1. INTRODUCTION:

Kaul [49] has defined and studied a four fold generalization to the hypersurface of a complex Finsler space of the curvature. This has identically led to the generalization of normal and secondary normal curvatures defined by Rund [99], the curvature of congruence obtained by Prakash [87] and also the normal curvature vector field given by Prakash [88]. Further, Singh [123*] has studied the curvature of a congruence relative to a vector field of a complex Finsler space and hypersurface and various cases in normal and secondary normal curvatures have been investigated. Also, Thomas [185] has defined and studied the variation of curvature in Riemann space of constant mean curvature and several results have been obtained by him. Next, Eliopoulos [33] has defined and studied the sub-
CHAPTER-IV

THE CURVATURE OF A CONGRUENCE RELATIVE TO A VECTOR FIELD OF A KAEHLERIAN SPACE AND RELATIVE ASSOCIATE CURVATURE OF A VECTOR FIELD IN HERMITIAN SPACES

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spaces of a generalized metric space and has proved several theorems. Further, Singh [121] has studied relative associate curvature of a vector field in Kaehlerian spaces and several results have been obtained by him.

In the present chapter, we have studied curvature of a congruence relative to a vector field in a Kaehlerian space. Further, we have studied the normal and secondary normal curvatures and the normal curvature vector of the hypersurface of a Kaehlerian spaces and several theorems have been investigated. Also, we have defined and studied the subspaces of a generalized metric space. In the end, we have discussed some basic concepts and fundamental formulae in Hermitian spaces as given by Eliopoulos [33].

By introducing two sets of vectors:

$$\left( M^i, \tilde{M}^i \right)$$ and $$\left( M^*i, \tilde{M}^*i \right)$$,

We have discussed the concepts of primary relative associate curvature vector and secondary relative associate curvature vector (both the vectors are considered along the same curve C). The necessary and sufficient condition that the derived vector be equal to the primary relative associate curvature vector has also been investigated.

Firstly, we shall mention some preliminary facts concerning Kaehlerian and Hermitian spaces.

A Kaehler space $K^n_c$ is an even dimensional Riemannian space, i.e., the space of dimension $n(=2m)$ with the structure tensor $F^h_i$ and the Riemannian metric $g_{ij}$, which satisfy the following conditions:
spaces of a generalized metric space and has proved several theorems. Further, Singh [121] has studied relative associate curvature of a vector field in Kaehlerian spaces and several results have been obtained by him.

In the present chapter, we have studied curvature of a congruence relative to a vector field in a Kaehlerian space. Further, we have studied the normal and secondary normal curvatures and the normal curvature vector of the hypersurface of a Kaehlerian spaces and several theorems have been investigated. Also, we have defined and studied the subspaces of a generalized metric space. In the end, we have discussed some basic concepts and fundamental formulae in Hermitian spaces as given by Eliopoulos [33].

By introducing two sets of vectors:
\[
\left\{ M^i, \bar{M}^{\bar{i}} \right\} \text{ and } \left\{ M^{*i}, \bar{M}^{*\bar{i}} \right\},
\]
We have discussed the concepts of primary relative associate curvature vector and secondary relative associate curvature vector (both the vectors are considered along the same curve C). The necessary and sufficient condition that the derived vector be equal to the primary relative associate curvature vector has also been investigated.

Firstly, we shall mention some preliminary facts concerning Kaehlerian and Hermitian spaces.

A Kaehler space \( K_n \) is an even dimensional Riemannian space, i.e., the space of dimension \( n(=2m) \) with the structure tensor \( F^h_i \) and the Riemannian metric \( g_{ij} \), which satisfy the following conditions:
\[ F^h_i F^i_j = -\delta^h_j \]  
\[ F_{ij} = -F_{ji}, \quad (F_{ij} \overset{\text{def}}{=} F^a_i g_{aj}) \]

and

\[ \nabla_j F^h_i = 0, \]

where the del (i.e., \( \nabla \)) followed by an index denotes the operator of covariant differentiation with respect to the Riemannian metric \( g_{ij} \).

The space \( K^c_n \) is said to be a Kaehlerian space of complex dimension \( n \), if the Hermitian metric, defined by

\[ ds^2 = g_{ij} dz^i dz^j, \]

satisfies the condition (Kaehler [48]):

\[ \Gamma^\lambda_{\mu \nu} = \Gamma^\lambda_{\nu \mu} = 0, \]

where \( \Gamma^\lambda_{\mu \nu} \) and \( \Gamma^\lambda_{\nu \mu} \) are nothing but the usual Christoffel's symbols.

A necessary and sufficient condition that a Hermitian space be Kaehlerian is that (Yano [198]):

\[ \nabla_i F^h_j \overset{\text{def}}{=} \partial_i F^h_j + F^i_j \Gamma^h_{ij} - F^h_i \Gamma^i_{ij} = 0 \]

and

\[ \nabla_i F_{jh} = 0, \]

*) Throughout the chapter we shall use the following variations of indices:

\[ i, j, k, \ldots, = 1, 2, \ldots, n \]
\[ \bar{i}, \bar{j}, \bar{k}, \ldots, = \bar{1}, \bar{2}, \ldots, \bar{n} \]
\[ \alpha, \beta, \gamma, \ldots, = 1, 2, \ldots, m \]
\[ \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \ldots, = \bar{1}, \bar{2}, \ldots, \bar{m} \]

where \( F_{jh} = F^l_j F^l_h \).

Hence the metric of Kaehler space is given by
\( (1.1) \quad F^h_i F^i_j = -\delta^h_j \), \\
\( (1.2) \quad F_{ij} = -F_{ji}, \quad (F_{ij} \overset{\text{def}}{=} F^a_ig_{aj}) \)

and

\( (1.3) \quad \nabla_j F^h_i = 0, \)

where the del (i.e., \( \nabla \)) followed by an index denotes the operator of covariant differentiation with respect to the Riemannian metric \( g_{ij} \).

The space \( K^c_n \) is said to be a Kaehlerian space of complex dimension \( n \), if the Hermitian metric, defined by

\[
\text{ds}^2 = g_{ij} dz^i dz^j ,
\]

satisfies the condition (Kaehler [48]):

\( (1.5) \quad \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} = 0, \)

where \( \Gamma^\lambda_{\mu\nu} \) and \( \Gamma^\lambda_{\nu\mu} \) are nothing but the usual Christoffel's symbols.

A necessary and sufficient condition that a Hermitian space be Kaehlerian is that (Yano [198]):

\( (1.6) \quad \nabla_i F^h_j \overset{\text{def}}{=} \partial_i F^h_j + F^h_l \Gamma^h_{lj} - F^h_i \Gamma^h_{lj} = 0 \)

and

\( (1.6^*) \quad \nabla_i F_{jh} = 0, \)

*) Throughout the chapter we shall use the following variations of indices:

\[ i, j, k, \ldots, = 1, 2, \ldots, n \]

\[ i, j, k, \ldots, = i, j, k, \ldots, = 1, 2, \ldots, n \]

\[ \alpha, \beta, \gamma, \ldots, = 1, 2, \ldots, m \]

\[ \alpha, \beta, \gamma, \ldots, = \alpha, \beta, \gamma, \ldots, = 1, 2, \ldots, m \]

where \( F_{jh} = F^l_j F_{lh}. \)

Hence the metric of Kaehler space is given by
(1.7) \[ ds^2 = 2 g_{\mu \lambda} dz^\mu dz^{\bar{\lambda}}. \]

Now, let us consider a Kaehler manifold of complex dimension \( n \) and real dimension \( 2n \), equipped with the coordinate system \( (z^i, \bar{z}^\bar{i}) \) and let the equations

(1.8) \[ \phi (z^1, z^2, \ldots, z^n) = 1 \]

and

(1.8*) \[ \bar{\phi} (z^{\bar{1}}, z^{\bar{2}}, \ldots, z^{\bar{n}}) = 1 \]

represent a hypersurface of the complex Kaehler manifold \( K_n^c \).

Also, let there be an analytic hypersurface \( K_m^c \) in the space \( K_n^c \). If \( (u^\alpha, \bar{u}^{\bar{\alpha}}) \equiv (u^1, u^2, \ldots, u^m; u^{\bar{1}}, u^{\bar{2}}, \ldots, u^{\bar{m}}) \) represents the coordinate system in the space \( K_m^c \), then the coordinates of a point \( P \) in two different systems are connected by the relation (Yano & Bochner [205]):

(1.9) \[ z^i = z^i (u^\alpha); \quad z^{\bar{i}} = z^{\bar{i}} (u^{\bar{\alpha}}), \]

which represents the equation of complex Kaehler hypersurface and we shall denote such a space by the symbol \( K_m^c \).

The Jacobian of transformation for the relation (1.9) may be written as

\[ \left| \frac{\partial z^i}{\partial u^\alpha} \right|, \]

and the \( n \) functions \( z^i \) are said to be independent, if their functional determinant does not vanishes identically, \( i.e., \left| \frac{\partial z^i}{\partial u^\alpha} \right| \neq 0. \)

Now, by putting

(1.10) \[ z^i = x^i + i x^{\bar{i}}; \quad z^{\bar{i}} = x^i - x^{\bar{i}}, \]

we may introduce a one to one correspondence between \( (z^i, z^{\bar{i}}) \cong (x^i) \), thus \( (z^i, z^{\bar{i}}) \) may be considered as the coordinates of a point \( P \) in \( K_n^c \).
(1.7) \[ ds^2 = 2 g_{\mu \lambda}^\flat \, dz^\mu \, dz^\lambda. \]

Now, let us consider a Kaehler manifold of complex dimension \( n \) and real dimension \( 2n \), equipped with the co-ordinate system \( (z^i, \bar{z}^i) \) and let the equations

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represent a hypersurface of the complex Kaehler manifold \( K_n^c \).

Also, let there be an analytic hypersurface \( K_m^c \) in the space \( K_n^c \). If

\[
(u^a, \tilde{u}^a) \equiv (u^1, u^2, \ldots, u^m; u^1, u^2, \ldots, u^m) \]

represents the co-ordinate system in the space \( K_m^c \), then the co-ordinates of a point \( P \) in two different systems are connected by the relation (Yano & Bochner [205]):

(1.9) \[ z^i = z^i (u^a); \quad \bar{z}^i = z^i (\tilde{u}^a), \]

which represents the equation of complex Kaehler hypersurface and we shall denote such a space by the symbol \( K_m^c \).

The jacobian of transformation for the relation (1.9) may be written as

\[
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(Yano & Bochner [205]). Hence the equation (1.9) takes the form:

\[(1.11) \quad z^i = z^i (u, \bar{u}); \quad \bar{z}^i = \bar{z}^i (u, \bar{u})\]

and the transformation between two different co-ordinate systems becomes

\[(1.12) \quad z^i = f^i (u^\alpha), \text{ or } \bar{z}^i = f^i (u, \bar{u}).\]

The jacobian of transformation is then seen to be \(\left| \frac{\partial z^i}{\partial u^\alpha} \right| = \text{real} > 0\). We shall denote such a complex analytic space of complex dimension \(n\) by \(C_n\) (Yano & Bochner [205]).

In this \(C_n\), let us introduce a fundamental function \(g(z, \bar{z})\) of \(4n\) independent variables \(z^i\) and \(\bar{z}^i\) positively homogeneous of degree one with respect to the variables \(u^\alpha\) and \(\bar{u}^\alpha\).

The fundamental function \(g(z, \bar{z}) \geq 0\), is such that

\[g[z^i, \rho z^i] = |\rho| g(z^i, \bar{z}^i)\]

The arc length of the arc \(z^i = z^i (t), \text{ for } t_1 \leq t \leq t_2\), is defined by (Cartan [23] and Yano [199]):

\[(1.13) \quad s = \int_{t_1}^{t_2} \left[ g_{ij} \left( \frac{dz^i}{dt} \cdot \frac{dz^j}{dt} \right) \right]^{1/2} dt,\]

where \(t\) is some parameter, used to represent the variation of arc length \(s\).

Such a complex analytic space is called Kaehlerian manifold (Prakash [89]).

The fundamental function \(g(z, \bar{z})\) is assumed to be invariant under the co-ordinate transformations.

Now, putting

\[(1.14) \quad G (z, \bar{z}) \overset{\text{def}}{=} \frac{1}{2} g^2 (z, \bar{z})\]

and
(Yano & Bochner [205]). Hence the equation (1.9) takes the form:

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The fundamental function \( g(z, \bar{z}) \) is assumed to be invariant under the co-ordinate transformations.

Now, putting

\[(1.14) \quad G(z, \bar{z}) \overset{\text{def}}{=} \frac{1}{2} g^2 (z, \bar{z}) \]

and
\( g_{ij}^{-}(z, \bar{z}) \) and that
\[
(g_{ij}^{-}(z, \bar{z}))^2 = g_{ij}^{-}(z, \bar{z}) \frac{\partial z^i}{\partial t} \frac{\partial \bar{z}^j}{\partial t}.
\]
Next, let the hypersurface \( K_m^c \) given by the equations 
\[
z^i = z^i(u^a);
\]
\( z^i = z^i(\bar{u}^\alpha) \) be immersed in an n-dimensional complex Kaehlerian manifold \( K_n^c \).
The components of the unit tangent vector of a curve C: 
\[
z^i = z^i(s);
\]
\( z^i = z^i(\bar{s}) \) of \( K_m^c \) are given by
\[
\frac{dz^i}{ds} = B^i_a \left( \frac{du^a}{ds} \right) \text{ and } \frac{dz^i}{ds} = B^i_\bar{a} \left( \frac{d\bar{u}^\alpha}{ds} \right),
\]
where \( B^i_a = \partial z^i / \partial u^a \) and \( B^i_\bar{a} = \partial z^i / \partial \bar{u}^\alpha \).
Let, \( g_{ij}^{-}(z, \bar{z}) \) be the metric tensor of \( K_n^c \), then the vectors \( (N^i, \bar{N}^\bar{i}) \)
consistent with the conditions:
\[
g_{ij}^{-}(z, \bar{z}) N^i \frac{B^j_\beta}{\beta} = 0 \quad \text{and} \quad g_{ij}^{-}(z, \bar{z}) \bar{N}^\bar{i} \frac{B^j_\bar{\alpha}}{\bar{\alpha}} = 0
\]
and
\[
g_{ij}^{-}(z, \bar{z}) N^i \bar{N}^{\bar{j}} = 1 \quad \text{and} \quad g_{ij}^{-}(z, \bar{z}) N^{\bar{i}} \bar{N}^{\bar{j}} = 1
\]
are called the primary normal vectors and the secondary normal vectors of
the hypersurface.
Now, suppose that \( g_{ij}^{-}(z, \bar{z}) \) is the fundamental metric tensor of \( K_n^c \), then
the metric tensors \( g_{ij}^{-}(u, \bar{u}) \) and \( g_{ij}^{-}(z, \bar{z}) \) are such that
\[
g_{ij}^{-}(z, \bar{z}) = g_{ij}^{-}(u, \bar{u}) \left( B^i_a \left( B^j_\beta \right) \right).
\]
and the induced connection parameter \( \Gamma^{\alpha}_{\beta \gamma} \) and \( \Gamma^{\alpha}_{\bar{\beta} \bar{\gamma}} \) of \( K_m^c \) are given by
\[
\Gamma^{\alpha}_{\beta \gamma} (u, \bar{u}) = B^i_a \left[ \left( \frac{\partial^2 z^i}{\partial u^\beta \partial u^\gamma} \right) + \Gamma^{i}_{h \bar{k}} B^h_\beta B^k_\gamma \right]
\]
and
we see that \( g_{ij} \) is a symmetric covariant tensor and that

\[
g^2(z, \bar{z}) = g_{ij}(z, \bar{z}) \frac{dz^i}{dt} \frac{dz^j}{dt}.
\]

Next, let the hypersurface \( K^c_m \) given by the equations \( z^i = z^i(u^a) \); \( \bar{z}^i = \bar{z}^i(\bar{u}^\alpha) \) be immersed in an n-dimensional complex Kaehlerian manifold \( K^c_n \).

The components \( (\frac{dz^i}{ds}) \) (or, \( \frac{dz^j}{ds} \)) and \( (\frac{du^a}{ds}) \) (or, \( \frac{du^\alpha}{ds} \)) of the unit tangent vector of a curve \( C: z^i = z^i(s) \); \( \bar{z}^i = \bar{z}^i(s) \), (where s is real) of \( K^c_m \) are given by

\[
\frac{dz^i}{ds} = B'_i(\frac{du^a}{ds}) \quad \text{and} \quad \frac{dz^j}{ds} = B'_j(\frac{du^\alpha}{ds}),
\]

where \( B'_i = \frac{\partial z^i}{\partial u^a} \) and \( \bar{B}'_i = \frac{\partial \bar{z}^i}{\partial u^\alpha} \).

Let, \( g_{ij}(z, \bar{z}) \) be the metric tensor of \( K^c_n \), then the vectors \( (N^i, \bar{N}^i) \) consistent with the conditions:

\[
(1.17) \quad g_{ij}(z, \bar{z}) N^i B^j_\beta = 0; \quad g_{ij}(z, \bar{z}) \bar{N}^j \bar{B}^i_\alpha = 0
\]

and

\[
(1.18) \quad g_{ij}(z, \bar{z}) N^i \bar{N}^j = 1; \quad g_{ij}(z, \bar{z}) N^i \bar{N}^i = 1
\]

are called the primary normal vectors and the secondary normal vectors of the hypersurface.

Now, suppose that \( g_{\alpha \beta} \) is the fundamental metric tensor of \( K^c_m \), then the metric tensors \( g_{\alpha \beta}(u, \bar{u}) \) and \( g_{ij}(z, \bar{z}) \) are such that

\[
g_{\alpha \beta}(u, \bar{u}) = g_{ij}(z, \bar{z}) \begin{bmatrix} B'_i & B^j_\beta \end{bmatrix},
\]

and the induced connection parameter \( \Gamma^\alpha_{\beta \gamma} \) and \( \bar{\Gamma}^\alpha_{\beta \gamma} \) of \( K^c_m \) are given by

\[
(1.19) \quad \Gamma^\alpha_{\beta \gamma}(u, \bar{u}) = B^a_i \left( \frac{\partial^2 z^i}{\partial u^\alpha \partial u^\beta} \right) + \Gamma^i_{hk} B^h_\beta B^k_\gamma
\]

and
\begin{align}
(1.19^\ast) \quad & \Gamma^\alpha_{\beta \gamma} (u, \bar{u}) = B^i_i \left[ \frac{\partial^2 \bar{z}^i}{\partial u^\alpha \partial u^\gamma} \right] + \Gamma^i_{hk} B^h_i B^k_i,
\end{align}

where

\begin{align}
B^\alpha_i = g^{\alpha \beta} (u, \bar{u}) g_{ij} (z, \bar{z}) B^j_i; \quad & B^\alpha_i = g^{\alpha \beta} (u, \bar{u}) g_{ij} (z, \bar{z}) B^j_i,
\end{align}

and \( \{ \Gamma^i_{hk}, \Gamma^i_{hk} \} \) are the connection coefficients of the embedding space \( K_n^c \) (Rund [98], page 71).

The parameter \( \{ \Gamma^\alpha_{\beta \gamma}, \Gamma^\alpha_{\beta \gamma} \} \) has been called the tensor (or, mixed) derivative and induced covariant derivative of the hypersurface. In particular, the tensor derivatives of \( B^\alpha_i \) and \( B^\alpha_i \) are given by

\begin{align}
(1.20) \quad & \nabla^\alpha B^\alpha_i = \frac{\partial^2 \bar{z}^i}{\partial u^\alpha \partial u^\alpha} - B^i_i \Gamma^\gamma_{\alpha \beta} + \Gamma^i_{h k} B^h_i B^k_ \beta
\end{align}

and

\begin{align}
(1.20^\ast) \quad & \nabla^\alpha B^\alpha_i = \frac{\partial^2 \bar{z}^i}{\partial u^\alpha \partial u^\alpha} - B^i_i \Gamma^\gamma_{\alpha \beta} + \Gamma^i_{h k} B^h_i B^k_ \beta
\end{align}

hence, we have

\begin{align}
(1.21) \quad & \Omega^\alpha_{\beta} = m^i \nabla^\beta B^i_k; \quad \Omega^\alpha_{\beta} = m^i \nabla^\beta B^i_k.
\end{align}

The components of the tensor, i.e., the components \( \{ \Omega^\alpha_{\beta}, \Omega^\alpha_{\beta} \} \) is called the second fundamental tensor of the hypersurface. Further, it can be expressed that

\begin{align}
(1.22) \quad & \nabla^\beta B^i_\alpha = \Sigma \Omega^\alpha_{\beta} m^* i; \quad \nabla^\beta B^i_\alpha = \Sigma \Omega^\alpha_{\beta} m^* i,
\end{align}

where \( \{ \Omega^\alpha_{\beta}, \Omega^\alpha_{\beta} \} \) are the components of secondary second fundamental tensors of the hypersurface.

The tensor derivatives of the vector \( \{ m^i, m^* i \} \) are such that

\begin{align}
(1.23) \quad & \nabla^\beta m^i = A^\alpha_{\beta} B^i_\alpha + \nu^\beta m^i
\end{align}

and
\[ \Gamma^\alpha_{\beta \gamma} (u, \bar{u}) = B^\alpha_i \left( \frac{\partial^2 \bar{z}^i}{\partial u^\theta \partial u^\nu} \right) + \frac{\Gamma^i_{hk}}{\beta} B^h B^k , \]

where

\[ B^\alpha_i = g^{\alpha \beta}(u, \bar{u}) g_{ij}(z, \bar{z}) B^j_i ; ~ \bar{B}^\alpha_i = \bar{g}^{\alpha \beta}(u, \bar{u}) \]

and \( \langle \Gamma^i_{hk} , \bar{\Gamma}^i_{hk} \rangle \) are the connection coefficients of the embedding space \( K^c_n \) (Rund [98], page 71).

The parameter \( \langle \Gamma^\alpha_{\beta \gamma} , \Gamma^\alpha_{\beta \gamma} \rangle \) has been called the tensor (or, mixed) derivative and induced covariant derivative of the hypersurface. In particular, the tensor derivatives of \( B^\alpha_i \) and \( \bar{B}^\alpha_i \) are given by

\[ \nabla_\beta B^\alpha_i = -\frac{\partial^2 \bar{z}^i}{\partial u^\theta \partial u^\nu} - B^j_i \Gamma^\gamma_{\alpha \beta} + \Gamma^i_{hk} B^h B^k , \]

and

\[ \nabla_\beta \bar{B}^\alpha_i = -\frac{\partial^2 \bar{z}^i}{\partial u^\theta \partial u^\nu} - \bar{B}^j_i \Gamma^\gamma_{\bar{\alpha} \bar{\beta}} + \bar{\Gamma}^i_{hk} \bar{B}^h \bar{B}^k \]

hence, we have

\[ \Omega^\alpha_{\beta} = m_i \nabla_\beta B^i ; \Omega^{\alpha \beta} = m_i \nabla_\beta \bar{B}^i . \]

The components of the tensor, i.e., the components \( \langle \Omega^\alpha_{\beta} , \Omega^{\alpha \beta} \rangle \) is called the second fundamental tensor of the hypersurface. Further, it can be expressed that

\[ \nabla_\beta B^i = \Sigma \Omega^*_{\alpha \beta} m^* ; \nabla_\beta \bar{B}^i = \Sigma \Omega^{* \alpha \beta} m^{* i} , \]

where \( \langle \Omega^*_{\alpha \beta} , \Omega^{* \alpha \beta} \rangle \) are the components of secondary second fundamental tensors of the hypersurface.

The tensor derivatives of the vector \( \langle m^i , m^{* i} \rangle \) are such that

\[ \nabla_\beta m^i = A^\alpha_\beta B^i + \nu_\beta m^i \]

and
\[ \nabla_\beta m^* i = A^\alpha_\beta B^i_\alpha + \nu_\beta \ m^* i, \]

where

\[ A^\gamma_\beta = -\xi^{\alpha \gamma} (\Omega^{\alpha \beta} + E^i_{i j \beta} B^i_\alpha m^* i); \quad \nu_\beta = m_i \nabla_\beta m^i \]

and

\[ A^{* \gamma}_\beta = -\xi^{\alpha \gamma} (\psi \Omega^*_{\alpha \beta} + E^i_{i j \beta} B^i_{\alpha} m^* i); \quad \nu^*_\beta = \frac{1}{\psi} m_i^* \nabla_\beta m^{* i}. \]

Here, we have defined

\[ \zeta^{\alpha \beta} \zeta_{\alpha \gamma} = \delta^{\beta}_{\gamma} = \xi^{\alpha \beta} \xi_{\alpha \gamma} \]

and

\[ E_{i j \beta} = \nabla_\beta \xi_{i j}(z, \bar{z}); \quad E^{* i j}_\beta = \nabla_\beta \zeta_{i j}(z, \bar{z}) \]

The normal curvature (Rund [99]) of the hypersurface in the direction of a curve C, is defined by

\[ K = m_i \frac{\delta z^j}{\delta s}. \]

where \( \frac{\delta}{\delta s} \) is defined by Rund [98] (page 59).

Again, let us consider an n-dimensional complex space \( C_n \), referred to an allowable co-ordinate system:

\[ (z^i, \bar{z}^i) \equiv (z^1, z^2, ..., z^n, \bar{z}^1, \bar{z}^2, ..., \bar{z}^n) \]

(Yano & Bochner [205]):

In the \( C_n \), we introduce the metric defined by positive definite Hermitian form:

\[ ds^2 = 2 g_{i j}(z, \bar{z}) dz^i \bar{dz}^j \ (E.Kaehler[48]). \]

Let us consider a subspace \( H_m \) of \( H_n \), given by the equation:

\[ z^i = z^i (u^\alpha); \quad \bar{z}^i = \bar{z}^i (\bar{u}^\alpha). \]

The relation between the fundamental tensor of \( H_m \) and \( H_n \) is given
(1.23*) \[ \nabla_\beta m^*_i = A^*_\beta A^i_\alpha B^i_\alpha + \nu^*_\beta m^*_i , \]

where

(1.24) \[ A^\gamma_\beta = -\xi^{\alpha\gamma} (\Omega_\alpha_\beta + E^{*}_{i,j \beta} B^{i}_{\alpha}m^{i}) ; \quad \nu_\beta = m_i \nabla_\beta m^i \]

and

(1.24*) \[ A^\gamma_\beta = -\xi^{\alpha\gamma} (\psi \Omega^{*}_{\alpha_\beta} + E^{*}_{i,j \beta} B^{i}_{\alpha}m^{*}_i) ; \quad \nu^*_\beta = \frac{1}{\psi} m^*_i \nabla_\beta m^* i. \]

Here, we have defined

\[ \zeta^{\alpha\beta} \zeta_{\alpha\gamma} = \delta^\beta_\gamma = \xi^{\alpha\beta} \xi_{\alpha\gamma} \]

and

\[ E_{i,j \beta} = \nabla_\beta \xi_{i,j}(z, \bar{z}) ; \quad E^*_{i,j \beta} = \nabla_\beta \xi^*_{i,j}(z, \bar{z}) \]

The normal curvature (Rund [99]) of the hypersurface in the direction of a curve C, is defined by

(1.25) \[ K = m_i \frac{\delta z^i}{\delta s} . \]

where \( \frac{\delta}{\delta s} \) is defined by Rund [98] (page 59).

Again, let us consider an n-dimensional complex space \( C_n \), referred to an allowable co-ordinate system:

\[ (z^i, \bar{z}^i) \equiv (z^1, z^2, \ldots, z^n, \bar{z}^1, \bar{z}^2, \ldots, \bar{z}^n) \]

(Yano & Bochner [205]):

In the \( C_n \), we introduce the metric defined by positive definite Hermitian form:

(1.26) \[ ds^2 = 2 g_{i,j}(z, \bar{z}) \, dz^i \, d\bar{z}^j \quad (E.Kaehler[48]). \]

Let us consider a subspace \( H_m \) of \( H_n \), given by the equation:

(1.27) \[ z^i = z^i (u^\alpha) ; \quad \bar{z}^i = \bar{z}^i (\bar{u}^\bar{\alpha}) . \]

The relation between the fundamental tensor of \( H_m \) and \( H_n \) is given
by the equation:

\( g_{\alpha\beta}(u, u') = g_{ij}(z, \bar{z}) B^i_{\alpha} B^j_{\beta} , \)

where

\[
B^i_{\alpha} = \frac{\partial z^i}{\partial u^\alpha}, \quad \bar{B}^i_{\alpha} = \frac{\partial \bar{z}^i}{\partial u^\alpha}.
\]

The vectors \( \left\{ m^i, \bar{m}^i \right\} \) satisfying the following conditions:

\( g_{ij} \bar{B}^i_{\alpha} m^j \nu = 0; \quad 2 g_{ij} m^i \bar{m}^j = 1, \)

are called the normal vectors of the subspace \( H_m \).

Consider a curve \( C: z^i = z^i(s), \bar{z}^i = \bar{z}^i(s) \), (where \( s \) is real) of \( H_m \). The components \( \left( \frac{dz^i}{ds} \right) \) (or, \( \frac{dz^i}{ds} \)) and \( \left( \frac{du^\alpha}{ds} \right) \) (or, \( \frac{du^\alpha}{ds} \)) of the unit tangent vector of \( C \) are given by

\[
\frac{dz^i}{ds} = B^i_{\alpha} \left( \frac{du^\alpha}{ds} \right)
\]

and

\[
\frac{d\bar{z}^i}{ds} = \bar{B}^i_{\alpha} \left( \frac{du^\alpha}{ds} \right).
\]

The induced connection parameter of \( H_m \) is given by

\( \Gamma^\alpha_{\beta\gamma}(u, \bar{u}) = B^i_{\alpha} \left( \frac{\partial^2 z^j}{\partial u^\beta \partial u^\gamma} + \Gamma^i_{h k} B^h_{\beta} B^k_{\gamma} \right) \)

and

\( \bar{\Gamma}^\alpha_{\beta\gamma}(u, \bar{u}) = \bar{B}^i_{\alpha} \left( \frac{\partial^2 \bar{z}^j}{\partial u^\beta \partial u^\gamma} + \Gamma^i_{h k} B^h_{\beta} B^k_{\gamma} \right) \)

where

\[
B^i_{\alpha} = g^{\alpha\beta}(u, \bar{u}) g_{ij} B^j_{\beta}, \quad \bar{B}^i_{\alpha} = g^{\alpha\beta}(u, \bar{u}) g_{ij} \bar{B}^j_{\beta}
\]

and

\( \{ \Gamma^i_{h k}, \bar{\Gamma}^i_{h k} \} \) are the connection coefficients of the embedding space \( H_n \).

In particular, the derivative of \( B^i_{\alpha} \) and \( \bar{B}^i_{\alpha} \) are given by

\( B^i_{\alpha; \beta} = B^i_{\alpha, \beta} = \frac{\partial^2 z^i}{\partial u^\alpha \partial u^\beta} - B^i_{\gamma} \Gamma^\gamma_{\alpha \beta} + \Gamma^i_{h k} B^h_{\alpha} B^k_{\beta} \)

and
by the equation:

\begin{equation}
(1.28) \quad g_{\alpha \beta} (u, \tilde{u}) = g_{ij} (z, \tilde{z}) B^i_\alpha B^j_\beta,
\end{equation}

where

\[
B^i_\alpha = \frac{\partial z^i}{\partial u^\alpha}, \quad B^i_\alpha = \frac{\partial \tilde{z}^i}{\partial u^\alpha}.
\]

The vectors \( \left\{ m^i, \tilde{m}^i \right\} \) satisfying the following conditions:

\begin{equation}
(1.29) \quad g_{ij} B^i_\alpha m^j = 0; \quad 2 g_{ij} m^i \tilde{m}^j = 1,
\end{equation}

are called the normal vectors of the subspace \( H_m \).

Consider a curve \( C: z^i = z^i (s), \tilde{z}^i = \tilde{z}^i (s) \), (where \( s \) is real) of \( H_m \). The components \( \left( \frac{dz^i}{ds} \right) \) (or, \( \frac{dz^i}{ds} \)) and \( \left( \frac{du^\alpha}{ds} \right) \) (or, \( \frac{du^\alpha}{ds} \)) of the unit tangent vector of \( C \) are given by

\begin{equation}
(1.30) \quad \frac{dz^i}{ds} = B^i_\alpha \left( \frac{du^\alpha}{ds} \right)
\end{equation}

and

\begin{equation}
(1.30^*) \quad \frac{d\tilde{z}^i}{ds} = B^i_\alpha \left( \frac{d\tilde{u}^\alpha}{ds} \right).
\end{equation}

The induced connection parameter of \( H_m \) is given by

\begin{equation}
(1.31) \quad \Gamma^{\alpha \gamma}_{\beta \gamma} (u, \tilde{u}) = B^{\alpha}_{i} \left( \frac{\partial^2 z^i}{\partial u^\alpha \partial u^\gamma} + \Gamma^{i \beta}_{h k} B^h_\beta B^k_\gamma \right)
\end{equation}

and

\begin{equation}
(1.31^*) \quad \Gamma^{\alpha \gamma}_{\beta \gamma} (u, \tilde{u}) = B^{\alpha}_{i} \left( \frac{\partial^2 \tilde{z}^i}{\partial \tilde{u}^\alpha \partial \tilde{u}^\gamma} + \Gamma^{i \beta}_{h k} B^h_\beta B^k_\gamma \right)
\end{equation}

where

\[
B^{\alpha}_{i} = g^{\alpha \beta} (u, \tilde{u}) g_{ij} B^j_\beta; \quad B^{\alpha}_{i} = g^{\alpha \beta} (u, \tilde{u}) g_{ij} \tilde{B}^j_\beta
\]

and

\[
\left\{ \Gamma^{i \beta}_{h k}, \Gamma^{i \beta}_{h k} \right\}
\]

are the connection coefficients of the embedding space \( H_n \).

In particular, the derivative of \( B^{i}_{\alpha} \) and \( B^{\alpha}_{i} \) are given by

\begin{equation}
(1.32) \quad B^{i}_{\alpha; \beta} = B^{\alpha}_{i, \beta} = \frac{\partial^2 z^i}{\partial u^\alpha \partial u^\beta} - B^{i}_{ij} \Gamma^{\gamma}_{\alpha \beta} + \Gamma^{i \beta}_{h k} B^h_\alpha B^k_\gamma
\end{equation}

and
(1.32*) \[
\begin{align*}
B_{\alpha; \beta}^i &= B_{i, \beta}^\alpha - \frac{\partial z_i}{\partial u^\beta} - \frac{\partial z_i}{\partial \omega^\beta} - B_{\gamma}^i \Gamma^{\gamma}_{\alpha \beta} + \Gamma^{i}_{h k} B_{h}^\alpha B_{k}^\beta
\end{align*}
\]

whence, we have
\[
\Omega_{\alpha \beta} = m_i B_{\alpha, \beta}^i ; \quad \Omega_{\alpha \beta} = m_i B_{\alpha, \beta}^i .
\]

The tensors with the components \(\Omega_{\alpha \beta}\) and \(\Omega_{\alpha \beta}\) are called the second fundamental tensors of the subspace.

It is further expressed that
\[
(1.33) \quad B_{\alpha, \beta}^i = \sum_{\nu} \Omega_{\alpha \beta}^{*} m_{\nu}^i
\]

and
\[
(1.33*) \quad B_{\alpha, \beta}^i = \sum_{\nu} \Omega_{\alpha \beta}^{*} m_{\nu}^i ,
\]

where \(\Omega_{\alpha \beta}^{*}\) and \(\Omega_{\alpha \beta}^{*}\) are the components of secondary second fundamental tensor of \(H_m\).

We have (Eliopoulos [33]):
\[
(1.34) \quad B_{\alpha, \beta}^i = B_{\gamma}^i \sum_{\nu} A_{\alpha \beta} M_{\gamma}^\nu + \sum_{\nu} A_{\alpha \beta} m_{\nu}^i
\]

and
\[
(1.34*) \quad B_{\alpha, \beta}^i = B_{\gamma}^i \sum_{\nu} A_{\gamma \beta} M_{\gamma}^\nu + \sum_{\nu} A_{\gamma \beta} m_{\nu}^i ,
\]

where
\[
\begin{align*}
\Omega_{\alpha \beta} &= \sum_{\nu} A_{\alpha \beta} m_{\nu}^i m_{\nu}^i ; \quad \Omega_{\alpha \beta} = \sum_{\nu} A_{\gamma \beta} m_{\nu}^i m_{\nu}^i ,
\end{align*}
\]

and
\[
\begin{align*}
M_{\gamma}^\nu &= -m_{\nu} B_{\gamma}^i ; \quad M_{\gamma}^\nu = m_{\nu} B_{\gamma}^i .
\end{align*}
\]

Let \(V_{\nu}^i\) be a vector field of the space \(H_m\), whose components \((\nu', \nu')\) are such that
(1.32*) \[ \overline{B}^{i}_{\alpha; \beta} = B^{\alpha}_{i, \beta} = \frac{\partial^{2} z^{i}}{\partial u^\beta \partial w^\gamma} - B^{\gamma}_{\gamma} \Gamma_{\alpha \beta}^{\gamma} + \Gamma_{h k}^{\alpha \beta} \overline{B}^{h}_{n} \overline{B}^{k}_{m} \]

whence, we have

\[ \Omega_{\alpha \beta}^{(v)} = m^{i} B^{i}_{\alpha, \beta}; \quad \overline{\Omega}_{\alpha \beta}^{(v)} = m^{i} \overline{B}^{i}_{\alpha, \beta}. \]

The tensors with the components \( \Omega_{\alpha \beta}^{(v)} \) and \( \overline{\Omega}_{\alpha \beta}^{(v)} \) are called the second fundamental tensors of the subspace.

It is further expressed that

(1.33) \[ B^{i}_{\alpha, \beta} = \Sigma \Omega_{\alpha \beta}^{(v)} m^{* i} \]

and

(1.33*) \[ \overline{B}^{i}_{\alpha, \beta} = \Sigma \overline{\Omega}_{\alpha \beta}^{(v)} m^{* i}, \]

where \( \Omega_{\alpha \beta}^{(v)} \) and \( \overline{\Omega}_{\alpha \beta}^{(v)} \) are the components of secondary second fundamental tensor of \( H_{m} \).

We have (Eliopoulos [33]):

(1.34) \[ B^{i}_{\alpha, \beta} = B^{i}_{\gamma} \Sigma A_{\alpha \beta} M^{\gamma} + \Sigma A_{\alpha \beta} m^{i} \]

and

(1.34*) \[ \overline{B}^{i}_{\alpha, \beta} = \overline{B}^{i}_{\gamma} \Sigma A_{\alpha \beta} M^{\gamma} + \Sigma A_{\alpha \beta} m^{i}, \]

where

\[ \Omega^{(v)}_{\alpha \beta} = \Sigma A_{\alpha \beta} m^{i} m^{i}; \quad \overline{\Omega}^{(v)}_{\alpha \beta} = \Sigma A_{\alpha \beta} m^{i} m^{i} \]

and

\[ M^{\gamma}_{(v)} = -m_{i} B^{\gamma}_{i}; \quad M^{\gamma}_{(v)} = m_{i} \overline{B}^{\gamma}_{i}. \]

Let \( V(v^{i}, \nu^{i}) \) be a vector field of the space \( H_{m} \), whose components \( (v^{i}, \nu^{i}) \) are such that
The components \( \left( \frac{\delta v^i}{\delta s} , \frac{\delta \tilde{v}^i}{\delta s} \right) \) of the derived vector (with respect to \( H_m \)) of \((v^j, \tilde{v}^i)\) in the direction of C, are given by

\[
(1.35) \quad \frac{\delta v^i}{\delta s} = (\frac{\delta v^a}{\delta s}) B^i_a + B^i_{a\beta} v^\alpha \frac{\delta u^\beta}{\delta s}
\]

and

\[
(1.35*) \quad \frac{\delta \tilde{v}^i}{\delta s} = (\frac{\delta \tilde{v}^a}{\delta s}) B^i\tilde{a} + B^i_{\tilde{a}\beta} \tilde{v}^\alpha \frac{\delta \tilde{u}^\beta}{\delta s},
\]

where \( (\frac{\delta v^a}{\delta s}, \frac{\delta \tilde{v}^a}{\delta s}) \) are the components of the induced associate curvature vector of the vector field.

If \( (\frac{\delta v^a}{\delta s}, \frac{\delta \tilde{v}^a}{\delta s}) = 0 \), the vector field is said to be parallel (along C) in \( H_n \).

On the other hand, if \( \left( B^i_{a\beta} v^\alpha \frac{\delta u^\beta}{\delta s} , B^i_{\tilde{a}\beta} \tilde{v}^\alpha \frac{\delta \tilde{u}^\beta}{\delta s} \right) = 0 \), the vector field is said to be conjugate with respect to C.

The equations (1.33), (1.33*), (1.34) and (1.34*) may be used to express the conditions for conjugacy in the form:

\[
(1.36) \quad \Omega^*_{\{v\} \alpha \beta} \, v^\alpha \frac{du^\beta}{ds} = 0
\]

and

\[
(1.36*) \quad \Omega^*_{\{\tilde{v}\} \tilde{\alpha} \tilde{\beta}} \, \tilde{v}^\tilde{\alpha} \frac{\tilde{du}^\tilde{\beta}}{ds} = 0.
\]

or, in the forms:

\[
(1.37) \quad A^*_{\{v\} \alpha \beta} \, v^\alpha \frac{du^\beta}{ds} = 0
\]

and

\[
(1.37*) \quad A^*_{\{\tilde{v}\} \tilde{\alpha} \tilde{\beta}} \, \tilde{v}^\tilde{\alpha} \frac{\tilde{du}^\tilde{\beta}}{ds} = 0.
\]

In the present chapter, we have introduced two sets of vectors:
\[ \nu^i = B^i_\alpha \nu^\alpha ; \quad \nu^\alpha = B^\alpha_\bar{\alpha} \nu^{\bar{\alpha}}. \]

The components \( \left( \frac{\delta \nu^i}{\delta s}, \frac{\delta \nu^\alpha}{\delta s} \right) \) of the derived vector (with respect to \( H_m \)) of \( (\nu^i, \nu^\alpha) \) in the direction of \( C \), are given by

\[ (1.35) \quad \frac{\delta \nu^i}{\delta s} = \left( \frac{\delta \nu^\alpha}{\delta s} \right) B^i_\alpha + B^i_{\alpha\beta} \nu^\alpha \frac{\delta u^\beta}{\delta s} \]

and

\[ (1.35^*) \quad \frac{\delta \nu^\alpha}{\delta s} = \left( \frac{\delta \nu^\alpha}{\delta s} \right) B^\alpha_{\bar{\alpha}} + B^\alpha_{\bar{\alpha}\beta} \nu^{\bar{\alpha}} \frac{\delta u^\beta}{\delta s}, \]

where \( \left( \frac{\delta \nu^\alpha}{\delta s}, \frac{\delta \nu^{\bar{\alpha}}}{\delta s} \right) \) are the components of the induced associate curvature vector of the vector field.

If \( \left( \frac{\delta \nu^\alpha}{\delta s}, \frac{\delta \nu^{\bar{\alpha}}}{\delta s} \right) = 0 \), the vector field is said to be parallel (along \( C \)) in \( H_n \).

On the other hand, if \( \left( B^i_{\alpha\beta} \nu^\alpha \frac{\delta u^\beta}{\delta s}, B^\alpha_{\bar{\alpha}\beta} \nu^{\bar{\alpha}} \frac{\delta u^\beta}{\delta s} \right) = 0 \), the vector field is said to be conjugate with respect to \( C \).

The equations (1.33), (1.33*), (1.34) and (1.34*) may be used to express the conditions for conjugacy in the form:

\[ (1.36) \quad \Omega^*_{(\nu)} \nu^\alpha \frac{du^\beta}{ds} = 0 \]

and

\[ (1.36^*) \quad \Omega^*_{(\nu)} \nu^{\bar{\alpha}} \frac{du^{\bar{\beta}}}{ds} = 0. \]

or, in the forms:

\[ (1.37) \quad A^\alpha_{(\nu)\beta} \nu^\alpha \frac{du^\beta}{ds} = 0 \]

and

\[ (1.37^*) \quad A^{\bar{\alpha}}_{(\nu)\bar{\beta}} \nu^{\bar{\alpha}} \frac{du^{\bar{\beta}}}{ds} = 0. \]

In the present chapter, we have introduced two sets of vectors:
\[ \left( M^i, M^i \right) \text{ and } \left( M^i, M^i \right) \]

The process leads to the concepts of primary relative associate curvature vector and secondary relative associate curvature vector, which are discussed in the sections (4) and (5).

2. CURVATURE OF A CONGRUENCE RELATIVE TO A VECTOR FIELD

Consider a vector \( \lambda_\ast(u) \) representing a congruence of a curve, or a family of curves in \( K_n^c \) and a vector field \( v \) of the hypersurface \( K_m^c \), we may write

\[
(2.1a) \quad \lambda^i_\ast = p^\alpha B^i_\alpha + r n^i
\]

and

\[
(2.1b) \quad v^i = v^\alpha B^i_\alpha.
\]

The vectors are normalized by the conditions:

\[
(2.2a) \quad g_{ij}(z, \bar{z}) \lambda^i_\ast \lambda^j_\ast = 1
\]

and

\[
(2.2b) \quad g_{ij}(z, \bar{z}) v^i v^j = g_{\alpha \beta}(u, \bar{u}) v^\alpha v^\beta = 1.
\]

we define

\[
(2.3a) \quad v K_{\lambda_\ast} = -v^j \frac{\delta \lambda^i_\ast}{\delta s} ,
\]

and

\[
(2.3b) \quad \bar{v} K_{\lambda_\ast} = -v_i \frac{\delta \lambda^i_\ast}{\delta s} ,
\]

where
The process leads to the concepts of primary relative associate curvature vector and secondary relative associate curvature vector, which are discussed in the sections (4) and (5).

2. CURVATURE OF A CONGRUENCE RELATIVE TO A VECTOR FIELD

Consider a vector \( \lambda_*(u) \) representing a congruence of a curve, or a family of curves in \( K_n^c \) and a vector field \( v \) of the hypersurface \( K_m^c \), we may write

\[
(2.1a) \quad \lambda_i^* = p^\alpha B^i_\alpha + r n^i
\]

and

\[
(2.1b) \quad v^j = v^\alpha B^j_\alpha.
\]

The vectors are normalized by the conditions:

\[
(2.2a) \quad g_{ij}(z, \bar{z}) \lambda_i^* \lambda_j^* = 1
\]

and

\[
(2.2b) \quad g_{ij}(z, \bar{z}) v^i v^j = g_{\alpha \beta}(u, \bar{u}) v^\alpha v^\beta = 1.
\]

we define

\[
(2.3a) \quad vK_{\lambda^*} = -v^j \frac{\delta \lambda^j}{\delta s},
\]

and

\[
(2.3b) \quad \bar{v}K_{\lambda^*} = -v_i \frac{\delta \lambda^i}{\delta s},
\]

where
\[(2.4a) \quad \lambda_{*i} = g_{ij}(z, \bar{z}) \lambda^j.\]

and

\[(2.4b) \quad v_i = g_{ij}(z, \bar{z}) v^j.\]

The scalar \(v K_{\lambda_*}\) will be called the \(P\)-curvature (with respect to \(C\)) of the congruence \(\lambda_*\) relative to vector field \(v\), while \(v \overline{K}_{\lambda_*}\) will be called the \(S\)-curvature (with respect to \(C\)) of the congruence \(\lambda_*\) relative to \(v\).

From equations (2.1a), (2.4a), (1.23) and the definition
\[
\nabla_\beta \bar{E}_{i j} \overset{\text{def}}{=} \nabla_\beta g_{i j}(z, \bar{z}),
\]
we get
\[(2.5) \quad \frac{\delta \lambda_{*i}}{ds} = \nabla_\beta \lambda_{*i} \frac{du^\beta}{ds} = (q_{i j \beta} B^i_\alpha \alpha + p_{i j \beta} m^j + g_{i j}(z, \bar{z}) p^\alpha B^j_\alpha) \frac{du^\beta}{sa},\]
where
\[
q_{i j \beta} = \bar{E}_{i j \beta} p^\alpha + g_{i j}(z, \bar{z}) (\nabla_\beta p^\alpha + ra^\alpha),
\]
and
\[
P_{i j \beta} = r \bar{E}_{i j \beta} + g_{i j}(z, \bar{z}) (\nabla_\beta r + r v_\beta).
\]
Making use of the equations (2.1b) and (2.5) in (2.3) and simplifying, we have
\[(2.6) \quad v K_{\lambda_*} = \hat{\Omega}_{\beta \gamma}(u, \bar{u}) v^\beta \frac{du^\gamma}{ds},\]
where
\[(2.7) \quad \hat{\Omega}_{\beta \gamma}(u, \bar{u}) = -(q_{i j \gamma} B^i_\alpha \alpha B^j_\beta + p_{i j \gamma} B^j_\beta m^j + g_{i j}(z, \bar{z}) p^\alpha B^j_\alpha B^i_\beta).\]
More generally, if \(v^j\) does not satisfy (2.2b), then we have
\[(2.8) \quad v K_{\lambda_*} = \left(\hat{\Omega}_{\beta \gamma} \frac{du^\gamma}{ds}\right) \left(g_{\alpha \beta}(u, \bar{u}) v^\alpha v^\beta\right)^{1/2}.\]

We shall, now, proceed to consider the following particular cases:

**CASE-I:** Suppose that the congruence \(\lambda_*(u)\) is the normal \(N(u)\) of
(2.4a) \[ \lambda_{*,i} = g_{i,j} (z, \bar{z}) \lambda^j. \]

and

(2.4b) \[ v_i = g_{i,j} (z, \bar{z}) v^j. \]

The scalar \( vK_{\lambda_*} \) will be called the P-curvature (with respect to C) of the congruence \( \lambda_* \) relative to vector field \( v \), while \( vK_{\lambda_*} \) will be called the S-curvature (with respect to C) of the congruence \( \lambda_* \) relative to \( v \).

From equations (2.1a), (2.4a), (1.23) and the definition

\[ \nabla_{\beta} E_{i,j} \overset{\text{def}}{=} \nabla_{\beta} g_{i,j} (z, \bar{z}), \]

we get

(2.5) \[ \frac{\delta \lambda_{*,i}}{ds} = \nabla_{\beta} \lambda_{*,i} \frac{du^\beta}{ds} = (q_{i,j,\beta}^\alpha B_{\alpha}^i + p_{i,j,\beta} m^j + g_{i,j} (z, \bar{z}) p^\alpha B_{\alpha,\beta}^i) \frac{du^\rho}{sa}, \]

where

\[ q_{i,j,\beta}^\alpha = E_{i,j,\beta} p^\alpha + g_{i,j} (z, \bar{z}) (\nabla_{\beta} p^\alpha + r A_{\alpha,\beta}^i) \]

and

\[ P_{i,j,\beta} = r E_{i,j,\beta} + g_{i,j} (z, \bar{z}) (\nabla_{\beta} r + r v_{\beta}). \]

Making use of the equations (2.1b) and (2.5) in (2.3) and simplifying, we have

(2.6) \[ vK_{\lambda_*} = \Omega_{\beta,\gamma} (u, \bar{u}) v^\beta \frac{du^\gamma}{ds}, \]

where

(2.7) \[ \Omega_{\beta,\gamma} (u, \bar{u}) = -(q_{i,j,\gamma}^\alpha B_{\alpha}^j B_{\beta}^i + p_{i,j,\gamma} m^j + g_{i,j} (z, \bar{z}) p^\alpha B_{\alpha,\gamma}^i B_{\beta}^i). \]

More generally, if \( v^j \) does not satisfy (2.2b), then we have

(2.8) \[ vK_{\lambda_*} = \frac{\left( \Omega_{\beta,\gamma} v^\beta \frac{du^\gamma}{ds} \right)^{1/2}}{(g_{\alpha,\beta} (u, \bar{u}) v^\alpha v^\beta)^{1/2}}. \]

We shall, now, proceed to consider the following particular cases:

**CASE-I:** Suppose that the congruence \( \lambda_* (u) \) is the normal \( N(u) \) of
the hypersurface. This gives $p^\alpha = 0$ and $r=1$. A simple calculation, based on (1.24) and (2.7), yields $\hat{\Omega}_{\beta \gamma} = \Omega_{\beta \gamma}$. Hence, we have

$$vK_{\lambda^a} = \Omega_{\beta \gamma} v^\beta \frac{du^\gamma}{ds},$$

which is normal curvature of the vector field (Prakash [86]).

If, in addition $v^i = z^i$, we have

$$vK_{\lambda^a} = \Omega_{\beta \gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds}.$$  This is the normal curvature (Rund [99]) of the hypersurface in the direction of the curve.

**CASE-II:** Let the vector field be tangent to the curve $C$, that is, $v^i = z^i$. Equation (2.6) reduces to

$$vK_{\lambda^a} = \hat{\Omega}_{\beta \gamma}(u, \overrightarrow{u}) \frac{du^\beta}{ds} \frac{du^\gamma}{ds},$$

which is the curvature of congruence (Prakash [87]).

**CASE-III:** Suppose that the space $K^c_n$ and $K^c_m$ are Riemannian, then we have (Kaul [49]):

$$\overline{E}_{ij\beta} = 0, \ v_\beta = 0, \ A^\alpha_{\beta} = -g^{\alpha \gamma} \Omega_{\beta \gamma}(Mishra [66]).$$

Substituting these values and simplifying with the help of equation (2.6), (2.7) and the relation given immediately after (2.5), we get

$$vK_{\lambda^a} = (r \Omega_{\alpha \beta} - \nabla_\beta p_\alpha) v^\alpha \frac{du^\beta}{ds},$$

which is the normal curvature of the vector field as given by (Kaul [49]).

From (2.1a), (1.23) and the relation given by (Rund [99]):

$$\nabla_\beta B^i_\alpha = \Omega_{\alpha \beta} m^i - \Omega_{\alpha \beta} m^h B^i_h B^i_\gamma,$$

we get

$$\frac{\delta \lambda^i_\alpha}{ds} = \nabla_\beta \lambda^i_\alpha \frac{du^\beta}{ds} = \left[(\nabla_\beta p^\alpha + r A^\alpha_{\beta} - p^\varepsilon \Omega_{\varepsilon \beta} m^h B^\varepsilon_h) \times B^i_\alpha + (\nabla_\beta r + r v_\beta + p^\varepsilon \Omega_{\varepsilon \beta}) m^i \right] \frac{du^\beta}{ds}.$$

We define
the hypersurface. This gives \( p^\alpha = 0 \) and \( r=1 \). A simple calculation, based on (1.24) and (2.7), yields 
\[
\hat{\Omega}_{\beta\gamma} = \Omega_{\beta\gamma}.
\]
Hence, we have
\[
(2.9) \quad v K_{\lambda^*} = \Omega_{\beta\gamma} v^\beta \frac{dv^\gamma}{ds},
\]
which is normal curvature of the vector field (Prakash [86]).

If, in addition \( v^i = z^i \), we have
\[
v K_{\lambda^*} = \Omega_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds}.
\]
This is the normal curvature (Rund [99]) of the hypersurface in the direction of the curve.

**CASE-II:** Let the vector field be tangent to the curve \( C \), that is, \( v^i = z^i \). Equation (2.6) reduces to
\[
(2.10) \quad v K_{\lambda^*} = \hat{\Omega}_{\beta\gamma}(u, \tilde{u}) \frac{du^\beta}{ds} \frac{du^\gamma}{ds},
\]
which is the curvature of congruence (Prakash [87]).

**CASE-III:** Suppose that the space \( K_n^c \) and \( K_m^c \) are Riemannian, then we have (Kaul [49]):
\[
\bar{E}_{i\beta} = 0, \quad v_\beta = 0, \quad A^\alpha_{\beta} = -g^{\alpha\gamma} \Omega_{\beta\gamma} \text{(Mishra [66])}.
\]
Substituting these values and simplifying with the help of equation (2.6), (2.7) and the relation given immediately after (2.5), we get
\[
(2.11) \quad v K_{\lambda^*} = (r \Omega_{\alpha\beta} - \nabla_\beta p_\alpha) v^\alpha \frac{du^\beta}{ds},
\]
which is the normal curvature of the vector field as given by (Kaul [49]).

From (2.1a), (1.23) and the relation given by (Rund [99]):
\[
\nabla_{\beta} B^{i}_{\alpha} = \Omega_{\alpha\beta} m^i - \Omega_{\alpha\beta} m^h B^{i}_{h} B^{i}_{\gamma},
\]
we get
\[
(2.12) \quad \frac{\delta \lambda^i}{\delta s} = \nabla_{\beta} \chi_{\lambda^*} \frac{du^\beta}{ds} = \left[ (\nabla_{\beta} p^\alpha + r A^a_{\beta} - p^\epsilon \Omega_{\epsilon\beta} m^h B^{i}_{h}) \right. \\
\times B^{i}_{\alpha} + (\nabla_{\beta} r + r v_\beta + p^\epsilon \Omega_{\epsilon\beta} m^i) \left. \frac{du^\beta}{ds} \right].
\]
We define
(2.13) \( g_{ij}(z, z) B^i_{\alpha} B^j_{\beta} = \tilde{\xi}_{\alpha\beta} \) and \( g_{ij}(z, z) B^i_{\alpha} m^j = \tilde{\xi}_{\alpha} \).

Making use of equations (2.3b) and (2.12) in (2.3b) and simplifying with the help of (2.13), we get

\[
(2.14) \quad v K_{\lambda*} = \overline{\Omega}_{\alpha\beta} \frac{du^\beta}{ds} v^\alpha,
\]

where

\[
(2.15) \quad \overline{\Omega}_{\alpha\beta}(u, \tilde{u}) = \left[ \tilde{\xi}_{\alpha\epsilon}(\Omega_{\epsilon\beta} p^\gamma m^h B_h^\epsilon - \nabla_{\beta} p^\gamma - r A_{\beta}^\epsilon) - \right.
\]

\[
- \tilde{\xi}_\alpha (\nabla_{\beta} r + r v_{\beta} + p^\gamma \Omega_{\gamma\beta}) \right].
\]

Suppose, in particular, that the space \( K_n^c \) and \( K_m^c \) are Riemannian, then we have

\[
\tilde{\xi}_{\alpha\beta} = g_{\alpha\beta}, \quad \tilde{\xi}_\alpha = 0, \quad A_{\beta}^\alpha = -g^{\alpha\gamma} \Omega_{\gamma\beta}, \quad v_{\beta} = 0,
\]

and the equation (2.15) gives

\[
\overline{\Omega}_{\alpha\beta} = (r \Omega_{\alpha\beta} - \nabla_{\beta} p_{\alpha}).
\]

Equation (2.14), thereby reduces to (2.11). Therefore, the scalar \( v K_{\lambda*} \) is another generalization of the curvature as given by Kaul [49].

The curvature \( vK_{\lambda*}(u, \tilde{u}) \) and \( v K_{\lambda*}(u, \tilde{u}) \), studied in this section, may be used to define two types of principal directions of a congruence relative to a vector field.

3. SECONDARY CURVATURES

Let us consider a vector \( \lambda_{**}(u, \tilde{u}) \) defining a congruence of a curve in the embedding space and a vector field \( v^* (u, \tilde{u}) \) of the hypersurface. The vector functions \( \lambda_{**}(u, \tilde{u}) \) and \( v^* (u, \tilde{u}) \) define, in general, two cones
\[(2.13) \quad g_{ij}(z, \bar{z}) B^i_{\alpha} B^j_{\beta} = \bar{\xi}_{\alpha \beta} \text{ and } g_{ij}(z, \bar{z}) B^i_{\alpha} m^j = \bar{\xi}_\alpha.\]

Making use of equations (2.3b) and (2.12) in (2.3b) and simplifying with the help of (2.13), we get
\[(2.14) \quad v K_{\lambda*} = \overline{\Omega}_{\alpha \beta} \frac{du^\beta}{ds} v^\alpha,\]

where
\[(2.15) \quad \overline{\Omega}_{\alpha \beta}(u, \bar{u}) = \left[\bar{\xi}_{\alpha \epsilon}(\Omega_{\epsilon \beta} p^\gamma m^h B^\epsilon_h - \nabla_\beta p^\epsilon - r A^\epsilon_{\beta}) - \bar{\xi}_\alpha (\nabla_\beta r + r v_\beta + p^\gamma \Omega_{\gamma \beta})\right].\]

Suppose, in particular, that the space \(K_n^c\) and \(K_m^c\) are Riemannian, then we have
\[\bar{\xi}_{\alpha \beta} = g_{\alpha \beta}, \quad \bar{\xi}_\alpha = 0, \quad A^\alpha_{\beta} = -g^{\alpha \gamma} \Omega_{\gamma \beta}, \quad v_\beta = 0,\]
\[\bar{\xi}_{\epsilon \alpha} B^\epsilon_h m^h = 0 \text{ (Mishra [66])}\]

and the equation (2.15) gives
\[\overline{\Omega}_{\alpha \beta} = (r \Omega_{\alpha \beta} - \nabla_\beta p_\alpha).\]

Equation (2.14), thereby reduces to (2.11). Therefore, the scalar \(v K_{\lambda*}\) is another generalization of the curvature as given by Kaul [49].

The curvature \(v K_{\lambda*}(u, \bar{u})\) and \(v \bar{K}_{\lambda*}(u, \bar{u})\), studied in this section, may be used to define two types of principal directions of a congruence relative to a vector field.

3. SECONDARY CURVATURES

Let us consider a vector \(\lambda_{**}(u, \bar{u})\) defining a congruence of a curve in the embedding space and a vector field \(v^*(u, \bar{u})\) of the hypersurface. The vector functions \(\lambda_{**}(u, \bar{u})\) and \(v^*(u, \bar{u})\) define, in general, two cones
of vectors at every point of $K^C_m$. However, in the sequel, only congruences are considered for which

$$u^\alpha = u^\alpha$$ at each point of $C$. We may write

$$\begin{aligned}
(a) & \quad \lambda^i_{**} = p^{*\alpha} B^i_{\alpha} + r^* m^{*i}, \\
(b) & \quad v^* i = v^{*\alpha} B^i_{\alpha}.
\end{aligned}$$

These vectors are normalized by the conditions:

$$\begin{aligned}
& g_{i j}(z, \bar{z}) \lambda^i_{**} \lambda^j_{**} = 1, \\
& g_{i j}(z, \bar{z}) v^* i v^* j = 1.
\end{aligned}$$

We define

$$\begin{aligned}
& v K^s_{\lambda} = -v^* i \frac{\delta \lambda^i_{**}}{\delta s} / \sqrt{\psi}, \\
& v K^s_{\lambda} = -v^* i \frac{\delta \lambda^i_{**}}{\delta s} / \sqrt{\psi}.
\end{aligned}$$

where,

$$\lambda^i_{**} = g_{i j}(z, \bar{z}) \lambda^j_{**}, v^* i = g_{i j}(z, \bar{z}) v^* j$$

and

$$\psi = g_{i j}(z, \bar{z}) m^{*i} m^{*j}.$$ 

The scalar $v K^s_{\lambda}(u, \bar{u})$ will be called $P$-secondary curvature (with respect to $C$) of $\lambda_{**}$ relative to $v^*$ and $v K^s_{\lambda}(u, \bar{u})$ will be called $S$-secondary curvature (with respect to $C$) of $\lambda_{**}$ relative to $v^*$.

From equations (3.3a), (3.1a) and (1.23*), we get

$$v K^s_{\lambda} = \hat{\Omega}^*_{\alpha \beta} = -\psi^{-1/2} [E^*_{j \beta} \lambda^j_{**} B^i_{\alpha} + g_{\alpha \gamma}(u, \bar{u})$$

$$\times (\nabla_\beta p^{*\gamma} + r^* A^*_{\beta \gamma})].$$

We, now, consider the following particular cases:

**CASE-I:** Suppose that the congruence $\lambda_{**}$ is along the secondary normal $m^*$ of the hypersurface, that is, $p^{*\alpha} = 0$. The condition (3.2a) implies $r \psi^{1/2} = 1$. Therefore, in view of (1.15), the equation (3.5), reduces to

$$\hat{\Omega}^*_{\alpha \beta} = \Omega^*_{\alpha \beta}$$ and, we have $v K^s_{\lambda} = \Omega^*_{\alpha \beta} v^{*\alpha} \frac{du^\beta}{ds}$, which is
of vectors at every point of $K_m^*$. However, in the sequel, only congruences are considered for which

$$u^\alpha = \tilde{u}^\alpha$$

at each point of C. We may write

$$\begin{cases} 
(a) \quad \lambda^i_{**} = p^\alpha \Omega_i^i - r^\alpha m^* i, \\
(b) \quad v^* i = v^\alpha B^i _{\alpha}.
\end{cases}$$

These vectors are normalized by the conditions:

$$\begin{cases} 
g_{ij}(z, \bar{z}) \lambda^j_{**} \lambda^i_{**} = 1, \\
g_{ij}(z, \bar{z}) v^* i v^* j = 1.
\end{cases}$$

We define

$$\begin{cases} 
v K^*_{\lambda} = -v^* i \frac{\delta \lambda^*_{*i}}{\delta s} / \sqrt{\psi}, \\
v K^*_{\bar{\lambda}} = -v^* i \frac{\delta \lambda^*_{*i}}{\delta s} / \sqrt{\psi}.
\end{cases}$$

where,

$$\lambda^*_{*i} = g_{ij}(z, \bar{z}) \lambda^j_{**}, v^* i = g_{ij}(z, \bar{z}) v^* j$$

and

$$\psi = g_{ij}(z, \bar{z}) m^* i m^* j.$$

The scalar $v K^*_{\lambda} (u, \tilde{u})$ will be called $P$-secondary curvature (with respect to C) of $\lambda_{**}$ relative to $v^*$ and $v K^*_{\bar{\lambda}} (u, \tilde{u})$ will be called $S$-secondary curvature (with respect to C) of $\lambda_{**}$ relative to $v^*$.

From equations (3.3a), (3.1a) and (1.23*), we get

$$v K^*_{\lambda} = \hat{\Omega}^* \alpha \beta = -\psi^{-1/2} [E^*_{ij} \lambda^j_{*i} B^i _{\alpha} + g_{\alpha \gamma}(u, \tilde{u}) \times (\nabla_{\beta} p^* \gamma + r^\gamma A^*_{\beta} \gamma)].$$

We, now, consider the following particular cases:

**CASE-I:** Suppose that the congruence $\lambda_{**}$ is along the secondary normal $m^*$ of the hypersurface, that is, $p^* \alpha = 0$. The condition (3.2a) implies $r \psi^{1/2} = 1$. Therefore, in view of (1.15), the equation (3.5), reduces to

$$\hat{\Omega}^* \alpha \beta = \Omega^* \alpha \beta$$

and, we have $v K^*_{\lambda} = \Omega^* \alpha \beta v^\alpha \frac{du^\beta}{ds}$, which is
the secondary normal curvature of the vector field (Prakash [88]). If, in addition, the vector \( v^* \) is tangent to the curve \( C \), we have

\[
v K^*_\lambda = \Omega^*_\alpha \beta \frac{du^\alpha}{ds} \frac{dv^\beta}{ds},
\]

which is the secondary normal curvature of the hypersurface (Rund [99]).

**CASE-II:** Let the space \( K^c_n \) and \( K^c_m \) be Riemannian, then we have

\[
E^*_{i j \beta} = 0, \; \psi = 1, \; \nu_\beta = 0 \text{ and } A^*_\beta \alpha = -\Omega^*_\nu \beta g^{\alpha \gamma}.
\]

Substituting these values in (3.5), we find that (3.4) reduces to the curvature given by Kaul [49] (refer (2.11)).

From (3.3b), we deduce

\[
v K^*_\lambda = \Omega^*_\alpha \beta \frac{dv^\beta}{ds},
\]

where

\[
\Omega^*_\alpha \beta = -\psi^{-1/2} g_{\alpha \gamma} (\nabla_\beta p^* \gamma + r^* A^*_{\beta \gamma}).
\]

It is easy to show that the curvature \( v K^*_\lambda \) is another generalization in \( K^c_m \) of the result of Kaul [49].

The two secondary curvature \( v K^*_\lambda \) and \( v K^*_\lambda \) may be used to define two types of asymptotic directions of the congruence relative to a vector field.

4. **PRIMATY RELATIVE ASSOCIATE CURVATURE VECTOR**

Consider (n-m) congruences of curves by unit vectors \( \left\langle \lambda^i, \lambda^j \right\rangle \), such that through each point of \( H_n \), there process one curve of each congru-
the secondary normal curvature of the vector field (Prakash [88]). If, in addition, the vector \( v^* \) is tangent to the curve \( C \), we have

\[
v K_\lambda^* = \Omega^*_{\alpha \beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds},
\]

which is the secondary normal curvature of the hypersurface (Rund [99]).

**CASE-II:** Let the space \( K_n^c \) and \( K_m^c \) be Riemannian, then we have

\[
E_{ij}^* = 0, \quad \psi = 1, \quad v_\beta = 0 \quad \text{and} \quad A_{\beta}^* = -\Omega_{\gamma \beta}^* g^{\alpha \gamma}.
\]

Substituting these values in (3.5), we find that (3.4) reduces to the curvature given by Kaul [49] (refer (2.11)).

From (3.3b), we deduce

\[
\Omega_{\alpha \beta}^* = -\psi^{-1/2} g_{\alpha \gamma} \left( \nabla_{\beta} p^* \gamma + r^* A_{\beta}^* \right).
\]

It is easy to show that the curvature \( v K_\lambda^* \) is another generalization in \( K_m^c \) of the result of Kaul [49].

The two secondary curvature \( v K_\lambda^* \) and \( v K_\lambda^* \) may be used to define two types of asymptotic directions of the congruence relative to a vector field.

### 4. PRIMATY RELATIVE ASSOCIATE CURVATURE VECTOR

Consider \( (n-m) \) congruences of curves by unit vectors \( \left\{ \lambda^i, \lambda^\mu \right\} \), such that through each point of \( H_n \), there process one curve of each congru-
ence.

Let the \((n-m)\) vectors \(\left(\lambda^i, \tilde{\lambda}^i\right)\) with any \(m\)-linearly independent vectors of \(H_m\) form a set of \(n\)-linearly independent vectors of \(H_n\), we may write

\[
\lambda^i = t^{\alpha} B^{\alpha \, i} + \sum_{\nu} C^{(\mu) (\nu)} \eta^i
\]

and

\[
\tilde{\lambda}^i = t^{\alpha} B^{\alpha \, i} + \sum_{\nu} C^{(\mu) (\nu)} \eta^i.
\]

It follows that vectors \(\left(\lambda^i, \tilde{\lambda}^i\right)\) with any \(m\)-linearly independent vectors of \(H_m\) form a set of \(n\)-linearly independent vectors of \(H_n\) iff, the determinants \(|C^{(\mu) (\nu)}|\) and \(|C^{(\mu) (\nu)}|\) are non-zero. The congruences thus given are called \(\lambda\)-congruences.

We define at each point of \(C\), a set of \((n-m)\) unit vectors \(\left(M^i, \tilde{M}^i\right)\) with the following properties:

(A): The components \(M^i\), for each \(\mu\) is a linear combination of \(\lambda^i\) and \(\frac{dz^i}{ds}\). Similarly, the components \(\tilde{M}^i\), for each \(\bar{\mu}\) is a linear combination of \(\tilde{\lambda}^i\) and \(\frac{dz^i}{ds}\).

(B): Every \(M^i\) is orthogonal to \(\frac{dz^i}{ds}\). In the same fashion, every \(\tilde{M}^i\) is orthogonal to \(\frac{dz^i}{ds}\).

Therefore, we have

\[
M^i = a \frac{dz^i}{ds} + b \lambda^i \quad \text{and} \quad g^{ij} \frac{dz^i}{ds} = 0
\]

and
ence.

Let the \((n-m)\) vectors \(\begin{bmatrix} \lambda^i_{(\mu)} & \tilde{\lambda}^i_{(\bar{\mu})} \end{bmatrix}\) with any \(m\)-linearly independent vectors of \(H_m\) form a set of \(n\)-linearly independent vectors of \(H_n\), we may write

\[
\lambda^i_{(\mu)} = t^\alpha \beta_{\alpha}^i + \sum_{(v)} C_{(\mu\nu)} \eta^i_{(v)}
\]

and

\[
\tilde{\lambda}^i_{(\bar{\mu})} = t^{\bar{\alpha}} \beta_{\bar{\alpha}}^i + \sum_{(\bar{v})} C_{(\mu\nu)} \eta^i_{(\bar{v})}.
\]

It follows that vectors \(\begin{bmatrix} \lambda^i_{(\mu)} & \tilde{\lambda}^i_{(\bar{\mu})} \end{bmatrix}\) with any \(m\)-linearly independent vectors of \(H_m\) form a set of \(n\)-linearly independent vectors of \(H_n\) iff, the determinants \(|C_{(\mu\nu)}|\) and \(|C_{(\mu\bar{\nu})}|\) are non-zero. The congruences thus given are called \(\lambda\)-congruences.

We define at each point of \(C\), a set of \((n-m)\) unit vectors \(\begin{bmatrix} M^i_{(\mu)} & M^i_{(\bar{\mu})} \end{bmatrix}\) with the following properties:

\((A)\): The components \(M^i_{(\mu)}\) for each \(\mu\) is a linear combination of \(\lambda^i_{(\mu)}\) and \(\frac{d\lambda^i}{ds}\). Similarly, the components \(M^i_{(\bar{\mu})}\) for each \(\bar{\mu}\) is a linear combination of \(\tilde{\lambda}^i_{(\bar{\mu})}\) and \(\frac{d\tilde{\lambda}^i}{ds}\).

\((B)\): Every \(M^i_{(\mu)}\) is orthogonal to \(\frac{d\lambda^i}{ds}\). In the same fashion, every \(M^i_{(\bar{\mu})}\) is orthogonal to \(\frac{d\tilde{\lambda}^i}{ds}\).

Therefore, we have

\[
M^i_{(\mu)} = a \frac{dz^i}{ds} + b \lambda^i_{(\mu)} \quad \text{and} \quad g_{ij} \frac{dz^j}{ds} = 0
\]

and
(4.2*) \[ \dot{M}^i = a \frac{dz^i}{ds} + b \lambda^i \] and \[ g^{-ij} \frac{dz^j}{ds} = 0. \]

These relations give

(4.3) \[ a = -b \lambda \quad \text{and} \quad b = \pm \left( \frac{\phi}{(\mu)} \right)^{1/2} / \left( \psi - \lambda^2 \right)^{1/2}, \]

where

\[ \lambda = g^{ij} \lambda^i \frac{dz^j}{ds}, \quad \psi = g^{ij} \lambda^i \lambda^j \]

and

\[ \bar{\phi} = g^{ij} M^i M^j \]

and

(4.3*) \[ a = -b \lambda \quad \text{and} \quad b = \pm \left( \frac{\phi}{(\bar{\mu})} \right)^{1/2} / \left( \psi - \lambda^2 \right)^{1/2}, \]

where

\[ \bar{\lambda} = g^{ij} \bar{\lambda}^i \frac{dz^j}{ds}, \quad \bar{\psi} = g^{ij} \bar{\lambda}^i \bar{\lambda}^j \]

and

\[ \bar{\phi} = g^{ij} \bar{M}^i \bar{M}^j. \]

**REMARK:** Here, the repeated indices \( \mu \) and \( \bar{\mu} \) do not stand for summation.

Making use of equations (4.1), (4.1*) and (4.3), we deduce

(4.4) \[ M^i = \pm \left( \int (\alpha - \frac{du^\alpha}{ds}) B_\alpha^j + \sum_{(v)} C_{(\nu v)} m^j \left( (\bar{\phi}) \right)^{1/2} / \left( \psi - \lambda^2 \right)^{1/2} \right. \]

Similarly, with the help of equations (4.1), (4.2*) and (4.3*), we get
(4.2*) \[ M^i = a \frac{dz^i}{ds} + b \lambda^i \quad \text{and} \quad g^{-1}_{i j} \frac{dz^j}{ds} = 0. \]

These relations give

(4.3) \[ a = -b \lambda \quad \text{and} \quad b = \pm \left( -\frac{\phi}{(\mu)} \right)^{1/2} \left( \psi - \lambda^2 \right)^{1/2}, \]

where

\[ \lambda = g_{i j} \lambda^i \frac{dz^j}{ds}, \quad \psi = g_{i j} \lambda^i \lambda^j \]

and

\[ \phi = g_{i j} M^i M^j. \]

and

(4.3*) \[ a = -b \lambda \quad \text{and} \quad b = \pm \left( -\frac{\phi}{(\mu)} \right)^{1/2} \left( \psi - \lambda^2 \right)^{1/2}, \]

where

\[ \lambda = g^{-1}_{i j} \lambda^i \frac{dz^j}{ds}, \quad \psi = g^{-1}_{i j} \lambda^i \lambda^j \]

and

\[ \phi = g^{-1}_{i j} M^i M^j. \]

**REMARK:** Here, the repeated indices \( \mu \) and \( \bar{\mu} \) do not stand for summation.

Making use of equations (4.1), (4.1*) and (4.3), we deduce

(4.4) \[ M^i = \pm \left[ \left( \alpha - \lambda \right)_\mu \frac{du^\alpha}{ds} \right] B_\alpha^i + \sum_{(v)} C_{\mu \nu} m^i_{(\nu)} \left( \frac{\phi}{(\mu)} \right)^{1/2} \left( \psi - \lambda^2 \right)^{1/2}. \]

Similarly, with the help of equations (4.1), (4.2*) and (4.3*), we get
\[
(4.4^*) \quad M_i^{\dagger} = \pm \left[ \left( t^{\alpha} - \lambda \frac{du^\alpha}{ds} \right) B_{\alpha}^i + \sum_{(v)} C_{(\mu\nu)} m_i^{\mu} \right] \left( \frac{\phi}{(\mu)} \right)^{1/2} \sqrt{\left( \psi - \lambda (\mu) \right)^2}.
\]

Using the fact that the components \( B_{\alpha}^i \) and \( m_i^{\mu} \) are linearly independent and \( | C_{(\mu\nu)} | \neq 0 \), it may be verified that
\[
\sum_{\alpha=1}^{m} p(\alpha) B_{\alpha}^i + \sum_{\mu=m+1}^{n} q(\mu) M_i^{\mu} = 0, \quad \text{implies} \quad p(\alpha) = 0, \quad \text{for} \quad \alpha = 1,2,3,\ldots,m \quad \text{and} \quad q(\mu) = 0, \quad \text{for} \quad \mu = m+1,\ldots,n.
\]

Similarly, it may be verified that
\[
\sum_{\alpha=1}^{m} p(\alpha) B_{\alpha}^{\dagger} - \sum_{\mu=m+1}^{n} q(\mu) M_i^{\mu} = 0, \quad \text{implies} \quad p(\alpha) = 0, \quad \text{for} \quad \alpha = 1,2,3,\ldots,m \quad \text{and} \quad q(\mu) = 0, \quad \text{for} \quad \mu = m+1,\ldots,n.
\]

Hence, the vectors \( \left\{ B_{\alpha}^i, B_{\alpha}^{\dagger} \right\} \) and \( \left\{ M_i^{\mu}, M_i^{\mu} \right\} \) form a set of \( n \)-linearly independent vectors in \( H_n \). Also, the \( (n-m) \) vectors \( \left\{ M_i^{\mu}, M_i^{\mu} \right\} \) are linearly independent.

Eliminating \( m_i^{\mu} \) with the help of equations (1.34*), (1.35*) and (4.4*), we get
\[
(4.5) \quad \left( \frac{\partial m_i^{\mu}}{\partial s} \right) = B_{\alpha}^i \nu^\alpha + \sum_{(v)} \sum_{(\mu)} \overline{C}_{(\mu\nu)} A_{(\alpha\beta)} V^\alpha \frac{du^\beta}{ds} \left[ \left( \frac{\phi}{(\mu)} \right)^{1/2} \sqrt{\left( \psi - \lambda (\mu) \right)^2} M_i^{\mu} \right],
\]

where
\[
\overline{C}_{(\mu\nu)} = \text{Co-factor of } C_{(\mu\nu)} \text{ in } | C_{(\mu\nu)} | / | C_{(\mu\nu)} |.
\]

and
Using the fact that the components $B^i_\alpha$ and $m^i_{(v)}$ are linearly independent and 
\[ |C_{(\mu\nu)}| \neq 0, \] it may be verified that 

\[ \sum_{\alpha=1}^{m} p(\alpha) B^i_\alpha + \sum_{\mu=m+1}^{n} q(\mu) M^i_{(\mu)} = 0, \text{ implies } p(\alpha) = 0, \text{ for } \alpha = 1,2,3,\ldots,m \text{ and } q(\mu) = 0, \text{ for } \mu = m+1,\ldots,n. \]

Similarly, it may be verified that

\[ \sum_{\bar{\alpha}=1}^{\bar{m}} p(\bar{\alpha}) B^{i\bar{\alpha}} + \sum_{\bar{\mu}=m+1}^{\bar{n}} q(\bar{\mu}) M^{i\bar{\mu}} = 0, \text{ implies } p(\bar{\alpha}) = 0, \text{ for } \bar{\alpha} = 1,2,3,\ldots,\bar{m} \text{ and } q(\bar{\mu}) = 0, \text{ for } \bar{\mu} = m+1,\ldots,\bar{n}. \]

Hence, the vectors \( \{B^i_\alpha, B^{i\bar{\alpha}}\} \) and \( \{M^i_{(\mu)}, M^{i\bar{\mu}}\} \) form a set of n-linearly independent vectors in $H_n$. Also, the (n-m) vectors \( \{M^i_{(\mu)}, M^{i\bar{\mu}}\} \) are linearly independent.

Eliminating $m^i_{(v)}$ with the help of equations (1.34*), (1.35*) and (4.4*), we get

\[(4.5) \quad \left( \delta m^i_{(v)} / \delta s \right) = B^i_\alpha v^\alpha + \sum_{(\mu)} \sum_{(v)} \bar{C}_{(\mu\nu)} A_{\alpha\beta} V^\alpha \frac{du^\beta}{ds} (\bar{\phi})^{1/2}\left( \psi - \lambda(\mu) \right)^{1/2} M^i_{(\mu)}, \]

where

\[ \bar{C}_{(\mu\nu)} = \text{Co-factor of } C_{(\mu\nu)} \text{ in } |C_{(\mu\nu)}| / |C_{(\bar{\mu}\bar{\nu})}| \]

and
\[(4.5^*)\quad V^\alpha = \frac{\partial \bar{v}^\alpha}{\partial s} + \sum_{(\mu)} \sum_{(\nu)} A_{\beta\gamma} \bar{V}^\beta \frac{du^\gamma}{ds} \times \]
\[\times \{\delta_{(\mu\nu)} M^\alpha_{(\mu)} - \bar{C}_{(\mu\nu)} t_{(\mu)}^\alpha - \lambda_{(\mu)} \frac{du^\alpha}{ds} \},\]

where \(\delta_{(\mu\nu)}\) being the Kronecker delta. Similarly, we have
\[(4.6)\quad \left( \frac{\delta m^i}{\delta s} \right) = B^{-i}_{\alpha} \bar{v}^\alpha + \sum_{(\mu)} \sum_{(\nu)} \bar{C}_{(\mu\nu)} A_{\alpha\beta} \bar{V}^\alpha \frac{du^\beta}{ds} \times \]
\[\times \left( \phi \right)^{1/2} \left( \psi - \lambda_{(\bar{\mu})} \right)^{1/2} M^i_{(\bar{\mu})} \]

and
\[(4.6^*)\quad \bar{V}^{\tilde{\alpha}} = \frac{\partial \bar{v}^{\tilde{\alpha}}}{\partial s} + \sum_{(\bar{\mu})} \sum_{(\bar{\nu})} A_{\tilde{\beta}\bar{\gamma}} \bar{V}^{\tilde{\beta}} \frac{du^{\bar{\gamma}}}{ds} \times \]
\[\times \{\delta_{(\bar{\mu}\bar{\nu})} M^{\tilde{\alpha}}_{(\bar{\mu})} - \bar{C}_{(\bar{\mu}\bar{\nu})} t_{(\bar{\mu})}^{\tilde{\alpha}} - \lambda_{(\bar{\mu})} \frac{du^{\tilde{\alpha}}}{ds} \},\]

where \(\delta_{(\bar{\mu}\bar{\nu})}\) being the Kronecker delta.

The vectors \(\langle V^\alpha, \bar{V}^{\tilde{\alpha}} \rangle\) given above is called the primary relative associate curvature (relative to \(\lambda\)-congruence) of the vector field in the direction of \(C\). The scalar curvature \(\overline{K}^2\) defined by
\[(4.7)\quad \overline{K}^2 = 2 g_{\alpha \beta}(u, \bar{u}) V^\alpha V^{\bar{\beta}}\]
is called the primary relative associate curvature of the vector field.

We, now, have the following:

**THEOREM (4.1):** A sufficient condition that the primary relative associate curvature vector \(\langle V^\alpha, \bar{V}^{\tilde{\alpha}} \rangle\), in the direction of a curve \(C\) be equal to its associate curvature vector \(\left( \frac{\partial \bar{v}^\alpha}{\partial s}, \frac{\partial \bar{v}^{\tilde{\alpha}}}{\partial s} \right)\) is that the vector field is conjugate with respect to \(C\).

The proof follows immediately from the equations (1.37), (1.37*), (4.6) and (4.6*).
\[ (4.5^*) \quad V^\alpha = \frac{\delta V^\alpha}{\delta s} + \sum_{(\mu) (\nu)} A_{\beta \gamma} V^\beta \frac{du^\gamma}{ds} \times \]
\[ \times \left\{ \delta_{(\mu\nu)} M^\alpha - \overline{C}_{(\mu\nu)} \left( t_{(\mu)} - \lambda_{(\mu)} \frac{du^\alpha}{ds} \right) \right\}, \]

where \( \delta_{(\mu\nu)} \) being the Kronecker delta. Similarly, we have
\[ (4.6) \quad \left( \frac{\delta m^i}{(\nu)} / \delta s \right) = B^{- i}_{\alpha} V^\alpha + \sum_{(\mu)} C^{- i}_{(\mu\nu)} A^{- \alpha}_{\beta \gamma} V^\alpha \frac{du^\beta}{ds} \]
\[ \times \left( \phi \right)^{1/2} \left( \psi - \lambda_{(\mu)} \right)^2 \left( M^i \right) \]
and
\[ (4.6^*) \quad V^\bar{\alpha} = \frac{\delta V^\bar{\alpha}}{\delta s} + \sum_{(\mu) (\nu)} A^{- \bar{\alpha}}_{\beta \gamma} V^\bar{\beta} \frac{du^\gamma}{ds} \times \]
\[ \times \left\{ \delta_{(\mu\nu)} M^\bar{\alpha} - \overline{C}_{(\mu\nu)} \left( t_{(\mu)} - \lambda_{(\mu)} \frac{du^\alpha}{ds} \right) \right\}, \]

where \( \delta_{(\mu\nu)} \) being the Kronecker delta.

The vectors \( \left( V^\alpha, V^\bar{\alpha} \right) \) given above is called the primary relative associate curvature (relative to \( \lambda \)-congruence) of the vector field in the direction of \( C \). The scalar curvature \( \overline{K}^2 \) defined by
\[ (4.7) \quad \overline{K}^2 = 2 \, g_{\alpha \beta} (u, \bar{u}) \, V^\alpha \, V^\bar{\beta} \]
is called the primary relative associate curvature of the vector field.

We, now, have the following:

**THEOREM (4.1):** A sufficient condition that the primary relative associate curvature vector \( \left( V^\alpha, V^\bar{\alpha} \right) \), in the direction of a curve \( C \) be equal to its associate curvature vector \( \left( \frac{\delta V^\alpha}{\delta s}, \frac{\delta V^\bar{\alpha}}{\delta s} \right) \) is that the vector field is conjugate with respect to \( C \).

The proof follows immediately from the equations (1.37), (1.37*), (4.6) and (4.6*).
THEOREM (4.2): A necessary and sufficient condition that the derived vector \( \left( \frac{\partial \alpha}{\partial s}, \frac{\partial \beta}{\partial s} \right) \) be equal to the primary relative associate curvature vector \( \left( v_\alpha B_\alpha^i, \bar{v}_\alpha B_\alpha^{-i} \right) \) (both the vectors considered along the same curve C) is that the vector field is conjugate with respect to C.

PROOF: If the vector field is conjugate with respect to C, then (1.37), (1.37*), (4.5) and (4.6) give
\[
\frac{\partial \nu^i}{\partial s} = v_\alpha B_\alpha^i \quad \text{and} \quad \frac{\partial \nu^\alpha}{\partial s} = \bar{v}_\alpha B_\alpha^{-i}.
\]

Conversely, let the above relation holds. Since, \( \left( M^i, M_{\mu}^{-i} \right) \) are linearly independent, we have from (2.5) and (2.6)
\[
(4.8) \quad \sum_{(\nu)} \overline{C}_{(\mu\nu)} A_{\alpha\beta} V^\alpha \frac{du^\beta}{ds} \left( \overline{\phi} \right)_{(\mu)}^{1/2} \left( \psi - \lambda (\mu) \right)^2 = 0
\]
and
\[
(4.8^*) \quad \sum_{(\nu)} \overline{C}_{(\mu\nu)} A_{\tilde{\alpha}\tilde{\beta}} V^{\tilde{\alpha}} \frac{du^{\tilde{\beta}}}{ds} \left( \overline{\phi} \right)_{(\mu)}^{1/2} \left( \psi - \lambda (\mu) \right)^2 = 0.
\]

From (4.3) and (4.3*), we have
\[
\pm \left( \overline{\phi} - \lambda^2 \right)^{-1/2} \left( \psi \right)_{(\mu)}^{-1/2} = \left( 1 / b_{(\mu)} \right) \neq 0
\]
and
\[
\pm \left( \overline{\phi} - \lambda^2 \right)^{-1/2} \left( \psi \right)_{(\tilde{\mu})^{-1/2}} = \left( 1 / b_{(\tilde{\mu})} \right) \neq 0.
\]

Hence the equations (4.8) and (4.8*) reduce to
\[
(4.9) \quad \sum_{(\nu)} \overline{C}_{(\mu\nu)} A_{\alpha\beta} V^\alpha \frac{du^\beta}{ds} = 0
\]
and
\[
(4.9^*) \quad \sum_{(\nu)} \overline{C}_{(\mu\nu)} A_{\tilde{\alpha}\tilde{\beta}} V^{\tilde{\alpha}} \frac{du^{\tilde{\beta}}}{ds} = 0.
\]

These are \((n-m)\) linearly homogeneous equations in \((n-m)\)
THEOREM (4.2): A necessary and sufficient condition that the derived vector \( \left( \frac{\partial \nu^\alpha}{\partial s}, \frac{\partial \nu^\beta}{\partial s} \right) \) be equal to the primary relative associate curvature vector \( \left( \nu^\alpha B_{\alpha}^i, \nu^\beta B_{\beta}^i \right) \) (both the vectors considered along the same curve C) is that the vector field is conjugate with respect to C.

PROOF: If the vector field is conjugate with respect to C, then (1.37), (1.37*), (4.5) and (4.6) give

\[
\frac{\partial \nu^i}{\partial s} = \nu^\alpha B_{\alpha}^i \quad \text{and} \quad \frac{\partial \nu^\alpha}{\partial s} = \nu^\beta B_{\beta}^i.
\]

Conversely, let the above relation holds. Since, \( \left( M^i, M^\beta \right) \) are linearly independent, we have from (2.5) and (2.6)

\[
(4.8) \quad \sum_{(\nu)} \bar{C}_{(\mu \nu)} A_{(\nu)}^\alpha \frac{d\nu^\beta}{ds} \left( \phi_{(\mu)} \right)^{1/2} \left( \psi_{(\mu)} - \lambda_{(\mu)} \right)^2 = 0
\]

and

\[
(4.8*) \quad \sum_{(\nu)} \bar{C}_{(\mu \nu)} A_{(\nu)}^\tilde{\alpha} \frac{d\nu^\tilde{\beta}}{ds} \left( \phi_{(\mu)} \right)^{1/2} \left( \psi_{(\mu)} - \lambda_{(\mu)} \right)^2 = 0.
\]

From (4.3) and (4.3*), we have

\[
\pm \left( \phi_{(\mu)} - \lambda_{(\mu)} \right)^{-1/2} \left( \psi_{(\mu)} \right)^{-1/2} = (1 / b_{(\mu)}) \neq 0
\]

and

\[
\pm \left( \phi_{(\tilde{\mu})} - \lambda_{(\tilde{\mu})} \right)^{-1/2} \left( \psi_{(\tilde{\mu})} \right)^{-1/2} = (1 / b_{(\tilde{\mu})}) \neq 0.
\]

Hence the equations (4.8) and (4.8*) reduce to

\[
(4.9) \quad \sum_{(\nu)} \bar{C}_{(\mu \nu)} A_{(\nu)}^\alpha \frac{d\nu^\beta}{ds} = 0
\]

and

\[
(4.9*) \quad \sum_{(\nu)} \bar{C}_{(\mu \nu)} A_{(\nu)}^\tilde{\alpha} \frac{d\nu^\tilde{\beta}}{ds} = 0.
\]

These are (n-m) linearly homogeneous equations in (n-m)
unknowns
\[ \left( A_{\alpha \beta} V^\alpha \frac{du^\beta}{ds}, A_{\tilde{\alpha} \tilde{\beta}} \tilde{V}^{\tilde{\alpha}} \frac{d\tilde{u}^{\tilde{\beta}}}{ds} \right), \]

for which
\[ | \overline{C}_{(\mu \nu)} | = \{ | C_{(\mu \nu)} | \}^{-1} \neq 0 \]

and
\[ | \overline{C}_{(\tilde{\mu} \tilde{\nu})} | = \{ | C_{(\tilde{\mu} \tilde{\nu})} | \}^{-1} \neq 0. \]

We have, therefore, a unique solution given by
\[ A_{\alpha \beta} V^\alpha \frac{du^\beta}{ds} = 0 \text{ and } A_{\tilde{\alpha} \tilde{\beta}} \tilde{V}^{\tilde{\alpha}} \frac{d\tilde{u}^{\tilde{\beta}}}{ds} = 0. \]

This completes the proof of the theorem.

**COROLLARY (4.1):** A necessary and sufficient condition that the three vectors, the derived vector \( \left( \frac{\partial V^\alpha}{\partial s}, \frac{\partial V^{\tilde{\alpha}}}{\partial \tilde{s}} \right) \), the associate curvature vector \( \left( \frac{\partial V^{\alpha}}{\partial s} B^i_{\alpha}, \frac{\partial V^{\tilde{\alpha}}}{\partial \tilde{s}} B^i_{\tilde{\alpha}} \right) \) and the primary relative associate curvature vector \( \left( \nu^\alpha B^i_{\alpha}, \tilde{\nu}^{\tilde{\alpha}} B^i_{\tilde{\alpha}} \right) \) be equal is such that the vector field is conjugate with respect to the curve \( C \).

The proof is immediate from the theorems (2.1) and (2.2).

5. SECONDARY RELATIVE ASSOCIATE CURVATURE VECTOR

Consider a set of \((n-m)\) congruences of curve (called \( \lambda^* \)-congruence) given by the unit vectors \( \left( \lambda_{(\mu)}^*, \lambda_{(\tilde{\mu})}^* \right) \), we may write

\[ \lambda_{(\mu)}^* = t_{(\mu)}^* B^i_{\alpha} + \Sigma C^*_{(\mu \nu)} m^*_{(\nu)} \]

and

\[ \lambda_{(\tilde{\mu})}^* = t_{(\tilde{\mu})}^* B^i_{\tilde{\alpha}} + \Sigma C^*_{(\tilde{\mu} \tilde{\nu})} m^*_{(\tilde{\nu})}. \]
unknowns

\[ \left( A_{a\beta} V^\alpha \frac{du^\alpha}{ds}, A_{\alpha\beta} V^\alpha \frac{du^\alpha}{ds} \right), \text{for which} \]

\[ | \overline{C}_{(\mu \nu)} | = \{ | C_{(\mu \nu)} | \}^{-1} \neq 0 \]

and

\[ | \overline{C}_{(\bar{\mu} \bar{\nu})} | = \{ | C_{(\bar{\mu} \bar{\nu})} | \}^{-1} \neq 0. \]

We have, therefore, a unique solution given by

\[ A_{a\beta} V^\alpha \frac{du^\alpha}{ds} = 0 \text{ and } A_{\alpha\beta} V^\alpha \frac{du^\alpha}{ds} = 0. \]

This completes the proof of the theorem.

**COROLLARY (4.1):** A necessary and sufficient condition that the three vectors, the derived vector \( \left( \frac{\partial \lambda^i}{\partial s}, \frac{\partial \lambda^i}{\partial \lambda} \right) \), the associate curvature vector \( \left( \frac{\partial \gamma^\alpha}{\partial s} B^{i, \alpha}, \frac{\partial \gamma^\alpha}{\partial s} B^{i, \bar{\alpha}} \right) \) and the primary relative associate curvature vector \( \left( \gamma_\alpha B_{\alpha}^i, \gamma_{\bar{\alpha}} B_{\bar{\alpha}}^\bar{i} \right) \) be equal is such that the vector field is conjugate with respect to the curve \( C \).

The proof is immediate from the theorems (2.1) and (2.2).

### 5. SECONDARY RELATIVE ASSOCIATE CURVATURE VECTOR

Consider a set of \((n-m)\) congruences of curve (called \( \lambda^* \)-congruence) given by the unit vectors \( \left( \lambda^*_i, \lambda^*_{\bar{i}} \right) \), we may write

\[ \lambda^*_i = t^{*\alpha} B^{i, \alpha} + \sum C^*_{(\mu \nu)} m^*_{\nu} \]

and

\[ \lambda^*_{\bar{i}} = t^{*\bar{\alpha}} B^{\bar{i}, \bar{\alpha}} + \sum C^*_{(\bar{\mu} \bar{\nu})} m^*_{\bar{\nu}}. \]
it is assumed that the vectors \( \left( \lambda^{*i}_{(\mu)}, \lambda^{*j}_{(\mu)} \right) \) with any m-linearly independent vectors of \( H_m \) form a set of n-linearly independent vectors in \( H_n \).

This implies that the determinants \( | C^*_{(\mu\nu)} | \) and \( | C^*_{(\mu\nu)} | \) are non-zero.

Defining unit vectors \( \left( \text{M}^{s}_{H_{mL}}, \text{M}^{s}_{I_{m-M}} \right) \) as

\[
\text{M}^{s}_{(\mu)} = A \frac{dz^i}{ds} + B \lambda^{*i}_{(\mu)} \text{ and } g_{t\bar{t}} \text{M}^{s}_{(\mu)} \frac{dz^j}{ds} = 0
\]

and

\[
\text{M}^{s}_{(\bar{\mu})} = A \frac{dz^i}{ds} + B \lambda^{*i}_{(\bar{\mu})} \text{ and } g_{t\bar{t}} \text{M}^{s}_{(\bar{\mu})} \frac{dz^j}{ds} = 0
\]

and simplifying (as in section 4), we get

\[
\text{M}^{s}_{(\mu)} = \pm \left( \phi^* \right)^{1/2}_{(\mu)} \left[ \left( t^{*\alpha} - t^{*} \frac{du^\alpha}{ds} \right) B_{\alpha}^* \right] + \sum \left( C^*_{(\mu\nu)} m_{(\nu)}^{*i} \right) \left( \psi^* - t^{*2}_{(\mu)} \right)^{-1/2}
\]

and

\[
\text{M}^{s}_{(\bar{\mu})} = \left( \phi^* \right)^{1/2}_{(\bar{\mu})} \left[ \left( t^{*\bar{\alpha}} - t^{*} \frac{du^\bar{\alpha}}{ds} \right) B_{\bar{\alpha}}^* \right] + \sum \left( C^*_{(\mu\nu)} m_{(\nu)}^{*i} \right) \left( \psi^* - t^{*2}_{(\mu)} \right)^{-1/2}
\]

where

\[
\psi^* = g_{ij} \lambda^{*i}_{(\mu)} \lambda^{*j}_{(\mu)} ; \quad \psi^* = g_{ij} \lambda^{*i}_{(\bar{\mu})} \lambda^{*j}_{(\bar{\mu})} ;
\]

\[
t^{*}_{(\mu)} = g_{\alpha\beta} t^{*\bar{\beta}} \frac{du^\alpha}{ds} ; \quad t^{*}_{(\bar{\mu})} = g_{\alpha\beta} t^{*\bar{\beta}} \frac{du^\alpha}{ds} ;
\]

\[
\phi^* = g_{ij} \text{M}^{s}_{(\mu)} \text{M}^{s}_{j} ; \quad \text{and } \phi^* = g_{ij} \text{M}^{s}_{(\bar{\mu})} \text{M}^{s}_{j}.
\]

The positive sign in (5.3) is taken if the determinant \( | C^*_{(\mu\nu)} | < 0 \).

Consequently, the positive sign in (5.3*) is taken if the determinant
it is assumed that the vectors $\left(\lambda^{*i}_{(\mu)}, \lambda^{*i}_{(\bar{\mu})}\right)$ with any m-linearly independent vectors of $H_m$ form a set of n-linearly independent vectors in $H_n$.

This implies that the determinants $|C^*_{(\mu\nu)}|$ and $|C^*_{(\bar{\mu}\bar{\nu})}|$ are non-zero.

Defining unit vectors $\left(M^i_{(\mu)}, M^i_{(\bar{\mu})}\right)$ as

$$M^*_{(\mu)} = A \frac{dz^i_{(\mu)}}{ds} + B \lambda^{*i}_{(\mu)} \text{ and } g_{i\bar{j}} M^*_{(\mu)} \frac{dz^i_{(\mu)}}{ds} = 0$$

and

$$M^*_{(\bar{\mu})} = A \frac{dz^i_{(\bar{\mu})}}{ds} + B \lambda^{*i}_{(\bar{\mu})} \text{ and } g_{i\bar{j}} M^*_{(\bar{\mu})} \frac{dz^i_{(\bar{\mu})}}{ds} = 0$$

and simplifying (as in section 4), we get

$$M^*_{(\mu)} = \pm \left(\phi^*_{(\mu)}\right)^{1/2} \left(\left(t^*_{(\mu)} - t^* \frac{du^*}{ds}\right) B_{\alpha^i} + \sum_{(\nu)} C^*_{(\mu\nu)} m^*_{(\nu)} \left(\psi^*_{(\mu)} - t^2_{(\mu)} \right)^{-1/2}\right)$$

and

$$M^*_{(\bar{\mu})} = \left(\phi^*_{(\bar{\mu})}\right)^{1/2} \left(\left(t^*_{(\bar{\mu})} - t^* \frac{du^*}{ds}\right) B_{\bar{\alpha}^i} + \sum_{(\bar{\nu})} C^*_{(\bar{\mu}\bar{\nu})} m^*_{(\bar{\nu})} \left(\psi^*_{(\bar{\mu})} - t^2_{(\bar{\mu})} \right)^{-1/2}\right)$$

where

$$\psi^*_{(\mu)} = g_{ij}^{(\mu)} \lambda^{*i}_{(\mu)} \lambda^{*j}_{(\mu)} ; \psi^*_{(\bar{\mu})} = g_{ij}^{(\bar{\mu})} \lambda^{*i}_{(\bar{\mu})} \lambda^{*j}_{(\bar{\mu})} ;$$

$$t^*_{(\mu)} = g_{\alpha\beta}^{(\mu)} \frac{du^*}{ds} ; \ t^*_{(\bar{\mu})} = g_{\alpha\beta}^{(\bar{\mu})} \frac{du^*}{ds} ;$$

$$\phi^*_{(\mu)} = g_{ij}^{(\mu)} M^*^i_{(\mu)} M^*^j_{(\mu)} ; \text{ and } \phi^*_{(\bar{\mu})} = g_{ij}^{(\bar{\mu})} M^*^i_{(\bar{\mu})} M^*^j_{(\bar{\mu})}.$$
\[ |C^*_\left(\mu\nu\right)\left(\mu\mu\right)| < 0.\] The equation (5.3) reduces to \(M^*_{\left(\mu\mu\right)} = m^*_{\left(\mu\mu\right)}\), if \(\lambda^*_{\left(\mu\mu\right)}\) is a linear combination of \(\frac{dx}{ds}_{\left(\mu\mu\right)}\) and \(m^*_{\left(\mu\mu\right)}\). Similarly, (5.3*) reduces to \(M^*_{\left(\mu\mu\right)} = m^*_{\left(\mu\mu\right)}\), if \(\lambda^*_{\left(\mu\mu\right)}\) is a linear combination of \(\frac{dx}{ds}_{\left(\mu\mu\right)}\) and \(m^*_{\left(\mu\mu\right)}\).

It follows that the vectors \(\left\{B^0_{\alpha^0}, B^0_{\alpha^0}^{-1}\right\}\) are linearly independent in \(H_n\) and also the vectors \(\left\{M^\beta, M^\beta\right\}\) are linearly independent.

Eliminating \(m^*_{\left(\mu\mu\right)}\) with the help of (1.33), (1.35) and (1.37), we have

\[(5.4)\]

\[
\left(\frac{d^V}{ds}\right) = V^{*\alpha} B^0_{\alpha^0} \pm \sum_{\left(\nu\right)\left(\mu\right)} C^*_{\left(\nu\nu\right)\left(\mu\mu\right)} \Omega_{\left(\nu\beta\right)\left(\mu\gamma\right)} V^\beta \frac{du^\gamma}{ds} \left(\phi_{\left(\mu\mu\right)}\right)^{-1/2} \times \left(\psi^* - t^*V\right) M^*_{\left(\mu\mu\right)},
\]

where

\[C^*_{\left(\nu\mu\right)} = \text{Cofactor of } C^*_{\left(\nu\mu\right)} \text{ in } |C^*_{\left(\mu\nu\right)}| / |C^*_{\left(\mu\nu\right)}|\]

and

\[(5.4^*)\]

\[V^{*\alpha} = \left(\frac{d^V}{ds}\right) - \sum_{\left(\nu\right)\left(\mu\right)} C^*_{\left(\nu\nu\right)\left(\mu\mu\right)} \Omega_{\left(\nu\beta\right)\left(\mu\gamma\right)} V^\beta \frac{du^\gamma}{ds} \times \left(\psi^* - t^*V\right) \frac{du^\gamma}{ds}.
\]

Similarly, in view of (1.33*), (1.35*) and (5.3*), we get

\[(5.5)\]

\[
\left(\frac{d^V}{ds}\right) = V^{*\alpha} B^0_{\alpha^0} \pm \sum_{\left(\nu\right)\left(\mu\right)} C^*_{\left(\nu\mu\right)\left(\mu\nu\right)} \Omega_{\left(\nu\beta\right)\left(\mu\gamma\right)} V^\beta \frac{du^\gamma}{ds} \times \left(\phi_{\left(\mu\mu\right)}\right)^{-1/2} \left(\psi^* - t^*V\right) M^*_{\left(\mu\mu\right)},
\]

where

\[\overline{C}^*_{\left(\mu\nu\right)} = \text{Cofactor of } C^*_{\left(\mu\nu\right)} \text{ in } |C^*_{\left(\mu\nu\right)}| / |C^*_{\left(\mu\nu\right)}|\]

and
\(|C^*_{(\mu\nu)}| < 0\). The equation (5.3) reduces to \(M^{*i} = m^{*i}\), if \(\lambda^{*i}\) is a linear combination of \(\frac{dv}{ds}\) and \(m^{*i}\). Similarly, (5.3*) reduces to \(M^{*i} = m^{*i}\), if \(\lambda^{*i}\) is a linear combination of \(\frac{dv}{ds}\) and \(m^{*i}\).

It follows that the vectors \(\{B_{\alpha}^i, B_{-\alpha}^{-i}\}\) are linearly independent in \(H_n\) and also the vectors \(\{M^i, M^{-i}\}\) are linearly independent.

Eliminating \(m^{*i}\) with the help of (1.33), (1.35) and (1.37), we have

\[
(5.4) \quad \left(\frac{\delta V^i}{\delta s}\right) = V^{*\alpha} B^{i}_{\alpha} \pm \sum_{(v)(\mu)} \bar{C}^*_{(\mu\nu)} \Omega^{\nu}_{(\nu)(\mu)} V^\beta \frac{du^\gamma}{ds} \left(\phi^{(\mu)}\right)^{-1/2} \times \left(\psi^{*} - t^{*2}\right) M^{*i},
\]

where

\[
\bar{C}^*_{(\mu\nu)} = \text{Cofactor of } C^*_{(\mu\nu)} \text{ in } |C^*_{(\mu\nu)}| / |C^*_{(\mu\nu)}|
\]

and

\[
(5.4*) \quad V^{*\alpha} = \left(\frac{\delta V^i}{\delta s}\right) - \sum_{(v)(\mu)} \bar{C}^*_{(\mu\nu)} \Omega^{\nu}_{(\nu)(\mu)} V^\beta \frac{du^\gamma}{ds} \times \left(\psi^{*} - t^{*2} \frac{du^\gamma}{ds}\right).
\]

Similarly, in view of (1.33*), (1.35*) and (5.3*), we get

\[
(5.5) \quad \left(\frac{\delta V^i}{\delta s}\right) = V^{*\alpha} B^{i}_{\alpha} \pm \sum_{(v)(\mu)} \bar{C}^*_{(\mu\nu)} \Omega^{\nu}_{(\nu)(\mu)} V^\beta \frac{du^\gamma}{ds} \times \left(\psi^{*} - t^{*2}\right) M^{*i},
\]

where

\[
\bar{C}^*_{(\mu\nu)} = \text{Co factor of } C^*_{(\mu\nu)} \text{ in } |C^*_{(\mu\nu)}| / |C^*_{(\mu\nu)}|
\]

and
\[(5.5^*) \quad V^s_\alpha = \left( \frac{\delta Y^\alpha}{\delta s} \right) - \sum_{(\nu)} \sum_{(\mu)} C^\gamma_{\beta \nu} \Omega_{\nu \gamma} V^\beta \frac{du^\gamma}{ds} \times \]

\[\times \left[ t^s_\mu - t^s_{\mu} \frac{du^\mu}{ds} \right].\]

The vectors \( \langle V^s_\alpha, V^s_{\alpha} \rangle \) given above is called the secondary relative associate curvature vector of \( \langle V_\alpha, V_{\alpha} \rangle \) in the direction of \( C \) and \( \overline{K}^s \) defined by

\[ \overline{K}^s = 2 g_{\alpha \beta} (u, \tilde{u}) V^s_\alpha V^s_\beta, \]

is called the secondary relative associate curvature of the vector field.

Let \( \Omega^* \) and \( T^* \) be the single rowed matrices given as

\[ \Omega^* = \left[ \Omega^s_{\gamma} V^\beta \frac{du^\gamma}{ds} \right], \quad T^* = \left[ t^s_\alpha - t^s_{\alpha} \frac{du^\alpha}{ds} \right] \]

and \( C^* \) be the non-singular matrix \( \left[ C^s_{\gamma \nu} \right] \) of rank \( (n-m) \).

The equation (5.4\( ^* \)) may be written as

\[(5.6) \quad V^s_\alpha = \left( \frac{\delta Y^\alpha}{\delta s} \right) - T^\alpha \overline{C}^s \Omega^* ', \]

where \( \Omega^* ' \) is the transpose of \( \Omega^* \) and \( T^\alpha \overline{C}^s \Omega^* ' \) in the above equation stands for the single element of this matrix product.

In the same fashion, let \( \overline{\Omega}^* \) and \( \overline{T}^* \) be the single rowed matrices given by

\[ \overline{\Omega}^* = \left[ \Omega^*_{\gamma} V^\beta \frac{du^\gamma}{ds} \right], \quad \overline{T}^* = \left[ t^s_\mu - t^s_{\mu} \frac{du^\mu}{ds} \right] \]

and \( \overline{C}^s \) be the non-singular matrix \( \left[ \overline{C}^s_{\gamma \mu} \right] \) of rank \( (n-m) \). The equation (5.5) may be written as

\[(5.6^*) \quad V^s_{\alpha} = \left( \frac{\delta Y^\alpha}{\delta s} \right) - T^\alpha \overline{C}^s \overline{\Omega}^* ', \]

where \( \overline{\Omega}^* ' \) is the transpose of \( \overline{\Omega}^* \) and \( T^\alpha \overline{C}^s \overline{\Omega}^* ' \) in the above equation stands for the single element of the matrix product.
(5.5*) \[ V^{*\alpha} = \left( \frac{\delta V}{\delta s} \right) - \sum_{(\nu)} \sum_{(\mu)} \bar{C}^{\nu}_{(\mu)} \Omega^{\beta\gamma} V_{\beta}^{\gamma} \frac{du^{\gamma}}{ds} \times \]

\[ \times \left( t^{*\alpha} - t^{*} \frac{du^{*}}{ds} \right). \]

The vectors \( \langle V^{*\alpha}, V^{*\tilde{\alpha}} \rangle \) given above is called the secondary relative associate curvature vector of \( \langle V^\alpha, V^{\tilde{\alpha}} \rangle \) in the direction of \( C \) and \( \bar{K}^* \) defined by

\[ \bar{K}^* = 2 g_{\alpha\beta} (u, \tilde{u}) V^{*\alpha} V^{*\beta}, \]

is called the secondary relative associate curvature of the vector field.

Let \( \Omega^* \) and \( T^* \) be the single rowed matrices given as

\[ \Omega^* = \left[ \Omega^*_{(\nu)\beta} V^\beta \frac{du^\gamma}{ds} \right], \quad T^* = \left[ t^{*\alpha} - t^{*} \frac{du^\alpha}{ds} \right] \]

and \( C^* \) be the non-singular matrix \( \left[ \bar{C}^*_{(\nu)\mu} \right] \) of rank (n-m).

The equation (5.4*) may be written as

(5.6) \[ V^{*\alpha} = \left( \frac{\delta V_{\alpha}}{\delta s} \right) - T^\alpha \bar{C}^{\beta} \Omega^{*\gamma}, \]

where \( \Omega^{*\gamma} \) is the transpose of \( \Omega^* \) and \( T^\alpha \bar{C}^{\beta} \Omega^{*\gamma} \) in the above equation stands for the single element of this matrix product.

In the same fashion, let \( \bar{\Omega}^* \) and \( \bar{T}^* \) be the single rowed matrices given by

\[ \bar{\Omega}^* = \left[ \Omega^*_{(\nu)\beta} V^\beta \frac{du^\gamma}{ds} \right], \quad \bar{T}^* = \left[ t^{*\alpha} - t^{*} \frac{du^\alpha}{ds} \right] \]

and \( \bar{C}^* \) be the non-singular matrix \( \left[ \bar{C}^*_{(\nu)\mu} \right] \) of rank (n-m). The equation (5.5) may be written as

(5.6*) \[ V^{*\tilde{\alpha}} = \left( \frac{\delta V_{\tilde{\alpha}}}{\delta s} \right) - T^{\tilde{\alpha}} \bar{C}^{\tilde{\beta}} \bar{\Omega}^{*\gamma}, \]

where \( \bar{\Omega}^{*\gamma} \) is the transpose of \( \bar{\Omega}^* \) and \( T^{\tilde{\alpha}} \bar{C}^{\tilde{\beta}} \bar{\Omega}^{*\gamma} \) in the above equation stands for the single element of the matrix product.