Chapter 5

Generalized Mathematical Model

5.1 Modelling

From the previous two-layer model, we generalized it with a three-layer model. Here it is assumed that the co-axial flows of serous, mucus and air are taking place in a circular tube model representing an airway. In the central core, air is assumed to flow under quasi-steady state turbulent condition, the mucus layer surrounding this central core is assumed to flow under steady laminar condition and the serous layer surrounding this mucus layer is assumed to flow under steady laminar condition. Mucus-Serous and Air-Mucus are the two interfaces in the Circular tube. The main aim is to study the effect of flow of mucus and serous fluid viscosity as well as effect of mucus and serous fluid layer thickness on mucus transport.

The flow of fluid is along the horizontal z-direction (Figure 9). The flow geometry in the figure has air flows in the region $0 \leq r \leq R_a$, mucus flows in the region $R_a \leq r \leq R_m$ and serous fluid flows in the region $R_m \leq r \leq R_s$.

![Figure 9. Tube model for mucus transport in the airway](image)

Here we can take same assumption as taken in the previous two layer model.

The means of steady state equations of serous and mucus in the laminar layer and quasi steady state equation in the turbulent layer can be written in cylindrical coordinates as follows:
Governing Equations with Initial, Matching and Boundary Conditions

**Region I:** Steady laminar flow of serous ($ R_m \leq r \leq R_s $):

\[
-\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r}(r \tau_s) = 0 \tag{20}
\]
\[
\tau_s = \mu_s \frac{\partial u_s}{\partial r} \tag{21}
\]

**Region II:** Steady laminar flow of mucus ($ R_s \leq r \leq R_m $):

\[
-\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r}(r \tau_m) = 0 \tag{22}
\]
\[
\tau_m = \mu_m \frac{\partial u_m}{\partial r} \tag{23}
\]

**Region III:** Quasi steady turbulent flow of air ($ 0 \leq r \leq R_a $):

\[
-\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r}(r \tau_a) = 0 \tag{24}
\]
\[
\tau_a = -\rho_a \ell^2 \left( -\frac{\partial u_a}{\partial r} \right)^2 \tag{25}
\]

Where $ r $ is the coordinate in the radial direction and perpendicular to fluid flows, $ R_m $ and $ R_a $ is the thickness up to mucus-serous and air-mucus interface respectively, $ u_s, u_m $ and $ u_a $ are the mean velocity components of serous, mucus and air in the z-direction respectively, $ \tau_s $ and $ \tau_m $ are the mean shear stresses in the serous and mucus layer respectively and $ \tau_a $ is the mean shear stress in the air, $ \rho_a $ is the density of air and $ \mu_s $ and $ \mu_m $ are viscosity of serous and mucus respectively.

The mixing length $ \ell $ is assumed as follows:

\[
\ell = \ell_0 (R_s - r) \tag{26}
\]

Where $ \ell_0 $ is proportionality constant and determined experimentally.

**Initial Conditions**

Initially there is no pressure gradient, one can assume that the velocities and stresses are zero, therefore, the initial conditions are

\[
u_a = u_m = u_s = 0, \tau_a = \tau_m = \tau_s = 0, \frac{\partial u_m}{\partial r} = 0 \tag{27}
\]
Matching Conditions
Since the velocities and stresses are continuous at the two interfaces $r = R_a$ and $r = R_m$. Therefore, the matching conditions are

$$u_a = u_m, \tau_a = \tau_m \text{ at } r = R_a \quad (28)$$

$$u_m = u_s, \tau_m = \tau_s \text{ at } r = R_m \quad (29)$$

Boundary Conditions
Due to symmetry at $r = 0$ and no slip at $r = R_s$, we have the boundary conditions as

$$\frac{\partial u_a}{\partial r} = 0 \text{ at } r = 0 \quad (30)$$

$$u_s = 0 \text{ at } r = R_s \quad (31)$$