Chapter 3

Mathematical Model

3.1 Modelling

It is assumed that the co-axial flows of mucus and air are taking place in a circular tube model representing an airway. In the central core, air is assumed to flow under quasi-steady state turbulent condition and the mucus layer surrounding this central core is assumed to flow under steady laminar condition. The main aim is to study the effect of flow of mucus viscosity and mucus layer thickness on mucus transport.

The flow of fluid is along the horizontal z-direction (Figure 4). The flow geometry in the figure have air flows in the region \(0 \leq r \leq R_a\), and mucus flows in the region \(R_a \leq r \leq R_m\).

Figure 4. Tube model for mucus transport in the airway

The cough is represented by a time dependent pressure gradient function. Therefore we assume that

\[
-\frac{\partial p}{\partial z} = \varphi_o (t) = \varphi_0 f (t)
\]

(1)

Where \(t\) is the time, \(p\) is the mean pressure which is constant across two layers, \(z\) is the axial coordinate of the circular tube in the direction of flow, \(\varphi_o\) is the strength of the cough, the magnitude of which depends upon the intensity of normal cough and as this increases flow rates also increases. \(f(t)\) is prescribed by the following expression:
\[ f(t) = \begin{cases} \frac{27t(T-t)^2}{4T^3}, & 0 < t < T \\ 0, & t > T \end{cases} \] (2)

Where \( T \) is the duration of cough. The function \( f(t) \) represents the coughing for \( T = 0.03 \text{sec} \) (Figure 5).

In view of the above considerations and using Prandtl mixing length theory, the means of steady state equations of mucus in the laminar layer and quasi-steady state equation in the turbulent layer can be written in cylindrical coordinates as follows:

3.2 Governing Equations with Initial, Matching and Boundary Conditions

Region I: Steady laminar flow of mucus \(( R_a \leq r \leq R_m)\):

\[ - \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_m \right) = 0 \] (3)

\[ \tau_m = \mu_m \frac{\partial u_m}{\partial r} \] (4)

Region II: Quasi-steady turbulent flow of air \(( 0 \leq r \leq R_a)\):
\[-\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r \tau_a) = 0 \]  

(5)

\[\tau_a = -\rho_a \ell^2 \left( -\frac{\partial u_a}{\partial r} \right)^2 \]  

(6)

Where \( r \) is the coordinate in the radial direction and perpendicular to fluid flows, \( R_a \) is the thickness up to air-mucus interface, \( u_m \) and \( u_a \) are the mean velocity components of mucus and air in the \( z \)-direction respectively, \( \tau_m \) is the mean shear stress in the mucus layer and \( \tau_a \) is the mean shear stress in the air, \( \rho_a \) is the density of air and \( \mu_m \) is viscosity of mucus.

The mixing length \( \ell \) is assumed as follows:

\[ \ell = \ell_0 (R_m - r) \]  

(7)

Where \( \ell_0 \) is proportionality constant and determined experimentally.

**Initial Conditions**

Initially there is no pressure gradient, one can assume that the velocities and stresses are zero, therefore, the initial conditions are

\[ u_a = u_m = 0, \tau_a = \tau_m = 0, \frac{\partial u_m}{\partial r} = 0 \]  

(8)

**Matching Conditions**

Since the velocities and stresses are continuous at the interface \( r = R_a \). Therefore, the matching conditions are

\[ u_a = u_m, \tau_a = \tau_m \] at \( r = R_a \)  

(9)

**Boundary Conditions**

Due to symmetry at \( r = 0 \) and no-slip at \( r = R_m \) we have the boundary conditions as

\[ \frac{\partial u_a}{\partial r} = 0 \] at \( r = 0 \)  

(10)

\[ u_m = 0 \] at \( r = R_m \)  

(11)