \times \Sigma \to E\) is called as \(B\)-restricted-accretive map with respect to argument \(\rho\), if \(A, B\) and \(A \wedge B : E \times \Sigma \to A(x(\rho), \rho) \wedge B(x(\rho), \rho) \in E\) all are comparative and comparative to each other with respect to the argument \(\rho\), and there exist two constant \(0 < \alpha_1, \alpha_2 \leq 1\) such that for any \(\rho \in \Sigma\) and arbitrary \(x(\rho), y(\rho) \in E\)

\[
(A(x(\rho), \rho) \wedge B(x(\rho), \rho) + I(x(\rho), \rho)) \oplus (A(y(\rho), \rho) \wedge B(y(\rho), \rho) + I(y(\rho), \rho)) \\
\leq \alpha_1\left((A(x(\rho), \rho) \wedge B(x(\rho), \rho)) \oplus (A(y(\rho), \rho) \wedge B(y(\rho), \rho))\right) + \alpha_2(x(\rho) \oplus y(\rho))
\]

holds.
Definition 1.2.21 ([52]). Let \( A : \mathcal{E} \to \mathcal{E} \) and \( M : \mathcal{E} \to 2^\mathcal{E} \) be single-valued and set-valued mappings respectively.

(i) \( M \) is called a weak-comparison map, if for \( t_x \in M(x) \), \( x \propto t_x \), and if \( x \propto y \), then \( \exists t_x \in M(x) \) and \( t_y \in M(y) \) such that \( t_x \propto t_y \), for all \( x, y \in \mathcal{E} \);

(ii) \( M \) is called an \( \alpha \)-weak-non-ordinary difference map associated with \( A \), if it is weak comparison and for each \( x, y \in \mathcal{E} \), \( \exists \alpha > 0 \) and \( t_x \in M(A(x)) \) and \( t_y \in M(A(y)) \) such that

\[
(t_x \oplus t_y) \oplus \alpha(A(x) \oplus A(y)) = \theta;
\]

(iii) \( M \) is said to be a weak comparison mapping with respect to \( A \), if for any \( x,y \in \mathcal{E} \), \( x \propto y \), then there exist \( v_x \in M(A(x)) \) and \( v_y \in M(A(y)) \) such that \( x \propto v_x \), \( y \propto v_y \), where \( v_x \) and \( v_y \) are said to be weak comparison elements, respectively;

(iv) \( M, \) a weak-comparison map is called an ordered \( (\alpha, \lambda) \)-weak-ANODM map, if it is an \( \alpha \)-weak-non-ordinary difference map and \( \lambda \)-order different weak-comparison map associated with \( A \), and \( (A + \lambda M)(\mathcal{E}) = \mathcal{E} \), for \( \alpha, \lambda > 0 \).

Definition 1.2.22 ([2, 48, 51]). Let \( A : \mathcal{H} \to \mathcal{H} \) be a strong comparison and \( \beta \)-ordered compression mapping and \( M : \mathcal{H} \to CH(\mathcal{H}) \) to be set-valued mapping. Then

(i) a comparison mapping \( M \) is said to be ordered rectangular, if for each \( x,y \in \mathcal{H} \), \( v_x \in M(x) \) and \( v_y \in M(y) \) such that

\[
\langle v_x \odot v_y, -(x \oplus y) \rangle = 0;
\]

(ii) a comparison mapping \( M \) is said to be a \( \gamma \)-ordered rectangular with respect to \( A \), if there exists a constant \( \gamma_A > 0 \) for any \( x,y \in \mathcal{E} \), there exist \( v_x \in M(A(x)) \) and \( v_y \in M(A(y)) \) such that

\[
\langle v_x \odot v_y, -(A(x) \oplus A(y)) \rangle \geq \gamma_A \|A(x) \oplus A(y)\|^2,
\]

holds, where \( v_x \) and \( v_y \) are said to be \( \gamma_A \)-elements, respectively;

(iii) \( M \) is said to be a \( \lambda \)-weak ordered different comparison mapping with respect to \( A \), if there exists a constant \( \lambda > 0 \) such that for any \( x,y \in \mathcal{H} \), there exist \( v_x \in M(A(x)) \) and \( v_y \in M(A(y)) \), \( \lambda(v_x - v_y) \propto (x - y) \) holds, where \( v_x \) and \( v_y \) are said to be \( \lambda \)-elements, respectively;
1.2. Basic definitions and results

(iv) a weak comparison mapping \( M \) is said to be a \((\gamma_A, \lambda)\)-weak ARD mapping with respect to \( A \), if \( M \) is a \( \gamma_A \)-ordered rectangular and \( \lambda \)-weak ordered different comparison mapping with respect to \( A \) and \((A + \lambda M)(H) = H\), for \( \lambda > 0 \) and there exist \( v_x \in M(A(x)) \) and \( v_y \in M(A(y)) \) such that \( v_x \) and \( v_y \) are \((\gamma_A, \lambda)\)-elements, respectively.

**Definition 1.2.23** ([52]). Let \( A : \mathcal{E} \to \mathcal{E} \) and \( M : \mathcal{E} \to 2^\mathcal{E} \) be a \( \gamma \)-order non-extended map and an \( \alpha \)-non-ordinary difference mapping with respect to \( A \), respectively. The resolvent operator \( R_{A,\lambda}^M : \mathcal{E} \to \mathcal{E} \) associated with both \( A \) and \( M \) is defined by

\[
R_{A,\lambda}^M(x) = (A + \lambda M)^{-1}(x), \quad \text{for all } x \in \mathcal{E},
\]

where \( \gamma, \alpha, \lambda > 0 \) are constants.

**Lemma 1.2.24** ([52]). If \( M : \mathcal{E} \to 2^\mathcal{E} \) and \( A : \mathcal{E} \to \mathcal{E} \) are an \( \alpha \)-weak-non-ordinary difference map associated with \( A \) and a \( \gamma \)-order non-extended map, respectively with \( \alpha \lambda \neq 1 \), then \( M_\theta = \{ \theta \oplus x \mid x \in M \} \) is an \( \alpha \)-weak-non-ordinary difference map associated with \( A \) and the resolvent operator \( R_{A,\lambda}^{M_\theta} = (A + \lambda M_\theta)^{-1} \) of \((A + \lambda M_\theta)\) is a single-valued for \( \alpha, \lambda > 0 \), i.e., \( R_{A,\lambda}^{M_\theta} : \mathcal{E} \to \mathcal{E} \) of \( M_\theta \) holds.

**Lemma 1.2.25** ([52]). Let \( M : \mathcal{E} \to 2^\mathcal{E} \) and \( A : \mathcal{E} \to \mathcal{E} \) be a \((\alpha_A, \lambda)\)-weak-ANODD set-valued map and a strongly comparison map associated with \( R_{A,\lambda}^M \), respectively. Then, the resolvent operator \( R_{A,\lambda}^M : \mathcal{E} \to \mathcal{E} \) is a comparison map.

**Lemma 1.2.26** ([52]). Let \( M : \mathcal{E} \to 2^\mathcal{E} \) be an ordered \((\alpha_A, \lambda)\)-weak-ANODD map and \( A : \mathcal{E} \to \mathcal{E} \) be a \( \gamma \)-ordered non-extended map associated with \( R_{A,\lambda}^M \), for \( \alpha_A > \frac{1}{\lambda} \), respectively. Then, the following relation holds:

\[
R_{A,\lambda}^M(x) \oplus R_{A,\lambda}^M(y) \leq \frac{1}{\gamma(\alpha_A \lambda - 1)}(x \oplus y), \quad \text{for all } x, y \in \mathcal{E}.
\]

**Definition 1.2.27** ([57]). A set-valued mapping \( A : \mathcal{E} \to CB(\mathcal{E}) \) is said to be \( \delta_A \)-Lipschitz continuous, if for each \( x, y \in \mathcal{E}, x \propto y \), there exists a constant \( \delta_A \) such that

\[
D(A(x), A(y)) \leq \delta_A \|x \oplus y\|, \quad \text{for all } x, y \in \mathcal{E}.
\]

**Definition 1.2.28** ([2]). Let \( M : \mathcal{H} \to CB(\mathcal{H}) \) be a set-valued mapping, \( A : \mathcal{H} \to \mathcal{H} \) be a single valued mapping and \( I : \mathcal{H} \to \mathcal{H} \) be an identity mapping. Then a weak comparison
mapping $M$ is said to be a $(\gamma', \lambda)$-weak-ARD mapping with respect to $(I - A)$, if $M$ is a $\gamma'$-ordered rectangular and $\lambda$-weak ordered different comparison mapping with respect to $(I - A)$ and $[(I - A) + \lambda M](H) = H$, for $\lambda > 0$ and there exist $v_x \in M((I - A)(x))$ and $v_y \in M((I - A)(y))$ such that $v_x$ and $v_y$ are called $(\gamma', \lambda)$-elements, respectively.

**Definition 1.2.29** ([2]). Let $C$ be a normal cone with normal constant $\lambda_C$ and $M : H \to CB(H)$ be a weak-ARD set-valued mapping. Let $I : H \to H$ be the identity mapping and $A : H \to H$ be a single-valued mapping. The relaxed resolvent operator $R_{M,\lambda}^{(I - A)} : H \to H$ associated with $I$, $A$ and $M$ is defined by

$$R_{M,\lambda}^{(I - A)}(x) = [(I - A) + \lambda M]^{-1}(x), \quad \text{for all } x \in H \text{ and } \lambda > 0. \quad (1.2.3)$$

The relaxed resolvent operator defined by (1.2.3) is single-valued, a comparison mapping as well as Lipschitz continuous.

**Proposition 1.2.30** ([2]). Let $A : H \to H$ be a $\beta$-ordered compression mapping and $M : H \to CB(H)$ be the set-valued ordered rectangular mapping. Then the resolvent $R_{M,\lambda}^{(I - A)} : H \to H$, is single-valued, for all $\lambda > 0$.

**Proposition 1.2.31** ([2]). Let $M : H \to CB(H)$ be a $(\gamma_A, \lambda)$-weak-ARD set-valued mapping with respect to $R_{M,\lambda}^{(I - A)}$. Let $A : H \to H$ be a strongly comparison mapping with respect to $R_{M,\lambda}^{(I - A)}$ and $I : H \to H$ be the identity mapping. Then the resolvent operator $R_{M,\lambda}^{(I - A)} : H \to H$ is a comparison mapping.

**Proposition 1.2.32** ([2]). Let $M : H \to CB(H)$ be a $(\gamma_A, \lambda)$-weak-ARD set-valued mapping with respect to $R_{M,\lambda}^{(I - A)}$. Let $A : H \to H$ be a strongly comparison and $\beta$-ordered compression mapping with respect to $R_{M,\lambda}^{(I - A)}$ with condition $\lambda \gamma_A > \beta + 1$. Then the following condition holds:

$$\|R_{M,\lambda}^{(I - A)}(x) \oplus R_{M,\lambda}^{(I - A)}(y)\| \leq \left(\frac{1}{\lambda \gamma_A - \beta - 1}\right)\|x \oplus y\|.

**Definition 1.2.33** ([39]). Let $H : H \to H$ be a single valued mapping. A mapping $T : H \to H$ is said to be

(i) monotone if

$$(Tx - Ty, x - y) \geq 0, \quad \text{for all } x, y \in H.$$
(ii) strictly monotone if \( T \) is monotone and
\[
\langle Tx - Ty, x - y \rangle = 0, \text{ if and only if } x = y;
\]

(iii) strongly monotone if there exists a constant \( r > 0 \) such that
\[
\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2, \text{ for all } x, y \in \mathcal{H};
\]

(iv) Lipschitz continuous if there exists a constant \( s > 0 \) such that
\[
\|Tx - Ty\| \leq s\|x - y\|, \text{ for all } x, y \in \mathcal{H}.
\]

(v) relaxed Lipschitz continuous if there exists a constant \( r > 0 \) such that
\[
\langle Tx - Ty, x - y \rangle \leq (-r)\|x - y\|^2, \text{ for all } x, y \in \mathcal{H}.
\]

**Definition 1.2.34 ([39]).** A multi-valued mapping \( M : \mathcal{H} \to 2^{\mathcal{H}} \) is said to be

(i) monotone if
\[
\langle x - y, u - v \rangle \geq 0, \forall \ u, v \in \mathcal{H} \text{ and } x \in M(u), y \in M(v);
\]

(ii) strongly monotone if there exists a constant \( \eta > 0 \) such that
\[
\langle x - y, u - v \rangle \geq \eta\|u - v\|^2, \forall \ u, v \in \mathcal{H} \text{ and } x \in M(u), y \in M(v);
\]

(iii) maximal monotone if \( M \) is monotone and \( (I + \lambda M)(\mathcal{H}) = \mathcal{H} \) holds for all \( \lambda > 0 \), where \( I \) is the identity operator on \( \mathcal{H} \);

(iv) maximal strongly monotone if \( M \) is strongly monotone and \( (I + \lambda M)(\mathcal{H}) = \mathcal{H} \) holds for all \( \lambda > 0 \).

### 1.3 Brief survey of variational inequalities

In this section, we discuss variational inequality problem and generalized equations and subsequently the relationship between these two notions. Furthermore, we present various forms of variational inequalities.
Let $\mathcal{H}$ be a Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{C} \subset \mathcal{H}$ be a non-void closed and convex subset of Hilbert space. Let $a(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form. Then the problem is to find $x \in \mathcal{C}$ such that

$$a(x, y - x) \geq \langle f, y - x \rangle, \quad \text{for all } y \in \mathcal{C} \text{ and } f \in \mathcal{H}^*.$$  \hspace{1cm} (1.3.1)

The problem (1.3.1) is known as variational inequality problem which characterizes the classical Signorini problem of elastostatics, i.e., the analysis of a linear elastic in contact with a rigid frictionless base. This problem was investigated by Lions and Stampacchia [44], by using the projection method.

Let the bilinear form $a(x, y)$ be continuous, which in turns, implies

$$a(x, y) = \langle Ax, y \rangle, \quad \text{for all } x, y \in \mathcal{H}, \text{ (by Riesz-Frèchet theorem)}$$

where $A : \mathcal{H} \to \mathcal{H}^*$ be a continuous linear mapping. Then the problem (1.3.1) is equivalent to the following problem:

Find $x \in \mathcal{C}$ such that

$$\langle Ax, y - x \rangle \geq \langle f, y - x \rangle, \quad \text{for all } y \in \mathcal{C}. \hspace{1cm} (1.3.2)$$

Since the general problem of equilibrium of elastic bodies in contact with rigid base, on which frictional forces are developed, is one of the most toughest problem in solid mechanics. Davaut and Lions [26], investigated the following variational inequality problem with friction:

For $f \in \mathcal{H}^*$, find $x \in \mathcal{C}$ such that

$$a(x, y - x) + \phi(y) - \phi(x) \geq \langle f, y - x \rangle, \quad \text{for all } y \in \mathcal{C}, \hspace{1cm} (1.3.3)$$

where $\phi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is convex, lower semi-continuous proper functional. The problem (1.3.3) characterizes the classical Signorini problem with friction force. A very useful generalization of the problem (1.3.3) is the following problem

Given $f \in \mathcal{H}^*$, find $x \in \mathcal{C}$ such that

$$a(x, y - x) + b(x, y) - b(x, x) \geq \langle f, y - x \rangle, \quad \text{for all } y \in \mathcal{C}, \hspace{1cm} (1.3.4)$$

where $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a suitable nonlinear form. This type of problem has been widely considered by many authors, see [26, 42, 59]. The problem (1.3.4) characterizes the fluid flow through porous media and the Signorini problems with non-local frictions.
Noor [60], in 1975, extended the problem (1.3.1) to a class of mildly nonlinear elliptic boundary value problems having constraints. For nonlinear mappings \( T, A : \mathcal{H} \to \mathcal{H}^* \), he considered the following problem:

Find \( x \in C \) such that

\[
\langle T(x), y - x \rangle \leq \langle A(x), y - x \rangle, \quad \text{for all } y \in C.
\]  

(1.3.5)

The problem (1.3.5) is called as mildly nonlinear variational inequality.

For a convex, lower semi-continuous proper non-differentiable functional \( \phi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \), the problem (1.3.5) has been generalized by Siddiqi et al. [73] in the setting of Banach space, i.e., to find \( x \in \mathcal{H} \) such that

\[
\langle T(x), y - x \rangle + \phi(y) - \phi(x) \geq \langle A(x), y - x \rangle, \quad \text{for all } y \in C.
\]  

(1.3.6)

If the underlying region in inequalities (1.3.4) or (1.3.6), is the whole space \( \mathcal{H} \) instead of a closed, convex set \( C \) of \( \mathcal{H} \), then such kind of inequality is known as variational inclusion problems provided the functional \( b(\cdot, \cdot) \) (or \( \phi(\cdot) \)) is differentiable.

For continuous mappings \( T, g : \mathcal{H} \to \mathcal{H} \), with \( \text{Im}(g) \cap \text{dom}(\partial \phi) \neq \emptyset \), the problem is to find \( x \in \mathcal{H} \) such that

\[
\langle T(x) - A(x), y - g(x) \rangle \geq \phi(g(x)) - \phi(y), \quad \text{for all } y \in \mathcal{H},
\]  

(1.3.7)

where \( A \) is a nonlinear continuous mapping on \( \mathcal{H} \), \( \partial \phi \) denotes the sub-differential of a proper, convex and lower semi-continuous function \( \phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\} \), \( \text{dom}(\partial \phi) \) denotes the domain of \( \partial \phi \). Problem (1.3.7) was introduced and studied by Hassouni and Moudafi [40] in 1994.

Let \( \phi = \delta_C \), the indicator function of a closed convex set \( C \) in \( \mathcal{H} \), be defined by

\[
\delta_C = \begin{cases} 
0, & \text{if } x \in C, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

Then the variational inclusion problem (1.3.7) reduces to the following strongly nonlinear variational inequality problem

\[
\langle T(x) - A(x), y - g(x) \rangle \geq 0, \quad \text{for all } y \in C,
\]  

(1.3.8)

which was studied by Noor [62].
Let $T : \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued maximal monotone operator, $S, A : \mathcal{H} \rightarrow \mathcal{H}$ be two mappings such that $\text{range}(S) \cap \text{dom}(T) \neq \emptyset$, and $B : \mathcal{H} \rightarrow \mathcal{H}$ is a nonlinear continuous mapping.

Then the problem, to find $x \in \mathcal{H}$ such that

$$0 \in (A - B)(x) + T(S(x)), \quad (1.3.9)$$

is known as general class of variational inclusion problems and was introduced and studied by S. Adly [4].

Let $E$ be a Banach space, $T : E \rightarrow E$ be a single-valued mapping and $M : E \rightarrow 2^E$ be a set-valued mapping. Then the problem, to find $x \in E$ such that

$$0 \in T(x) + M(x), \quad (1.3.10)$$

is called as generalized variational inclusion problem and was Zou and Huang [88].

Contrary to the study of variational inequality (inclusion) systems in which a lot of work has been done, the study of ordered variational inequality (inclusion) systems on an ordered Banach spaces is in its starting phase (see citations [7, 29, 49, 51, 53, 54, 35, 36]). Some new and interesting problems for systems of variational inequalities (inclusions) have been introduced and studied in this field, since 1999, e.g., the approximation solution for general nonlinear ordered variational inequalities and ordered equations [53, 54] and a nonlinear ordered inclusion problem [49, 51] have been studied by Li in an ordered Banach space.

Let $\mathcal{E}$ be a real ordered Banach space with a norm $\|\cdot\|$, a zero element $\theta$, a normal cone $C$, a normal constant $N$ of $C$ and a partial ordered relation “$\leq$” be defined by the cone $C$. Let $F_{ij}, g, f : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ be single-valued nonlinear ordered compression mappings, and $F_{ij}$ be a Lipschitz continuous mapping ($i, j = 1, 2$), and for any $x, y \in \mathcal{E}$, range($f$) $\cap$ dom($F_{11}(\cdot, y)$) $\cap$ range($g$) $\cap$ dom($F_{22}(x, \cdot)$) $\neq \emptyset$, Li et al. [46]. Then the problem,

for $u, v \in \mathcal{E}$, to find $x, y \in \mathcal{E}$ such that

$$\begin{cases}
u \in F_{11}(f(x), y) + F_{12}(y, x), \\
v \in F_{21}(x, y) \oplus F_{22}(x, g(y)),
\end{cases} \quad (1.3.11)$$

is called a generalized nonlinear mixed ordered variational inequality system (GNM ordered variational inequality system) with ordered Lipschitz continuous mappings in an ordered
Banach space. Obviously, system (1.3.11) belongs to a new class of generalized nonlinear mixed ordered variational inequality systems with the $\oplus$ calculation.

For a suitable choice of the mappings $u, v, f, g, F_{ij}$ ($i, j = 1, 2$) and the space $\mathcal{E}$, a number of known classes of ordered variational inequalities, have been studied by the authors as special cases of system (1.3.11) in ordered Banach space (see [53, 54, 74]).

With all these motivation, I, under the supervision of Prof. Md. Kalimuddin Ahmad, have written my doctoral thesis.