1. Introduction

Characterization results for the exponential distribution based on different properties of order statistics are discussed by several authors. The normalized spacing of order statistics (when the random variables are iid and from exponential distribution) was introduced by Sukhatme (1937). He considered the transformation \( D_{1n} = n X_{1n} \) and \( D_{r,n} = (n-r+1)(X_{r,n} - X_{r-1,n}) \), \( 2 \leq r \leq n \) to prove that \( D_{r,n} \), \( 1 \leq r \leq n \) are iid from exponential distribution with scale parameter \( \theta \). Utilizing the above transformation, it can be shown that \( X_{r,n} - X_{r,n} \) and \( X_{s-r,n-r} \) are identically and exponentially distributed for all \( 1 \leq r < s < n \). Puri and Rubin (1970) characterized the exponential distribution based on absolute difference of two iid random variables. Ahsanullah (1978, 1981 a) and Gajek and Gather (1989) considered the identical distribution of \( D_{r,n} \) and \( D_{s,n} \) as well as weaker condition for some integers \( r \) and \( s \) with \( 1 \leq r < s < n \).

For a detailed survey on characterization results based on order statistics, one may refer to Ahsanullah (1975), Gather (1989), Ahsanullah and Nevzorov (2002) and David and Nagaraja (2003) and references cited therein.

Tata (1969) has shown that the independence of \( X_{U(2)} - X_{U(1)} \) and \( X_{U(1)} \) is a characteristic property of exponential distribution, where \( X_{U(r)} \) is the \( r^{th} \) upper record. The related results of characterization of exponential distribution for record values through spacing were shown by Ahsanullah (1981 b) and Iwińska (1986). Ahsanullah (1981 b) has shown that if

\[
E(X_{U(n)} - X_{U(m)}) = E(X_{U(n-m)}), \quad n > m,
\]

then the random variable \( X \) is exponentially distributed.

This chapter is divided into four sections. In Section 2, we have characterized the exponential distribution through normalized spacing of order statistics,
while Section 3 of the present chapter deals with characterization through
distributional properties of record statistics using normalized spacing. Here
the distribution of difference of two non-adjacent records is being considered
to characterize exponential distribution. Similar characterization result is
obtained by Ahsanullah (1987) but approach used in this chapter is entirely
different. In Section 4, a simulation study is carried out to construct the
confidence interval for scale parameter of exponential distribution based on
upper records and compared it with asymptotic confidence interval.

2. Characterization of exponential Distribution based on order statistics

In this section, we have characterized the exponential distribution by
considering the difference of two order statistics. Before the proof of the
characterization theorem, the following lemmas are given.

**Lemma 2.1:** Let \( X_{r,n} \) and \( X_{r-1,n} \) be two adjacent order statistics, then

\[
F_{r,n}(x) - F_{r-1,n}(x) = \frac{n!}{(r-1)!(n-r+1)!} \left[ F(x) \right]^{-1} \left[ F(x) \right]^{n-r+1} 
\]  

(2.1)

where \( F(x) = 1 - F(x) \).

**Proof:** The proof is immediate, but for completeness we prove it here.

\[
F_{r,n}(x) = \frac{n!}{(r-1)!(n-r)!} \int_{s}^{x} \left[ F(u) \right]^{-1} \left[ F(u) \right]^{n-r} f(u) \, du .
\]

Integrating by parts, considering \( [F(u)]^{-1} \) as first part and \( [F(u)]^{n-r} f(u) \) as second part we get

\[
F_{r,n}(x) = -\frac{n!}{(r-1)!(n-r)!} \left[ F(u) \right]^{-1} \left[ F(u) \right]^{n-r+1} \bigg|_{x}^{\infty} 
\]

\[
+ \frac{n!}{(r-2)!(n-r+1)!} \int_{-\infty}^{x} [F(u)]^{-2} [F(u)]^{n-r+1} f(u) \, du .
\]
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\[
\begin{align*}
\frac{n!}{(r-1)!(n-r+1)!} [F(x)]^{r-1} & [F(x)]^{n-r+1} \\
& \quad + \frac{n!}{(r-2)!(n-r+1)!} \int_{u}^{\infty} [F(u)]^{r-2} [F(u)]^{n-r+1} f(u) \, du \\
\]

\[
F_{r,n}(x) = \frac{n!}{(r-1)!(n-r+1)!} [F(x)]^{r-1} [F(x)]^{n-r+1} + F_{r-1,n}(x)
\]

Hence the Lemma.

**Lemma 2.2:** Let \( U_{r,n} \) be the \( r^{th} \) order statistics from a uniform population over the support \((0,1)\) following \( B(r, n-r+1) \). Further the pdf \( f_r(u) \) of \( U_{r,n} \) is unimodal and mode occurs at \( u = \frac{r-1}{n+1} \). Thus, the pdf increases in \( \left(0, \frac{r-1}{n+1}\right) \) and decreases in \( \left(\frac{r-1}{n+1}, 1\right) \). However, in case of the first order statistics \( U_{1,n} \) the maxima occur at zero and then \( f_1(u) \) is strictly decreasing thereafter.

For unimodality of order statistics one may refer to Dharmadhikari and Joag-dev (1988, section 9.4).

**Theorem 2.1:** Let \( X_1, X_2, \ldots, X_n \) be \( n \) iid non-negative random variables with strictly increasing df \( F(x) \) and pdf \( f(x) \) over the support \((0, \infty)\), then for \( 1 \leq r < s \leq n \),

\[
X_{t,n} - X_{r,n} \quad \overset{d}{\sim} \quad X_{t-r,n-r}, \quad l = s-1, s
\]

if and only if \( X \) has an exponential distribution with df \( F(x) = 1 - e^{-\frac{x}{\theta}} \). Here \( \overset{d}{\sim} \) denotes identical in distribution.

**Proof:** To prove the necessary part, using normalized spacing of order statistics and using the transformation
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\[ D_1 = \sum_{i=1}^{r} (X_{i:n} - X_{i-1:n}) \quad D_i = (n-i+1)(X_{i:n} - X_{i-1:n}), \quad i = 2, ..., n. \]  

(2.3)

Assuming that \( X_{0:n} = 0 \), it may be noted that \( D_i \)'s are iid and follow exponential distribution with scale parameter \( \theta \). Further

\[
X_{r:n} = \sum_{i=1}^{r} (X_{i:n} - X_{i-1:n}) \frac{D_i}{(n-i+1)}
\]

and

\[
X_{s:n} - X_{r:n} = \sum_{i=r+1}^{n} \frac{D_i}{(n-i+1)}
\]

(2.4)

\[
X_{s:n} - X_{r:n} \equiv \sum_{i=1}^{r} \frac{D_{r+i}}{(n-r+i+1)}
\]

\[ i.e. \quad X_{s:n} - X_{r:n} \equiv {X}_{s-r:n-r}. \]

This proves the necessary part.

To prove the sufficiency part, we have

\[
P[X_{s:n} - X_{r:n} \geq u] = C_{r,s:n} \int_{0}^{\infty} \int_{0}^{\infty} [F(x)]^{s-1}[F(y) - F(x)]^{r-1}[F(y)]^{n-s} f(x) f(y) dy dx
\]

(2.5)

where

\[
A(x, y) = \int_{x+u}^{\infty} [F(y) - F(x)]^{r-1}[F(y)]^{n-s} f(y) dy.
\]

(2.6)

Now integrating (2.6) by parts w.r.t. \( y \), taking \([F(y) - F(x)]^{r-1}\) as first part and \([F(y)]^{n-s} f(y)\) as second part of the integral we get,

\[
A(x, y) = -[F(y) - F(x)]^{r-1}[F(y)]^{n-s} \int_{x+u}^{\infty} \frac{f(y)}{(n-s+1)} dy
\]

(2.7)
Putting the value of $A(x,y)$ obtained in (2.7) into (2.5), we have

\[
P[X_{\alpha,n} - X_{\rho,n} \geq u]
= \frac{C_{r,n}}{(n-s+1)} \int_{0}^{\infty} [F(x)]^{r-1} \left[ \frac{F(x+u) - F(x)}{F(x)} \right]^{n-s+1} f(x) dx
+ \frac{C_{r,n}}{(n-s+1)} \int_{0}^{\infty} \left[ \frac{F(x) - F(x+u)}{F(x)} \right]^{r-2} \left[ \frac{F(x+u)}{F(x)} \right]^{n-s+1} f(y) f(x) dy dx.
\]  
(2.8)

Now in view of (2.4) and (2.8), we get

\[
P[X_{\alpha,n} - X_{\rho,n} \geq u] - P[X_{s-\alpha,n} - X_{\rho,n} \geq u]
= \frac{C_{r,n}}{(n-s+1)} \int_{0}^{\infty} \left[ \frac{F(x) - F(x+u)}{F(x)} \right]^{r-1} \left[ \frac{F(x+u)}{F(x)} \right]^{n-s+1} f(x) dx.
\]  
(2.9)

Now using Lemma 2.1 and using the fact that $f_{r,n}(x)$ is a pdf, we have

\[
P[X_{s-\alpha,n} - X_{\rho,n} \geq u] - P[X_{s-\alpha,n} - X_{\rho,n} \geq u]
= \frac{(n-r)!}{(s-r-1)!(n-s+1)!} [F(u)]^{r-1} [\bar{F}(u)]^{n-s+1}
= \frac{(n-r)!}{(s-r-1)!(n-s+1)!} [F(u)]^{r-1} [\bar{F}(u)]^{n-s+1} \int_{0}^{\infty} f_{r,n}(x) dx
= \frac{c_{r,n}}{(n-s+1)} [F(u)]^{r-1} [\bar{F}(u)]^{n-s+1} \int_{0}^{\infty} [F(x)]^{r-1} [\bar{F}(x)]^{n-r} f(x) dx.
\]  
(2.10)
(2.9) and (2.10) will be equal if and only if

\[
\int_0^\infty \left[ \frac{F(x) - F(x + u)}{F(x)} \right]^{s-1} \left[ \frac{F(x + u)}{F(x)} \right]^{n-s+1} - [F(u)]^{s-1}[\bar{F}(u)]^{n-s+1} \times [F(x)]^{-r}[\bar{F}(x)]^{p-r} f(x) dx = 0.
\]

As \([F(x)]^{r-1}[\bar{F}(x)]^{p-r} f(x) > 0\), the above integral will be zero if

\[
\left[ \frac{F(x) - F(x + u)}{F(x)} \right]^{s-1} \left[ \frac{F(x + u)}{F(x)} \right]^{n-s+1} = [1 - F(u)]^{s-1}[\bar{F}(u)]^{n-s+1}
\]

or,

\[
\left[ 1 - \frac{F(x + u)}{F(x)} \right]^{s-1} \left[ \frac{F(x + u)}{F(x)} \right]^{n-s+1} = [1 - F(u)]^{s-1}[\bar{F}(u)]^{n-s+1}. \tag{2.11}
\]

Now in view of Lemma 2.2, the function \([1 - F(u)]^{s-1}[\bar{F}(u)]^{n-s+1}\) is unimodal and mode depends on the values of \(r, s\) and \(n\).

Thus, the left-hand side and right-hand side will be equal at maxima only, that leads us to conclude that \(F(x + u) = \bar{F}(u) F(x)\), i.e., random variable \(X\) satisfies the memoryless property. The only continuous random variable over the support \((0, \infty)\) which satisfy memoryless property is exponential distribution. This implies that the random variable \(X\) follows exponential distribution, and hence the theorem.

### 3. Characterization of exponential distribution based on records

In this section, we have characterized the exponential distribution based on record statistics. Before the proof of the main result, the following two lemmas are given, which are used in the proof of the theorem.

**Lemma 3.1:** Let \(x_{U(r-1)}\) and \(x_{U(r)}\) be the two adjacent upper records, then
\[
\bar{F}_{X_{U(r)}}(u) - \bar{F}_{X_{U(r-1)}}(u) = \frac{[-\ln \bar{F}(u)]^{r-1}}{(r-1)!} \bar{F}(u). \tag{3.1}
\]

**Proof:**

\[
\bar{F}_{X_{U(r)}}(u) = \int_u^\infty \frac{[-\ln \bar{F}(x)]^{r-1}}{(r-1)!} f(x)dx \tag{3.2}
\]

Integrating by parts, considering \([-\ln \bar{F}(x)]^{r-1}\) as first part and \(f(x)\) as second part, we get

\[
\bar{F}_{X_{U(r)}}(u) = \frac{(-1)}{(r-1)!} \left[\frac{[-\ln \bar{F}(x)]^{r-1}}{(r-1)!} \bar{F}(x)\right]^\infty_u + \frac{(r-1)}{(r-1)!} \int_u^\infty \frac{[-\ln \bar{F}(x)]^{r-2}}{\bar{F}(x)} \bar{F}(x) f(x) dx
\]

\[
= \frac{1}{(r-1)!} \left[-\ln \bar{F}(u)\right]^{r-1} \bar{F}(u) + \frac{1}{(r-2)!} \int_u^\infty [-\ln \bar{F}(x)]^{r-2} f(x) dx
\]

\[
\bar{F}_{X_{U(r)}}(u) = \frac{1}{(r-1)!} [-\ln \bar{F}(u)]^{r-1} \bar{F}(u) + \bar{F}_{X_{U(r-1)}}(u).
\]

Rearranging the terms, we get the result.

**Lemma 3.2:** The function \(h(x) = [-\ln \bar{F}(x)]^{r-1} \bar{F}(x)\) is strictly increasing with maxima occurring at point \(x = F^{-1}[1-e^{-(r-1)}]\) and then strictly decreasing.

**Proof:** We have the function

\[
h(x) = [-\ln \bar{F}(x)]^{r-1} \bar{F}(x).
\]

For maxima, differentiating \(h(x)\) w.r.t. \(x\) and equating to zero, we get

\[
h'(x) = (r-1)[-\ln \bar{F}(x)]^{r-2} f(x) - [-\ln \bar{F}(x)]^{r-1} f(x) = 0
\]

\[
(r-1) = -\ln \bar{F}(x)
\]

or \(x = F^{-1}[1-e^{-(r-1)}]\).

Second derivative of \(h(x)\) at \(x = F^{-1}[1-e^{-(r-1)}]\) is negative. Hence the Lemma.

**Theorem 3.1:** Let \(X\) be a non-negative continuous random variable with strictly increasing df \(F(x)\) over the support \((0, \infty)\), then for, \(1 \leq r < s - 1 \leq n - 1\),
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\[ X_{U(l)} - X_{U(r)} \leq d \ X_{U(l-r)}, \ l = s-1, s \]

if and only if \( F(x) = 1 - e^{-\frac{x}{\theta}} \).

**Proof:** For the proof of the necessary part, using the normalized spacing of record values from exponential distribution \( i.e. \ Y_1 = X_{U(l)} \) and \( Y_i = X_{U(l)} - X_{U(i-1)}, \ i = 2, 3, ..., \) it can be seen that \( Y_1, Y_2, ... \) are iid random variables from exponential distribution with scale parameter \( \theta \) (Ahsanullah, 1979). Thus,

\[ X_{U(l)} - X_{U(r)} = \sum_{i=r+1}^{l} Y_i, \ l = s-1, s. \]

Then \( X_{U(l)} - X_{U(r)} \) is gamma distributed with shape parameter as \( l-r \) and scale parameter \( \theta \). Further, when random variable \( X \) follows exponential distribution with scale parameter \( \theta \), then distribution of \( X_{U(l-r)} \) is also gamma with shape parameter as \( l-r \) and scale parameter \( \theta \). This implies that \( X_{U(l)} - X_{U(r)} \) \( d \ X_{U(l-r)}, \ l = s-1, s. \)

Now to prove the sufficient part, we have for any positive and finite \( u \)

\[ P[X_{U(s)} - X_{U(r)} \geq u] \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} \frac{[-\ln F(x)]^{r-1} [-\ln F(y) + \ln F(x)]^{r-1}}{(r-1)! (s-r-1)!} \frac{f(x)}{F(x)} f(y)dy dx. \]  \( (3.3) \)

Let \( A(x, y) = [-\ln F(y) + \ln F(x)]^{r-1} f(y) \).

Integrating \( A(x, y) \) \( w.r.t. \ y \) over \( (x+u, \infty) \), we have

\[ \int_{x+u}^{\infty} A(x, y)dy = \int_{x+u}^{\infty} [-\ln F(y) + \ln F(x)]^{r-1} f(y)dy \]

\[ = [-\ln F(y) + \ln F(x)]^{r-1} F(y) \bigg|_{x+u}^{\infty} \]
Substituting the value of \( \int_{x+u}^{\infty} A(x,y) \, dy \), obtained in (3.4) in (3.3), we have

\[
P[X_{U(s)} - X_{U(r)} \geq u] = \frac{1}{(r-1)! (s-r-2)!} \int_{0}^{\infty} \int_{x+u}^{\infty} \left[ -\ln F(x) \right]^{r-1} \left[ -\ln F(y) + \ln F(x) \right]^{s-r-2} \frac{f(y)}{F(x)} \, dy \, dx
\]

or,

\[
P[X_{U(s)} - X_{U(r)} \geq u] - P[X_{U(s-1)} - X_{U(r)} \geq u] = \frac{1}{(r-1)! (s-r-1)!} \int_{0}^{\infty} \left[ -\ln F(x) \right]^{r-1} \left[ -\ln F(x) + \ln F(x + u) \right]^{s-r-1} \frac{f(x)}{F(x)} \, dx .
\]

(3.5)

Now in view of Lemma 3.1 and utilizing the fact that \( f_{X_{U(r)}}(x) \) is a pdf, we have

\[
P[X_{U(s-r)} \geq u] - P[X_{U(s-r-1)} \geq u] = \frac{\bar{F}(u) \left[ -\ln \bar{F}(u) \right]^{s-r-1} \int_{0}^{\infty} f_{X_{U(r)}}(x) \, dx}{(s-r-1)!}
\]

\[
= \frac{\bar{F}(u) \left[ -\ln \bar{F}(u) \right]^{s-r-1} \int_{0}^{\infty} \left[ \ln F(x) \right]^{s-r-1} f(x) \, dx}{(r-1)! (s-r-1)!}.
\]

(3.6)

Equation (3.5) and (3.6) will be equal if and only if

\[
\frac{1}{(r-1)! (s-r-1)!} \int_{0}^{\infty} \left[ \ln F(x) \right]^{s-r-1} \frac{F(x + u)}{F(x)} \left[ -\ln \bar{F}(u) \right]^{s-r-1} \bar{F}(u) \, dx \times \left[ -\ln \bar{F}(x) \right]^{s-r-1} f(x) \, dx = 0 .
\]
Since \([-\ln \tilde{F}(x)]^{r-1}\) and \(f(x)\) both are positive, using the generalization of Muntz-Swartz theorem (see Hwang Lin; 1984), the above integral will be zero if and only if

\[
[\ln \tilde{F}(x+u) - \ln \tilde{F}(x)]^{r-1} \frac{\tilde{F}(x+u)}{\tilde{F}(x)} = [\ln \tilde{F}(u)]^{r-1} \tilde{F}(u)
\]

or,

\[
\left[ \ln \left( \frac{\tilde{F}(x+u)}{\tilde{F}(x)} \right) \right]^{r-1} \frac{\tilde{F}(x+u)}{\tilde{F}(x)} = [\ln \tilde{F}(u)]^{r-1} \tilde{F}(u).
\]

Now in view of Lemma 3.2, the function \(h(x) = [-\ln \tilde{F}(x)]^{r-1} \tilde{F}(x)\) is strictly increasing and has maxima at point \(x = F^{-1}[1-e^{-(r-1)}]\) (see Basak and Basak, 2002; Cramer et al., 2004). Thus, left hand side and right hand side are equal only at maxima, which is satisfied only with the condition \(\tilde{F}(x+u) = \tilde{F}(x)\tilde{F}(u)\), which is nothing but a memoryless property. The only continuous distribution with the boundary condition \(\tilde{F}(0) = 0\) and \(\tilde{F}(\infty) = 1\), which satisfies

\[
\tilde{F}(x+u) = \tilde{F}(x)\tilde{F}(u)
\]

is exponential distribution i.e. \(\tilde{F}(x) = 1 - e^{-\frac{x}{\theta}}\). This implies that the random variable \(X\) follows exponential distribution with scale parameter \(\theta\). Hence the theorem.

### 4. Simulation study

Here we have followed the procedure adopted by Nadarajah et al. (2014) and carried out the simulation to construct the confidence interval for scale parameter \(\theta\) of exponential distribution.

Let \(X_1, X_2, \ldots, X_n\) be a random sample from pdf \(f(x) = \theta^{-1} \exp\left( -\frac{x}{\theta} \right)\), \(x > 0, \theta > 0\). Based on maximum likelihood estimator of exponential
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distribution, 100(1 − α)% (0 < α < 1) asymptotic confidence interval

\[
CI_{(Asymptotic)} = \left( \frac{\overline{X}}{1 + \frac{Z_{\alpha/2}}{\sqrt{n}}}, \frac{\overline{X}}{1 - \frac{Z_{\alpha/2}}{\sqrt{n}}} \right)
\]  \hspace{1cm} (4.1)

where \(\overline{X}\) is sample mean of random variables and \(Z_{\alpha/2}\) is a positive constant satisfying the relation \(\Phi(Z_{\alpha/2}) - \Phi(-Z_{\alpha/2}) = 1 - \alpha\) and \(\Phi(.)\) is the cumulative distribution function of the standard normal distribution. Using Theorem 3.1, for record statistics, we have \(X_{U(s)} - X_{U(r)} \overset{d}{=} X_{U(s-r)}\). As \(Y = X_{U(s-r)} \sim G(s-r, \theta)\), based on \(Y\) a 100(1 − α)% confidence interval for \(\theta\) is given by

\[
CI = \left( \frac{2Y}{\chi^2_{2n,1-\alpha/2}}, \frac{2Y}{\chi^2_{2n,\alpha/2}} \right)
\]  \hspace{1cm} (4.2)

To construct the asymptotic confidence intervals from (4.1), we have simulated 10,000 samples of size \(n\) from \(F(x) = 1 - \exp\left(-\frac{x}{\theta}\right)\) for \(n = 4, 5, \ldots, 100\) and \(\theta = 0.2, 0.5, 1, 2\). Similarly, we have also generated 10,000 samples from \(G(s-r, \theta)\), for \(s = 4, 5, \ldots, 100\), \(r = 1\) and \(\theta = 0.2, 0.5, 1, 2\), to construct the exact confidence interval of \(\theta\) based on \(Y\) as given in (4.2). For each sample, we have calculated the limits of confidence interval given in (4.1) and (4.2). The plot of coverage lengths versus \(n = 4, 5, \ldots, 100\), is shown in figures 1, 2, 3 and 4.
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**Fig 1:** Plot of confidence interval for $\theta = 0.2$

**Fig 2:** Plot of confidence interval for $\theta = 0.5$
From the figures, it can be seen that the width of confidence interval (WCI) obtained from (4.2) is smaller than the asymptotic width of confidence interval (AWCI) obtained from (4.1) for $n \leq 60$. Thus, we get an improved estimator based on $X_{U(x-r)}$. 

**Fig 3**: Plot of confidence interval for $\theta = 1$.

**Fig 4**: Plot of confidence interval for $\theta = 2$. 

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