In this dissertation we have considered DFV methods for the approximation of optimal control problems governed by semilinear elliptic, parabolic and hyperbolic equations, and also by Brinkman equations. The main emphasis was on theoretical and computational aspects of the proposed methods for investigated problems. The main ingredient in the analysis part was the derivation of \textit{a priori} error estimates in suitable norms for the unknown variables that appeared in the formulation. Moreover, several numerical experiments were presented for validation of theoretical error estimates. Now, we summarize the main findings obtained in each Chapter of the thesis and describe the general conclusions based on these findings. Furthermore, we present possible extensions of this work.

\section{Summary}

Chapter 1 dealt with review and applications of optimal control problems governed by a class of PDEs. In this Chapter, we have clearly mentioned the suitability and advantages of the proposed method in comparison with the other existing numerical schemes such as FE, FV and DG methods. Here, two different strategies: \textit{optimize-then-discretize} and \textit{discretize-then-optimize}–generally, used for solvability of optimal control problems was discussed, and justification of employing \textit{optimize-then-discretize} approach was mentioned.

In Chapter 2, we have studied DFV approximations for semilinear elliptic optimal control problems. In this Chapter, first we have considered linear elliptic problem, because of the following reasons. First, these kind of problems have numerous applications and second, there are contributions which dealt with DFV approximations for linear elliptic problems. Also, the analysis presented for this case can be easily extended to semilinear and Brinkman optimal control problems—which are the problems of our interest. In this Chapter, by following the analysis of \cite{49}, we have established
optimal a priori error estimates for DFV methods applied to linear and semilinear elliptic optimal control problems. Moreover, for numerical solution of nonlinear algebraic equations—obtained after DFV discretization of semilinear elliptic equation, an interpolated coefficient method was employed. It has been shown that this idea has computational advantages (in terms of computation of the jacobian) compared with the standard Newton method.

Chapter 3 is devoted to the development of DFV methods for semilinear parabolic optimal control problems. Both semidiscrete and fully discrete scheme were discussed in detail, and existence of a unique local optimal control was examined. Error analysis for both the schemes in mesh and time dependent norms has been carried out for all three discretization of control variables. Numerical examples were presented by considering the applications of these problems related to controlled heating of a body and to justify the theoretical findings. In order to solve the resulting nonlinear system of equations, the idea of interpolated coefficient was exploited.

Since the treatment of hyperbolic problems is similar to the parabolic problems except the time derivative, in Chapter 4, we have extended the analysis of Chapter 3 to the semilinear hyperbolic optimal control problems. The time derivative was approximated by applying an implicit difference scheme, and for discretization of space, linear DFV methods was used. Error estimates for semi-discrete scheme are derived which were analogous to the estimates for parabolic case. In order to demonstrate the real life applications of these problems, in our numerical experiments, a membrane problem is considered.

Considering the applications of fluid control problems and FV methods in computational fluid dynamics, Chapter 5 is dedicated to describe the DFV approximations of optimal control problems governed by Brinkman equations. By following the analysis of [52] which dealt with DFV approximations of Stokes equations, a detailed error analysis has been carried out. Numerical examples consisting lid driven cavity problem and a cylindrical flow were presented in order to illustrate the performance of the proposed method and validate the predicted rate of convergence. Moreover, through our numerical experiments, a comparison study with other existing classical schemes in terms of accuracy and efficiency was made.
6.2 Concluding Remarks

We would like to make the following comments on theoretical and computational aspects of DFV approximations applied to optimal control problems listed in Chapters 2 to 5.

- The derivation of the optimal error estimate of $O(h^2)$ for state and costate variables in the $L^2$ norm with variational discretization of control, was straightforward and was achieved by decomposing the errors and using the estimate of control. However, using the similar arguments for the case of piecewise linear and piecewise constant discretizations of control leads to suboptimal order of convergence for state and costate variables. In order to obtain the optimal error estimates, we have used the duality arguments.

- In the case of variational discretization, we were able to derive $O(h^2)$ convergence rate for control by following the standard arguments. But, we could obtain only $O(h^{3/2})$ and $O(h)$ convergence order for control when piecewise linear and constant discretization techniques are used, respectively. Hence, theoretically this approach has advantages over the others. However, there would be some computational difficulties with this scheme (variational discretization), and this can be explained as follows. Since here the control set is not discretized explicitly but discretized by a projection of the discrete costate variables, we observed that the discrete control does not belong to the finite dimensional space associated with mesh and hence one would need to handle nonstandard numerical algorithms and require some advanced tools in order to set the stopping criteria.

- We would also like to mention in order to resolve nonlinear problems, Newton method is employed in general. This requires the computation of Jacobian matrix (which involves derivative) at each iteration which is very time consuming and expensive. For tackling these difficulties, we have utilized the idea of interpolated coefficients together with DFV methods to approximate semilinear elliptic, parabolic and hyperbolic optimal control problems. It was observed that with the introduction of interpolated coefficients the computation was cheap and the Jacobian matrix was computed in a simple way, as the derivative of nonlinear term involved direct multiplication with mass matrix and Jacobian matrix was updated.
in each iteration of the Newton method.

- Computationally DFV methods would be advantageous compared to classical FE, FV and DG methods. This is because the size of the control volume in DFV methods is almost half of the control volume used in continuous FV methods and test space is piecewise constant (same as in continuous FV). Moreover, this proposed method enjoy desirable features of both DG and FV methods.

6.3 Future Directions

Considering the applications of DFV methods, in immediate future, we will concentrate on the DFV approximations of the optimal control problems subject to partial differential equations governing flow-based phenomena. We aim at the development of specialized solution techniques and mathematical analysis that can allow us to put into proper perspective the framework for studying these processes. On the lines of our work presented in this thesis, we will exploit the essential advantages of both FVM and DG methods, now in the context of more application based optimal control problems. We outline a few milestones as follows:

6.3.1 DFV methods for convection-dominated diffusion optimal control problems

Optimal control for convection-diffusion equation is widely met in real life applications such as the shape optimization of technological devices, the identification of parameters in environmental processes and flow control problems. In environmental science, some phenomena modeled by linear convection-diffusion partial differential equations are often studied to investigate the distribution forecast of pollutants in water or in atmosphere. In this context, it might be of interest to regulate the source term of the convection-diffusion equation so that the solution is as near as possible to a desired one, e.g. to operate the emission rates of industrial plants to keep the concentration of pollutants near (or below) a desired level. This problem can be conveniently accommodated in the optimal control framework for convection diffusion equation. For our future study, we will consider the following distributed optimal control problem...
governed by the unsteady time dependent diffusion-convection reaction equation with control constraints

$$\min_{u \in U_{ad}} J(y, u) := \frac{1}{2} \int_0^T \left( \| y - y_d \|_{0, \Omega}^2 + \frac{\lambda}{2} \| u \|_{0, \Omega}^2 \right) dt,$$

subject to

$$\begin{align*}
\partial_t y - \varepsilon \Delta y + \vec{b} \cdot \nabla y + ay &= Bu + f \quad \text{in} \quad (0, T) \times \Omega, \\
y(t, x) &= 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \\
y(x, 0) &= y_0(x), \quad x \in \Omega.
\end{align*}$$

with the set of admissible controls defined by

$$U_{ad} = \{ u \in U = L^2(L^2) : u_a \leq u(x) \leq u_b, \ a.e. \ in \ \Omega \},$$

with bounds $u_a, u_b \in \mathbb{R}$ that fulfill $u_a < u_b$. The domain $\Omega$ is bounded, open and convex in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\partial \Omega$. The source function $f$ and the desired state $y_d \in L^2(L^2)$. The initial condition $y_0(x) \in H^1_0(\Omega)$. Here, $a > 0$ is the reaction coefficient, $0 < \varepsilon << 1$ is a small positive number. The given velocity field $\vec{b} \in W^{1,\infty}(\Omega)^2$ satisfies the incompressibility condition, i.e $\nabla \cdot \vec{b} = 0$. We also assume that the following coercivity condition holds:

$$a - \frac{1}{2} \nabla \cdot \vec{b} \geq 0 > 0.$$

For this problem, we will focus on the following:

- Development of suitable DFV schemes.
- Convergence analysis of the proposed scheme.
- Efficient implementation and numerical solution of problems with application-based interest.
6.3.2 DFV approximations for the optimal control problems governed by coupled flow-transport equations

We expect the advantages of DFV methods to be much more evident in presence of more complicated domain heterogeneities, high solution gradients, nonlinearities, and coupling with other transport phenomena modelling, e.g. thermal effects or sedimentation-consolidation of small particles within viscous fluids. In view of these, we consider the following optimal control problem:

$$\min_{u \in U_{ad}} J(u) := \frac{1}{2} \| \phi - \phi_d \|^2_{0, \Omega} + \frac{\lambda}{2} \| u \|^2_{0, \Omega},$$

subject to the following transport equation together with Stokes problem:

$$\partial_t \phi - \text{div}(\kappa(\phi) \nabla \phi) + \mathbf{y} \cdot \nabla \phi = B u + \nabla \cdot \mathbf{f}(\phi) \quad \text{in} \ (0, T) \times \Omega,$$

$$-\text{div}(\mu(\phi)\varepsilon(\mathbf{y}) - p \mathbf{I}) - \phi \mathbf{g} = 0 \quad \text{in} \ (0, T) \times \Omega,$$

$$\text{div} \mathbf{y} = 0 \quad \text{in} \ (0, T) \times \Omega,$$

$$\mathbf{y} = 0 \quad \text{on} \ (0, T) \times \partial \Omega,$$

$$\phi = 0 \quad \text{on} \ (0, T) \times \partial \Omega,$$

$$\phi(0) = \phi_0 \quad \text{on} \ \{0\} \times \Omega.$$

Here, the desired concentration field $\phi_d$ is assumed to be from $C^{0, \sigma}(\bar\Omega)^d, \sigma \in (0, 1)$ and $U_{ad} := \{u(t, x) \in U = L^\infty(L^\infty) : a \leq u(t, x) \leq b, \ a.e. \ (t, x) \in (0, T) \times \Omega; \ a, b \in \mathbb{R}\}$.

The above model problem describe the motion of an incompressible mixture and the evolution of the solids concentration. The primal unknowns are the volume average flow velocity of the mixture $\mathbf{y}$, the solids concentration $\phi$, and the pressure field $p$ and $u$ which is the control variable. In addition, $\mu(\phi)\varepsilon(\mathbf{y}) - p \mathbf{I}$ is the Cauchy stress tensor, $\varepsilon(\mathbf{y}) = \frac{1}{2}(\nabla \mathbf{y} + \nabla \mathbf{y}^T)$ is the infinitesimal rate of strain, and $\mu = \mu(\phi)$ is the concentration-dependent viscosity.

For the numerical approximation, we will pay close attention to the following:

- Development and a priori error analysis of DFV schemes for above mentioned problem.
• Derivation of a priori error estimates.

• Implementation of mesh adaptivity and validation using benchmark solutions.

• To investigate the exploitability of smart preconditioners and efficient solvers.

6.3.3 DFV methods for optimal control of Brinkman flows with pressure based optimality conditions

In future, we would also like to investigate about the optimal control with different formulations for Brinkman flows (including e.g. vorticity-based systems), along with unconstrained and pressure-based optimality conditions. In this direction, we will consider DFV approximations for the following optimal control problem:

\[
\min_{y, p, u} \frac{1}{2} \| y - y_d \|^2_{0, \Omega} + \frac{\delta}{2} \| p - p_d \|^2_{0, \Omega} + \frac{\lambda}{2} \| u \|^2_{0, \Omega},
\]

governed by the Brinkman equations

\[
\begin{aligned}
K^{-1}(x)y - \text{div}(\mu(x) \varepsilon(y) - pI) &= u \quad \text{in } \Omega, \\
\nabla \cdot y &= 0 \quad \text{in } \Omega, \\
y &= w \quad \text{on } \partial \Omega.
\end{aligned}
\]

Here, \( u \) denotes the forcing term on the right hand side, which is known as control. \( \lambda > 0 \) is the Tikhonov regularization parameter, \( \delta > 0 \) is a constant added in front of the desired pressure to enable us to penalize the pressure. The idea is to choose the forcing term \( u \) such that the velocity \( y \) and pressure \( p \) are as close as possible to \( y_d \) and \( p_d \) in some sense, while still satisfying the Brinkman equations.

So far we have considered only distributed optimal control problems. As a part of our future work we are also interested in the investigation of more applied control setting by considering boundary control problems.
6.3.4 DFV methods for optimal Dirichlet boundary control for the Navier-Stokes equations

We would like to extend DFV approximations for the Dirichlet boundary control problem governed by Navier-Stokes equations. We will investigate the following velocity tracking problem:

$$\min_{y,u} \frac{1}{2} \int_{\Omega} |y - y_d|^2 \, dx + \frac{\lambda}{2} \int_{\partial \Omega} |u|^2 \, ds,$$

governed by Navier-Stokes equations

$$\begin{align*}
-\Delta y + (y \cdot \nabla) y + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot y &= 0 \quad \text{in } \Omega, \\
y &= u \quad \text{on } \partial \Omega.
\end{align*}$$

The main idea of the model problem is to influence and eventually drive the velocity vector field $y$ to a given target field $y_d$, by using a control function $u$ on the boundary of the domain $\Omega$. 