Chapter 1

Introduction

1.1 Motivation and Survey of Literature

The Classical theorems of Korovkin impressed several mathematicians since their discovery for the simplicity and the potential. Positive approximation process play a fundamental role in the approximation theory and it appears in a very natural way in several problems dealing with the approximation of continuous functions and qualitative properties such as monotonicity, convexity, shape preservation and so on.

Korovkin [28,29] in 1953 proved the most powerful and simplest criterion to decide whether a given sequence $(\phi_n)_{n \in \mathbb{N}}$ of positive linear operators on the space of continuous functions $C([0,1])$ is an approximation process, that is $\phi_n(f) \to f$ uniformly on $[0,1]$ for every $f \in C[0,1]$. In fact it is sufficient to verify that $\phi_n(f) \to f$ uniformly on $[0,1]$ only for $f \in \{1, x, x^2\}$. The set $\{1, x, x^2\}$ is called a Korovkin’s set or a test
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A considerable amount of research extended the Korovkin's theorems to the setting of different function spaces or more general abstract spaces such as Banach spaces, Banach algebras, Banach lattices, $C^*$-algebras and so on during last fifty years. At the same time, strong and fruitful connections of Korovkin’s theory have been revealed not only with classical approximation theory but also with other fields such as functional analysis, measure theory, harmonic analysis, partial differential equations, probability theory and so on.

Another major advancement was the discovery of geometric theory of Korovkin's sets by Saskin [46] in 1966 and Wulbert [53] in 1968. A detailed survey of the most of these developments can be found in the survey article of Berens and Lorentz [11] in 1975. A selected part of the theory is already documented in the monograph of Altomare and Campiti [2] and survey article of Altomare [3].

Priestley [41] in 1976 initiated the study of Korovkin’s theorem in $C^*$-algebras. Priestley proved that for a $C^*$-algebra $A$ with identity $I$, if $\{\phi_n\}_{n \in \mathbb{N}}$ is a sequence of positive linear maps from $A$ into $A$ satisfying $\phi_n(I) \leq I$ for all $n$, then

$$C = \{a \in A : a = a^*, \phi_n(a) \to a, \phi_n(a^2) \to a^2\}$$

is a $J^*$-algebra (i.e, a norm closed Jordan algebra of self adjoint elements of $A$). Recall that a Jordan algebra in $A$ is a linear subspace of $A$ closed under the Jordan product $a \circ b = (ab + ba)/2$. The theorem holds for the operator norm convergence, the weak operator convergence and the strong operator convergence. Also, Priestley established
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the above results in the trace norm convergence when \( \{\phi_n\} \) acts on the trace class operators on \( B(H) \).

Robertson [42] in 1977 generalized Priestley’s results to complex \( C^* \)-algebras using ideas of Palmer [37] for large class of positive linear operators and obtained that the set \( C \) is actually a \( C^* \)-algebra. Robertson proved that if \( \{\phi_n\}_{n \in \mathbb{N}} \) is a sequence of Schwarz maps for a \( C^* \)-algebra \( A \) such that \( \phi_n(I) \leq I \) for all \( n \), then the set

\[
D = \{ a \in A : \|\phi_n(x) - x\| \to 0 \text{ for } x = a, a^*a, aa^* \}
\]

is a \( C^* \)-algebra. Meanwhile, in 1979 Takahasi [50] improved Priestley’s results in \( C^* \)-algebras considering norm convergence and without the assumption \( a = a^* \).

Limaye and Namboodiri [30] in 1982 generalized the results of Priestley and Robertson and obtained the following result. Let \( A \) and \( B \) be complex \( C^* \)-algebras with identity, let \( \{\phi_n\}_{n \in \mathbb{N}} \) be a sequence of positive linear maps from \( A \) into \( B \) and satisfying \( \phi_n(I) \leq I \) for all \( n \) and \( \phi \) is a \( C^* \)-homomorphism from \( A \) to \( B \). Then

\[
E = \{ a \in A : \phi_n(a) \to \phi(a), \phi_n(a^* \circ a) \to \phi(a^* \circ a) \}
\]

is a norm-closed *-subspace of \( A \) and is closed under the Jordan product. If all \( \phi_n \) and \( \phi \) are Schwarz maps, then \( E \) is a \( C^* \)-subalgebra of \( A \). The theorem holds for the operator norm convergence, the weak operator convergence and the strong operator convergence. A slight modification of this theorem for the convergence in the trace norm is also proved in [30].
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Arveson [4] in 1969 introduced the notion of boundary representation which is a non-commutative counterpart of a point in the Choquet boundary for function system in $C(X)$. The Choquet boundary of a function system is the set of points with unique representing measures. Let $A$ be an abstract $C^*$-algebra and $S$ be a linear subspace of $A$. An irreducible representation $\pi : A \rightarrow B(H)$ is called boundary representation for $S$ if the only completely positive map from $A$ to $B(H)$ which agrees with $\pi$ on $S$ is $\pi$ itself; that is boundary representation has the unique completely positive extension from its restrictions to $S$.

Arveson introduced boundary representations to study to what extent does a subspace of operators on a Hilbert space determine the structure of the $C^*$-algebra it generates. Arveson proposed that there should be sufficiently many boundary representations, so that their direct sum recovers the norm on $M_n(A)$ for all $n \geq 1$. The $C^*$-algebra generated by this direct sum enjoys universal property and provides a realization of the $C^*$-envelop.

The existence of boundary representations has nice relation with the non-commutative Silov boundary. The first goal was achieved by Arveson [5] in 1972 by giving several concrete examples and developing applications to operator theory. However, the existence of boundary representations and the Silov boundary were left open in general.

Tensor products of operator spaces (linear subspaces) of $C^*$-algebras and operator spaces of tensor product of $C^*$-algebras where explored by Hopenwasser in [23] and [24] to study boundary representations. In [23] it was shown that boundary representations of an operator subspace of a $C^*$-algebra $A \otimes M_n(\mathbb{C})$ under certain condi-
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tions are parametrised by the boundary representations of an operator subspace of the
\(C^*-\)algebra \(A\) which is given by an operator subspace in \(A \otimes M_n(C)\). In [24] it was
proved that if one of the \(C^*-\)algebras of the tensor product is a GCR algebra, then the
boundary representations of the tensor product of \(C^*-\)algebras correspond to products
of boundary representations.

Hamana [21,22] in 1979 was able to establish the existence of the non-commutative
Silov boundary by using his theory of injective envelopes. Hamana’s work made no
reference to boundary representations and left untouched the question of existence.

Muhly and Solel [34] in 1998 gave an algebraic characterization of boundary rep-
resentations in terms of Hilbert modules, but used a generalized version of bound-
ary representation by dropping the irreducibility condition. Muhly and Solel proved
that boundary representations of operator algebras may be characterized as those com-
pletely contractive representations that determine modules that are simultaneously or-
thogonally projective and orthogonally injective. However, their arguments used
Hamana’s techniques and therefore the results did not lead to a new construction of
the \(C^*-\)envelop.

Dritschel and McCullough [19] in 2005 took a major step forward by showing that
every unital completely positive map of an operator system into \(B(H)\) can be dilated
to a completely positive map with the unique extension property. This provided a new
proof of the existence of the non-commutative Silov boundary that makes no use of
injectivity. The motivation for Dritschel and McCullough was the work of Agler [1]
on a model theory for representations of non self-adjoint operator algebras. But their
results seem to give no information about the existence of boundary representations.
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Arveson [8] in 2008 settled the problem of the existence of boundary representations using the ideas of Dritschel and McCullough in the separable case. He used the disintegration theory of $C^*$-algebras and established that there exist sufficiently many boundary representations to completely norm it. That is, every separable operator system $S \subseteq C^*(S)$ has sufficiently many boundary representations in the sense that for every $n \geq 1$ and every $n \times n$ matrix $[s_{ij}]$ with components $s_{ij} \in S$, one has

$$||[s_{ij}]|| = \sup_\pi ||\pi([s_{ij}])||$$  \hspace{1cm} (1.1)

the supremum on the right hand side is taken over all boundary representations $\pi$ for $S$.

Kleski [26] in 2014 established some closely related results in the separable case. He proved that from equality 1.1 “sup” can be replaced by “max”. This implies that the Choquet boundary for a separable operator system is a boundary in the classical sense.

Finally, Davidson and Kennedy [16] in 2015 completely solved the existence of boundary representations using ideas of Arveson [4] and recent work of Dritschel and McCullough [19]. In particular their arguments neither require any disintegration theory nor they require separability. Therefore, every operator system in a $C^*$-algebra has sufficiently many boundary representations to completely norm it and hence they generate the $C^*$-envelop.

Saskin [46] in 1966 discovered an important geometric formulation of Korovkin’s theorem. In the classical case he explored the relation between the Korovkin sets and
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Choquet boundary as follows. Let $G$ be a subset of the continuous functions on the compact Hausdorff space $C(X)$ such that $G$ separates points of $X$, contains the constant function 1. Then $G$ is a Korovkin set if and only if the Choquet boundary of $S(= \text{linear span}(G))$ is whole of $X$.

Arveson [10] in 2011 tried to prove the non-commutative analogue of Saskin’s theorem using theory of non-commutative Choquet boundary for unital completely positive maps on $C^*$-algebras. For this purpose Arveson [10] introduced the non-commutative counterpart of the Korovkin’s set which he named as hyperrigid set. Arveson defined a hyperrigid set as follows: Let $G$ be a finite or countably infinite set that generates the abstract $C^*$-algebra $A = C^*(G)$. The set $G$ is said to be hyperrigid if for every faithful representation of $A$ on a Hilbert space $H$ and every sequence of unital completely positive maps $\phi_1, \phi_2, \ldots$ from $B(H)$ to itself

$$\lim_{n \to \infty} ||\phi_n(g) - g|| = 0, \forall \ g \in G \Rightarrow \lim_{n \to \infty} ||\phi_n(a) - a|| = 0, \forall \ a \in A.$$ 

Arveson [10] proved that if the separable operator system is hyperrigid in the $C^*$-algebra then every irreducible representation of $C^*$-algebra is a boundary representation for the operator system. The converse to this result is called ‘hyperrigidity conjecture’: that is if every irreducible representation of a $C^*$-algebra is a boundary representation for a separable operator system then the operator system is hyperrigid. Arveson [10] gave partial answer to the hyperrigidity conjecture. He showed that hyperrigidity conjecture is true for $C^*$-algebras with countable spectrum.
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Kleski [27] in 2014 established the hyperrigidity conjecture for all type-I $C^*$-algebras with additional assumptions on the co-domain. Kleski used the idea that every non-degenerate representation of the type-I $C^*$-algebras can be written as the direct integral of irreducible representations. Davidson and Kennedy [17] proved the conjecture for function systems. The hyperrigidity conjecture is still open for general $C^*$-algebras.

Bishop [12] in 1959 introduced the notion of peak points in the commutative case to study the generalization of the Choquet boundary based on slightly different ideas. Let $S$ be a linear subspace of $C(X)$, a point $x \in X$ is a peak point of $G$ if there exist a $f \in S$ such that $f(x) = ||f||$ and $|f(y)| < ||f||$, $x \neq y$. Suppose that $S$ separates the points of $X$ and contains the constant function $1$. Then the set of peak points of $S$ is a subset of the Choquet boundary of $S$ and also the Choquet boundary of $S$ is a subset of the closure of the set of peak points of $S$.

Arveson [10] in 2011 introduced the notion of peaking representation and strongly peaking representation which he used to improve his boundary theorem [5] which is as follows: Let $S$ be a separable operator system in $B(H)$ and let $A$ be the $C^*$-algebra generated by $S$. Let $K \neq 0$ be the ideal of compact operators in $A$ and let $\hat{K}$ be the set of unitary equivalence classes of irreducible representations of $A$ that live on $K$. Then $\hat{K}$ contains boundary representations for $S$ if and only if the quotient map $x \in A \mapsto \hat{x} \in A/K$ is not completely isometric on $S$. Assuming that is the case, then among the irreducible representations of $\hat{K}$, the boundary representations for $S$ are precisely the strongly peaking ones.
Peaking representations are a non-commutative generalization of peak points to operator systems. Like the classical case it is natural to enquire about the relation between the peaking representations and boundary representations. Arveson [9] proved that for a finite dimensional $C^*$-algebra all peaking representations are equivalent to the boundary representations. Kleski [26] established that for a separable operator system every peaking representation is a boundary representation.

Limaye and Namboodiri [32] in 1984 introduced the notion of weak Korovkin set in $B(H)$ using weak convergence of completely positive maps. Weak Korovkin set is a non-commutative analogue of the classical Korovkin set. They proved that an irreducible set in $B(H)$ is a weak Korovkin set if and only if the identity representation is a boundary representation for the irreducible set.

Namboodiri [36] in 2012, inspired by the work of Arveson [10] on hyperrigidity. He redefined the notion of weak Korovkin set as the weak hyperrigid set and explored the relation between the weak hyperrigid operator systems and boundary representations in [36]. Namboodiri gave a brief survey of the developments in non-commutative Korovkin-type theory in [35]. Uchiyama [52] proved the Korovkin type theorem for Schwarz maps using operator monotone functions in $C^*$-algebras.
1.2 Organisation of the Thesis

The notion of hyperrigidity introduced by Arveson [10] proved to be a very important idea connecting various directions of research in non-commutative approximation theory. Here we study the relation of hyperrigid operator systems to Hilbert modules, tensor product of hyperrigid operator systems, quasi hyperrigid operator systems, weak boundary representations and weak peak points.

In Chapter 1, we give the motivation and survey of various work about the classical Korovkin’s theorem, Choquet boundary and peak points. The developments of the non-commutative analogue to these notions such as hyperrigidity, boundary representations and peaking representations are explained.

In Chapter 2, we gather the preliminary ideas that we need in our study of hyperrigid operator systems in $C^*$-algebras making the thesis self-contained as much as possible. In Section 2.1, as a prerequisite, we require a basic knowledge of the theory of $C^*$-algebras, von Neumann algebras, representations of $C^*$-algebras, various types of $C^*$-algebras, operator spaces and operator algebras. In section 2.2, we provide the classical notion of Choquet boundary, Shilov boundary and peak points. A couple of theorems relating peak points and Choquet boundary are explained. In Section 2.3, we describe the classical Korovkin’s theorem, Korovkin set and Saskin’s theorem relating the Korovkin set and Choquet boundary. In Section 2.4, we discuss the concept of completely positive maps on $C^*$-algebras and Stinespring’s theorem for completely positive maps. In Section 2.5, we provide the concepts of boundary representation,
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unique extension property and $C^*$-envelope. We illustrate the developments in proving existence of boundary representations. In Section 2.6, we explain the notion of hyper-rigidity and hyperrigidity conjecture. We provide various partial answers available in the literature to the hyperrigidity conjecture. In Section 2.7, we describe the notion of peaking representations and explain the relation between peaking representations and boundary representations.

In Chapter 3, we study the algebraic characterization of hyperrigid operator systems in terms of Hilbert modules. In Section 3.1, we provide the notion of Hilbert modules over the operator algebras, short exact isometric sequence, orthogonally projective Hilbert module and orthogonally injective Hilbert module. We illustrate the theorem relating boundary representations and orthogonality properties of Hilbert modules. In Section 3.2, we discuss the theorems due to Arveson [4] concerning extensions of contractive linear maps on unital subspaces of $C^*$-algebras. In Section 3.3, for an operator algebra $\mathcal{A}$ and the operator system $\mathcal{S} = \mathcal{A} + \mathcal{A}^*$, we show that the unique extension property of the restriction to $\mathcal{S}$ of a representation of $C^*(\mathcal{S})$ is equivalent to the Hilbert modules over $\mathcal{A}$ corresponding to the representation being simultaneously orthogonally projective and orthogonally injective. This result leads to an algebraic characterization of hyperrigidity of the operator system $\mathcal{A} + \mathcal{A}^*$ in terms of the orthogonality properties of Hilbert modules over $\mathcal{A}$.

In Chapter 4, we study the tensor product of hyperrigid operator systems. In Section 4.1, we discuss the tensor product of $C^*$-algebras, tensor product of non-degenerate representations, spatial $C^*$-norm, maximal $C^*$-norm and nuclear $C^*$-algebras. In Section 4.2, we illustrate the work of Hopenwasser [23], [24] about the tensor product of
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boundary representations. In Section 4.3, we study hyperrigidity of operator systems in $C^*$-algebras in the context of tensor products of $C^*$-algebras. The question of whether tensor product of hyperrigid operator systems are hyperrigid is addressed here. By a result of Hopenwasser [24], tensor product of boundary representations of $C^*$-algebras for operator systems is a boundary representation if one of the constituent $C^*$-algebras is a GCR algebra. Since hyperrigidity implies that all irreducible representations are boundary representations, we will be able to deduce Hopenwasser’s result as a special case if we can prove a similar result for hyperrigidity. We achieve this by establishing first that unique extension property for unital completely positive maps on operator systems carry over to tensor product of those maps defined on the tensor product of operator systems in the spatial tensor product of $C^*$-algebras.

In Chapter 5, we study the notions of quasi hyperrigidity, weak boundary representations, weak peak points and their relations. In Section 5.1, we introduce weak boundary representations and study the relation between boundary representations and weak boundary representations for operator systems of $C^*$-algebras. We prove that irreducible finite representations of an operator system are equivalent to weak boundary representations. We introduce quasi hyperrigid sets in $C^*$-algebras and observe that hyperrigid sets are quasi hyperrigid but quasi hyperrigid sets need not be hyperrigid. We prove an analogue of Saskin’s theorem relating quasi hyperrigid operator systems and weak boundary representations for operator systems of $C^*$-algebras with countable spectrum. In Section 5.2, we introduce the notion of weak unique extension property. For type I $C^*$-algebras with an assumption on the co-domain of irreducible representations, we show that if an irreducible representation is a weak boundary representation
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for operator systems, then the operator system is quasi hyperrigid. In Section 5.3, we introduce the notion of weak peak points for operator systems in a $C^*$-algebra and prove that if an irreducible representation is a weak boundary and weak peak, then it is a boundary representation.

In Chapter 6, We discuss some problems for further research. The problems are described briefly.