CHAPTER 5

Optimal control problem governed by Brinkman equations

In this Chapter, we describe discontinuous finite volume approximations for optimal control problems governed by the Brinkman equations written in terms of velocity and pressure. An additional force field is sought that produces a velocity close to a desired known value. The discretization of state and costate velocity and pressure follows a lowest order DFV scheme, whereas three different approaches are used for the control approximation: variational discretization, element-wise constant and element-wise linear functions. Here also we have employed optimize-then-discretize approach, and the resulting discrete formulation is nonsymmetric. We derive a priori error estimates for velocity, pressure and control in natural norms. A set of numerical examples is finally presented to illustrate the performance of the method and to confirm the predicted accuracy of the state, costate and control approximations under various scenarios including 2D and 3D cases.

5.1 Introduction

Fluid control problems are highly important in the field of science and engineering. They are often useful to minimize drag, to increase mixing properties, to reduce turbulent kinetic energy, and several other features.

Theoretical aspects of these control problems can be found in the classical works [1, 54]. Regarding their numerical solution, the literature is abundant, especially if associated to FE methods (see e.g., [11, 34, 38, 69, 74, 78] and the references therein). Most contributions in the context of Stokes and Navier-Stokes approximation employ conforming discretizations for state, costate and control variables. In this case, it has been found that the convergence rate of the control approximation is of $O(h)$ and $O(h^{3/2})$ for piecewise constant and piecewise linear discretizations, respectively. On the other hand,
using the so-called variational discretization approach (cf. [42], in which the control set is not discretized explicitly but recovered by a projection), an improved convergence of \( O(h^2) \) was obtained. A similar result holds if using graded meshes instead of uniform partitions [68]. A few results are also available for finite differences [33], spectral [22], mimetic [3], fully-mixed [7], and DG [18, 19, 23] methods applied to flow control problems. On the other hand, motivated by local conservation properties, a priori error estimates of FVE approximations of linear elliptic and parabolic optimal control problems have been established in [59, 60], employing a variational discretization approach.

We recall that two main strategies are available for the numerical solution of optimal control problems: the so-called optimize-then-discretize approach and discretize-then-optimize. It is well-known that for non-symmetric discrete formulations, these two approaches may lead to different discrete adjoint equations and the solutions may not coincide. In general, finite volume (FV) and related schemes are not necessarily symmetric and a choice of the appropriate strategy should be based on both theoretical and computational considerations. For instance, FVE methods were employed in [59, 60] together with optimize-then-discretize approach for the approximation of elliptic and parabolic optimal control problems. Here we will adopt optimize-then-discretize strategy.

In contrast with the condensed review given above, here we will focus on DFV methods for the approximation of optimal control problems. We also recall the fact that in DFV methods, discontinuous piecewise linear functions conform the trial space, whereas piecewise constant test functions are used in a finite volume fashion. The application of DFV methods in the approximation of Stokes and related fluid problems can be found in e.g. [12, 32, 50, 52, 82]. In this Chapter our objective is to apply DFV schemes to the case of velocity control for the linear Brinkman equations. For the approximation of the control variable, we will discuss three alternatives: a variational discretization approach, element-wise constant and element-wise linear discretization.

**Notations:** Let \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), be a bounded convex polygonal domain with boundary \( \partial \Omega \). The outward unit normal vector to \( \Omega \) is denoted by \( n \). Standard terminology will be employed for Sobolev spaces: \( H^1(\Omega) = H^1(\Omega)^d \) and \( H^1_0(\Omega) := \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \} \). The corresponding norms will be denoted by \( \| \cdot \|_{1, \Omega} \). We also consider the space of integrable functions with zero mean: \( L^2_0(\Omega) = \{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \} \) and...
\( L^2(\Omega) = L^2(\Omega)^d \). The notation \((\cdot, \cdot)_{0,\Omega}\) stands for the scalar product in \( L^2(\Omega) \) and we use \( \| \cdot \|_{0,\Omega} \) to denote the associated norm. These notations will be frequently used along the Chapter. By \( \text{div} \) we will denote the usual divergence operator \( \text{div} \) applied row-wise to a tensor, \( I \) stands for \( d \times d \) identity matrix, and \( 0 \) will be used as a generic null vector.

### 5.1.1 The Brinkman model problem

Optimal control problems governed by Brinkman equations describe the controlled motion of an incompressible viscous fluid within an array of porous particles. We investigate the following optimization problem with control variable \( u \), fluid velocity \( y \) and pressure field \( p \):

\[
\min_{u \in U_{ad}} J(u) := \frac{1}{2} \| y - y_d \|^2_{0,\Omega} + \frac{\lambda}{2} \| u \|^2_{0,\Omega},
\]

(5.1)
governed by the Brinkman equations

\[
\begin{aligned}
K^{-1}(x)y - \text{div} \left( \mu(x)\varepsilon(y) - pI \right) &= u + f, \quad \text{in } \Omega, \\
\text{div } y &= 0, \quad \text{in } \Omega, \\
y &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(5.2)

The set of feasible controls \( U_{ad} \) is defined by

\[
U_{ad} = \{ u \in L^2(\Omega) : u_{a_j} \leq u_j \leq u_{b_j} \text{ a.e. in } \Omega \}.
\]

for \(-\infty \leq u_{a_j} < u_{b_j} \leq \infty, \ j = 1, \ldots, d\). The quantity \( \mu(x)\varepsilon(y) - pI \) is the Cauchy (true stress) tensor, where \( \varepsilon(y) = \frac{1}{2}(\nabla y + \nabla y^T) \) is the infinitesimal rate of strain, \( \mu(x) \) is the dynamic viscosity of the fluid, and \( K(x) \) stands for the permeability tensor of the medium (typically rescaled by the viscosity). As before, \( \lambda > 0 \) denotes a given Tikhonov regularization (or control cost) parameter. The desired velocity \( y_d \) and the applied body force \( f \) are known data with regularity \( L^2(\Omega) \) or \( H^1(\Omega) \), depending on the specific case. The whole idea of this problem is to identify an additional force \( u \) giving rise to a velocity \( y \) close to a known desired or target velocity \( y_d \).

We assume that \( K \) is symmetric, uniformly bounded and positive definite; and that
viscosity and permeability satisfy

$$\exists \gamma_1, \mu_{\text{min}}, \mu_{\text{max}} > 0 : \forall s \in \mathbb{R}_+; \mu_{\text{min}} < \mu(s) < \mu_{\text{max}}, |\mu'(s)| \leq \gamma_1,$$

$$\exists k_1, k_2 > 0 : 0 < k_1 \leq K^{-1}(x) \leq k_2 \quad \forall x \in \Omega,$$

(5.3)

the last inequalities being understood component-wise. The standard weak formulation of the state equations (5.2) is given by: find $$(y, p) \in H^1_0(\Omega) \times L^2_0(\Omega)$$ such that

$$\begin{aligned}
a(y, v) + c(y, v) + b(v, p) &= (f + u, v)_{0, \Omega} \quad \forall v \in H^1_0(\Omega), \\
b(y, q) &= 0 \quad \forall q \in L^2_0(\Omega),
\end{aligned}$$

(5.4)

where the bilinear forms $a(\cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$, $c(\cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ and $b(\cdot, \cdot) : H^1_0(\Omega) \times L^2_0(\Omega) \to \mathbb{R}$ are defined as:

$$\begin{aligned}
a(y, v) &= \int_{\Omega} K^{-1}(x) y \cdot v \, dx, \\
c(y, v) &= \int_{\Omega} \mu(x) \varepsilon(y) : \varepsilon(v) \, dx, \\
b(v, q) &= -\int_{\Omega} q \text{div} v \, dx,
\end{aligned}$$

for all $y, v \in H^1_0(\Omega)$ and $q \in L^2_0(\Omega)$. Problem (5.4) satisfies the following Babuška-Brezzi condition (see [71], for example): there exists $\xi > 0$ such that

$$\inf_{q \in L^2_0(\Omega)} \sup_{0 \neq v \in H^1_0(\Omega)} \frac{b(v, q)}{\|v\|_{1, \Omega} \|q\|_{0, \Omega}} \geq \xi,$$

and its unique solvability is ensured [71].

The optimality condition can be formulated as

$$J'(u)(\tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad},$$

which can be rewritten in the form:

$$\begin{aligned}
(w + \lambda u, \tilde{u} - u)_{0, \Omega} &\geq 0 \quad \forall \tilde{u} \in U_{ad}, \\
\end{aligned}$$

(5.5)
where $w$ is the velocity associated to the adjoint equation

\[
\begin{align*}
K^{-1}(x)w - \text{div}(\mu(x)\varepsilon(w) + rI) &= y - y_d \quad \text{in } \Omega, \\
\text{div } w &= 0 \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

The variational inequality (5.5) can be equivalently recast in component-wise manner

\[
u_j(x) = P_{[u_{a_j}, u_{b_j}]}\left(-\frac{1}{\lambda} w_j(x)\right) \quad \text{a.e. in } \Omega, j = 1, \ldots, d,
\]

where the operator $P_{[u_{a_j}, u_{b_j}]}$ denotes a projection defined for a generic scalar function $z$ as

\[P_{[u_{a_j}, u_{b_j}]}(z(x)) = \max(u_{a_j}, \min(u_{b_j}, z(x))), \quad \text{a.e. in } \Omega, j = 1, \ldots, d,
\]

and if $z \in W^{1,\infty}(\Omega)$, it further satisfies

\[
\|\nabla P_{[a_j,b_j]}(z)\|_{L^\infty(\Omega)} \leq \|\nabla z\|_{L^\infty(\Omega)}. \tag{5.6}
\]

The remainder of this Chapter is structured in the following manner. In Section 5.2 we formulate the DFV scheme of the considered optimal control problem. Section 5.3 focuses on the development of a priori error estimates for different types of control discretizations. Finally, in Section 5.4 we summarize the solution algorithm and illustrate our theoretical error bounds and performance of the method by a set of numerical experiments.

### 5.2 Discretization

We first recall the construction of control volumes in DFV scheme as presented before in Section 2.2 of Chapter 2.

#### 5.2.1 Meshes, discrete spaces, and interpolation properties

Let $\mathcal{T}_h$ be a regular, quasi-uniform partition of $\bar{\Omega} \subset \mathbb{R}^d$, $d = 2, 3$, into closed triangles (or tetrahedra if $d = 3$). By $h_T$ we denote the diameter of a given element $T \in \mathcal{T}_h$, and
the global meshsize by \( h = \max_{T \in \mathcal{T}_h} h_T \). Moreover, let \( \mathcal{E}_h \) and \( \mathcal{E}_h^d \) denote, respectively, the set of all faces and boundary faces in \( \mathcal{T}_h \) (edges and boundary edges if \( d = 2 \)), and the symbol \( h_e \) represents the length of the edge \( e \) (or the area of the face \( e \) if \( d = 3 \)). It follows from the definitions of \( h_e \), \( h_T \) and \( h \) that
\[
h_e \leq h_T^{d-1} \leq h^{d-1}.
\] (5.7)

In addition to \( \mathcal{T}_h \), we introduce a dual partition in the following way. Each element \( T \in \mathcal{T}_h \) is split into three sub triangles (or four sub-tetrahedra if \( d = 3 \)) \( T_i^*, i = 1, \ldots, d+1 \), by connecting the barycenter of the element to its corner nodes (see a schematic for \( d = 2 \) and \( d = 3 \) in Figure 5.1). The set of all these elements generated by barycentric subdivison will be denoted by \( \mathcal{T}_h^* \) and will be called the dual partition of \( \mathcal{T}_h \).

We recall the definition of jump and average defined in Chapter 2. Let \( e \) be an interior face shared by two elements \( T_1 \) and \( T_2 \) in \( \mathcal{T}_h \). By \( n_1 \) and \( n_2 \) we will denote unit normal vectors on \( e \) pointing outwards \( T_1 \) and \( T_2 \), respectively. Then the average \( \langle \cdot \rangle \) and jump \( \llbracket \cdot \rrbracket \) operators defined on \( e \) for generic scalar and vector fields \( q, v \), respectively, are:
\[
\langle q \rangle = \frac{1}{2} (q|_{\partial T_1} + q|_{\partial T_2}), \quad \llbracket q \rrbracket = q|_{\partial T_1} - q|_{\partial T_2},
\]
\[
\langle v \rangle = \frac{1}{2} (v|_{\partial T_1} + v|_{\partial T_2}), \quad \llbracket v \rrbracket = v|_{\partial T_1} - v|_{\partial T_2}.
\]

If \( e \in \mathcal{E}_h^d \), then we simply take \( \langle q \rangle = q \) and \( \llbracket v \rrbracket = v \). Notice that jump and averages are defined so that they preserve the dimension of the argument.

As usual, we denote by \( \mathcal{P}_m(T) \) the space of polynomials of degree less or equal than \( m \), defined on the element \( T \), and \( \mathcal{P}_m(T) \) will denote its vectorial counterpart. Then, a finite dimensional trial space (that will be used for the state and costate velocity approximation) associated with the primal partition \( \mathcal{T}_h \) is given by
\[
V_h = \{ v_h \in \mathbf{L}^2(\Omega) : v_h|_T \in \mathcal{P}_1(T), \forall T \in \mathcal{T}_h \}.
\]
The finite dimensional test space for velocities and corresponding to the dual partition \( \mathcal{T}_h^* \) is
\[
V_h^* = \{ v_h \in \mathbf{L}^2(\Omega) : v_h|_{T^*} \in \mathcal{P}_0(T^*), \forall T^* \in \mathcal{T}_h^* \}.
\]
and the discrete space for state and costate pressure approximation is defined as

$$Q_h = \{ q_h \in L^2_0(\Omega) : q_h|_T \in P_0(T), \forall T \in T_h \}.$$

In addition, we define a space with higher regularity

$$V(h) = V_h + [H^2(\Omega) \cap H^1_0(\Omega)],$$

and the connection between spaces associated to the two different meshes is characterized by the transfer operator $\gamma : V(h) \rightarrow V_h^*$, defined in the following manner:

$$\gamma v|_{T^*} = \frac{1}{h_e} \int_{e} v|_{T_e} \, ds, \quad \text{for } T^* \in T_h^*.$$

Some useful properties of this map are collected in the following result.

**Lemma 5.2.1.** Let $\gamma$ be a transfer operator defined as in (5.8). Then

i) $\gamma$ is self-adjoint with respect to the $L^2$-inner product, i.e.

$$(v_h, \gamma z_h )_{0,\Omega} = (z_h, \gamma v_h )_{0,\Omega}, \quad \forall v_h, z_h \in V_h.$$

ii) If $\|v_h\|_{0,h}^2 := (v_h, \gamma v_h )_{0,\Omega}$, then $\|\cdot\|_{0,h}$ and $\|\cdot\|_{0,\Omega}$ are equivalent, with equivalence constants being independent of $h$. 

Figure 5.1: Left: sketch of a single primal element $T$ in $T_h$, and sub-elements $T_i^*$ belonging to the dual partition $T_h^*$. Right: its three-dimensional counterpart.
iii) The operator $\gamma$ is stable with respect to the norm $\| \cdot \|_{0,\Omega}$, that is

$$\| \gamma v_h \|_{0,\Omega} = \| v_h \|_{0,\Omega} \quad \forall v_h \in V_h.$$  \hfill (5.9)

iv) For all $v \in V(h)$ and $T \in T_h$, we have

$$\int_T (v - \gamma v) \, dx = 0, \quad \int_T [v - \gamma v]_e \, ds = 0,$$

$$\int_e (v - \gamma v) \, ds = 0, \quad \| v - \gamma v \|_{0,T} \leq C h_T \| v \|_{1,T}.$$

Proof. Different proofs can be found in e.g. \cite{9, 10, 49}.

Let $v_h \in V_h$. We proceed to test the state equation (5.2) against $\gamma v_h \in V_h^*$ and $\phi_h \in Q_h$, and after integrating by parts the momentum equation on each dual element and the mass equation on each primal element we end up with the following DFV scheme: Find $(y_h, p_h) \in V_h \times Q_h$ such that

$$\hat{A}_h(y_h, v_h) + c_h(y_h, v_h) + C_h(v_h, p_h) = (u_h + f, \gamma v_h)_0, \quad \forall v_h \in V_h, \quad \hfill (5.10)$$

$$B_h(y_h, \phi_h) = 0, \quad \forall \phi_h \in Q_h, \quad \hfill (5.11)$$

where the discrete bilinear forms $\hat{A}_h(\cdot, \cdot), c_h(\cdot, \cdot), C_h(\cdot, \cdot)$ and $B_h(\cdot, \cdot)$ are defined in the following manner (see also \cite{12}):

$$\hat{A}_h(w_h, v_h) := (K^{-1}(x) w_h, \gamma v_h)_0,$$

$$c_h(w_h, v_h) := - \sum_{T \in T_h} \sum_{j=1}^{d+1} \int_{f_j} \mu(x) \varepsilon(w_h) \cdot \gamma v_h \, ds - \sum_{e \in E_h} \int_e \langle \mu(x) \varepsilon(w_h) \rangle \, ds - \sum_{e \in E_h} \int_e \langle \mu(x) \varepsilon(w_h) \rangle \, ds,$$

$$C_h(v_h, q_h) := \sum_{T \in T_h} \sum_{j=1}^{d+1} \int_{f_j} q_h \n \cdot \gamma v_h \, ds + \sum_{e \in E_h} \int_e \langle q_h \n \rangle \cdot [\gamma v_h] \, ds,$$

$$B_h(v_h, q_h) := b(v_h, q_h) - \sum_{e \in E_h} \int_e \langle q_h \n \rangle \cdot [\gamma v_h] \, ds,$$

for all $w_h, v_h \in V_h$ and $q_h \in Q_h$. Here, $\alpha_d$ and $\delta$ are penalty parameters independent of $h$. In general, $\delta = (d - 1)^{-1}$ is commonly used in interior penalty methods.

For the sake of our forthcoming analysis, we introduce the following discrete norms
in $\mathbf{V}(h)$, which are naturally associated with the bilinear form $c_h(\cdot, \cdot)$:

$$\|v_h\|_{1,h}^2 := \sum_{T \in T_h} |v_h|^2_T + \sum_{e \in E_h} h_e^{-\delta} \|v_h\|_{0,e}^2,$$

$$\|v_h\|_{2,h}^2 := \|v_h\|_{1,h}^2 + \sum_{T \in T_h} h_T^2 |v_h|^2_T,$$

and we note that $\|\cdot\|_{1,h}$ and $\|\cdot\|_{2,h}$ are equivalent on $\mathbf{V}_h$. Moreover, we also have the following discrete Poincaré-Friedrichs inequality (see [12, pp. 457])

$$\|v_h\|_{0,\Omega} \leq C \|v_h\|_{2,h} \quad \forall v_h \in \mathbf{V}_h,$$

(5.12)

and as in e.g. [12], we can use Cauchy-Schwarz inequality and the definition of $\gamma$ to readily obtain

$$\frac{1}{h_e^{\delta/2}} \|\gamma v_h\|_{0,e} \leq \left( \frac{1}{h_e^{\delta/2}} \int_{e} [v_h]^2_e \, ds \right)^{1/2}.$$  (5.13)

**Lemma 5.2.2.** The bilinear forms defined above possess the following properties:

i) The bilinear form $\hat{A}_h(\cdot, \cdot)$ is continuous and coercive, that is, there exists a constant $C > 0$, independent of $h$, such that

$$|\hat{A}_h(v, w)| \leq C \|v\|_{0,\Omega} \|w\|_{0,\Omega}, \quad \forall v, w \in \mathbf{V}(h),$$

(5.14)

$$\hat{A}_h(v_h, v_h) \geq C \|v_h\|_{0,\Omega}^2, \quad \forall v_h \in \mathbf{V}_h.$$  (5.15)

In addition, we have the following estimate

$$|\hat{A}_h(v_h, z_h) - \hat{A}_h(z_h, v_h)| \leq C h \|v_h\|_{2,h} \|z_h\|_{2,h}, \quad \forall v_h, z_h \in \mathbf{V}_h.$$  (5.16)

ii) For the non-symmetric bilinear form $c_h(\cdot, \cdot)$ it holds that

$$|c_h(v, w)| \leq C \|v\|_{2,h} \|w\|_{2,h} \quad \forall v, w \in \mathbf{V}(h),$$

(5.17)

$$c_h(v_h, v_h) \geq C \|v_h\|_{2,h}^2 \quad \forall v_h \in \mathbf{V}_h,$$  (5.18)

$$c_h(v_h, z_h) - c_h(z_h, v_h) \leq C h \|v_h\|_{2,h} \|z_h\|_{2,h}, \quad \forall v_h, z_h \in \mathbf{V}_h.$$  (5.19)

where for (5.18), $\alpha_d > 0$ is assumed sufficiently large.

iii) The choice of approximation spaces $\mathbf{V}_h$ and $Q_h$ for velocity and pressure, respec-
tively, yields the following inf-sup condition

\[
\sup_{v_h \in \mathbf{V}_h \setminus \{0\}} \frac{B_h(v_h, q_h)}{\|v_h\|_{2,h}} \geq \beta_1 \|q_h\|_{0,\Omega},
\]  

(5.20)

where \(\beta_1 > 0\) is independent of \(h\).

iv) The bilinear form \(C_h(\cdot, \cdot)\) satisfies for all \(v, w \in \mathbf{V}(h)\), \(q \in L^2_0(\Omega)\), and \(q_h \in Q_h\)

\[
|C_h(v, q)| \leq C \|v\|_{2,h} \left(\|q\|_{0,\Omega} + \sum_{T \in T_h} h^2 |q_h|^2_{1,T} \right)^{1/2},
\]  

(5.21)

\[
C_h(v, q_h) = -B_h(v, q_h).
\]  

(5.22)

Proof. For i) it suffices to apply the definition of \(\hat{A}_h(\cdot, \cdot)\), together with relation (5.3), and the norm equivalence between \(\|\cdot\|_{0,h}\) and \(\|\cdot\|_{0,\Omega}\). Results in ii) have been established in [52] and [9], whereas proofs for iii)-iv) can be found in [82].

5.2.2 Control discretization

Let \(U_h \subseteq L^2(\Omega)\) denote the discrete control space, and let us introduce the discrete admissible space for the control field as \(U_{h,\text{ad}} = U_h \cap U_{\text{ad}}\). Three approaches are outlined in what follows.

Variational discretization. We recall that in the so-called variational approach (cf. [42]), control variables are not discretized explicitly, that is, one simply takes \(U_h = L^2(\Omega)\) and in this case the discrete and continuous admissible spaces \(U_{h,\text{ad}}\) and \(U_{\text{ad}}\) coincide. Induced discretization errors using this method will be postponed to Section 5.3.1.

In contrast, it is possible to fully discretize the control field. We focus on the two lowest order cases.

Piecewise linear control discretization. Here we approximate the control variable with the similar elements as those employed for the state and co-state velocity approximation. That is,

\[
U_{h,1} = \{ u_h \in L^2(\Omega) : u_h|_T \in P_1(T) \quad \forall T \in T_h \}.
\]
We note that the state velocity space $V_h$ coincides with the control space $U^1_h$ in the case of homogeneous Neumann boundary conditions, whereas for Dirichlet boundary data, we have $V_h \subset U^1_h$.

**Piecewise constant discretization.** In this case, the discrete control space is defined as

$$U^0_h = \{ u_h \in L^2(\Omega) : u_h|_T \in P_0(T) \quad \forall T \in T_h \}.$$ 

The convergence properties associated to the above two approaches (piecewise linear and piecewise constant) will be derived in Section 5.3.2, but already at this point we can apply Lemma 5.2.2 along with the Babuška-Brezzi theory for saddle point problems to ensure the unique solvability of (5.10)-(5.11), for a fixed control $u_h$.

Using relation (5.22), the DFV approximation of the continuous optimal system (5.1)-(5.2) can be summarized as: Find $(y_h, p_h, w_h, r_h, u_h) \in V_h \times Q_h \times V_h \times Q_h \times U_h, \text{ad}$ such that

$$\hat{A}_h(y_h, v_h) + c_h(y_h, v_h) - B_h(v_h, p_h) = (u + f, \gamma v)_0, \forall v_h \in V_h, \quad (5.23)$$

$$B_h(y_h, \phi_h) = 0, \forall \phi_h \in Q_h, \quad (5.24)$$

$$\hat{A}_h(w_h, z_h) + c_h(w_h, z_h) + B_h(z_h, r_h) = (y_h - y_d, \gamma z)_0, \forall z_h \in V_h, \quad (5.25)$$

$$B_h(w_h, \psi_h) = 0, \forall \psi_h \in Q_h, \quad (5.26)$$

$$(w_h + \lambda u_h, \tilde{u}_h - u_h)_0 \geq 0, \forall \tilde{u}_h \in U_{h, \text{ad}}. \quad (5.27)$$

### 5.3 Error estimates

In this Section, we provide *a priori* error estimates for DFV approximations of the state and adjoint equations, and for the three control discretization approaches outlined in Section 5.2.2.

For a given arbitrary $u$, let the pair $(y_h(u), p_h(u))$ be the solution of the following auxiliary problem for all $v_h \in V_h$ and $\phi_h \in Q_h$:

$$\hat{A}_h(y_h(u), v_h) + c_h(y_h(u), v_h) - B_h(v_h, p_h(u)) = (u + f, \gamma v_h)_0, \quad (5.28)$$

$$B_h(y_h(u), \phi_h) = 0. \quad (5.29)$$
Similarly, we assume that for an arbitrary $y$, let $(w_h(y), r_h(y))$ be the solution of

$$
\hat{A}_h(w_h(y), z_h) + c_h(w_h(y), z_h) + B_h(z_h, r_h(y)) = (y - y_d, \gamma z_h)_{0, \Omega},
$$

(5.30)

$$
B_h(w_h(y), \psi_h) = 0,
$$

(5.31)

for all $z_h \in V_h$ and $\psi_h \in Q_h$. We then proceed to decompose the errors $y - y_h$, $w - w_h$, $p - p_h$ and $r - r_h$ in the following manner:

$$
y - y_h = y - y_h(u) + y_h(u) - y_h, w - w_h = w - w_h(y) + w_h(y) - w_h, \quad (5.32)
$$

$$
p - p_h = p - p_h(u) + p_h(u) - p_h, \quad r - r_h = r - r_h(y) + r_h(y) - r_h. \quad (5.33)
$$

Noting that $y_h = y_h(u_h), p_h = p_h(u_h), w_h = w_h(y_h)$, and $r_h = r_h(y_h)$, the following intermediate result is established.

**Lemma 5.3.1.** Let $(y_h(u), p_h(u))$ and $(w_h(y), r_h(y))$ be the solutions of equations (5.28)-(5.29) and (5.30)-(5.31), respectively. Then there exists a positive constant $C$ independent of mesh size $h$ such that the following estimates hold

$$
\|y_h(u) - y_h\|_{2,h} + \|p_h(u) - p_h\|_{0,\Omega} \leq C \|u - u_h\|_{0,\Omega}, \quad (5.34)
$$

$$
\|w_h(y) - w_h\|_{2,h} + \|r_h(y) - r_h\|_{0,\Omega} \leq C \|y - y_h\|_{0,\Omega}. \quad (5.35)
$$

**Proof.** Subtracting equations (5.23) and (5.24) from (5.28) and (5.29), respectively, we have that the following relations hold for all $v_h \in V_h$ and $\phi_h \in Q_h$

$$
\hat{A}_h(y_h(u) - y_h, v_h) + c_h(y_h(u) - y_h, v_h) - B_h(v_h, p_h(u) - p_h)
$$

$$
= (u - u_h, \gamma v_h)_{0,\Omega}, \quad (5.36)
$$

$$
B_h(y_h(u) - y_h, \phi_h) = 0.
$$

Adding the above two equations after choosing $v_h = y_h(u) - y_h$ and $\phi_h = p_h(u) - p_h$ implies that

$$
\hat{A}_h(y_h(u) - y_h, y_h(u) - y_h) + c_h(y_h(u) - y_h, y_h(u) - y_h)
$$

$$
= (u - u_h, \gamma (y_h(u) - y_h))_{0,\Omega}.
$$

In turn, using the coercivity of $\hat{A}_h(\cdot, \cdot)$ and $c_h(\cdot, \cdot)$ in combination with (5.9) and (5.12),
we obtain
\[
\|\mathbf{y}_h(u) - \mathbf{y}_h\|^2_{0,\Omega} + \|\mathbf{y}_h(u) - \mathbf{y}_h\|^2_{2,h} \leq C(\|u - u_h\|_{0,\Omega} \|\mathbf{y}_h(u) - \mathbf{y}_h\|_{0,\Omega} ,
\]
\[
\leq C \|u - u_h\|_{0,\Omega} \|\mathbf{y}_h(u) - \mathbf{y}_h\|_{0,\Omega} ,
\]
\[
\leq C \|u - u_h\|_{0,\Omega} \|\mathbf{y}_h(u) - \mathbf{y}_h\|_{2,h}.
\]

which on dropping the first term of above relation, readily yields the bound
\[
\|\mathbf{y}_h(u) - \mathbf{y}_h\|_{2,h} \leq C \|u - u_h\|_{0,\Omega}.
\] (5.37)

On the other hand, applying the inf-sup condition (5.20), using (5.36), the boundedness of \(\hat{A}_h(\cdot, \cdot)\) and \(c_h(\cdot, \cdot)\), along with (5.37), we realize that
\[
\|p_h - p_h(u)\|_{0,\Omega}
\]
\[
\leq \frac{1}{\beta_1} \sup_{\mathbf{v}_h \in \mathbf{V}_h \{ 0 \}} \frac{B_h(\mathbf{v}_h, p_h - p_h(u))}{\|\mathbf{v}_h\|_{2,h}} ,
\]
\[
= \frac{1}{\beta_1} \sup_{\mathbf{v}_h \in \mathbf{V}_h \{ 0 \}} \frac{\hat{A}_h(\mathbf{y}_h(u) - \mathbf{y}_h, \mathbf{v}_h) + c_h(\mathbf{y}_h(u) - \mathbf{y}_h, \mathbf{v}_h) + (u_h - u, \gamma \mathbf{v}_h)_{0,\Omega}}{\|\mathbf{v}_h\|_{2,h}}
\]
\[
\leq C \|u - u_h\|_{0,\Omega}.
\] (5.38)

Notice that relations (5.37) and (5.38) imply, in particular, that (5.34) holds. Next, we subtract equations (5.25) and (5.26) from (5.30) and (5.31), respectively, and test the result against \(z_h = w_h(y) - w_h\) and \(\psi_h = r_h(y) - r_h\), which yields the second desired result (5.35) after repeating the same steps as above.

\[\square\]

**Lemma 5.3.2.** Under the assumptions \(\mu \in W^{2,\infty}(\Omega)\) and \(u, f, y_d \in H^1(\Omega)\), we have that
\[
\|\mathbf{y} - y_h(u)\|_{2,h} + \|p - p_h(u)\|_{0,\Omega} = O(h),
\] (5.39)
\[
\|w - w_h(y)\|_{2,h} + \|r - r_h(y)\|_{0,\Omega} = O(h),
\] (5.40)
\[
\|\mathbf{y} - y_h(u)\|_{0,\Omega} + \|w - w_h(y)\|_{0,\Omega} = O(h^2).
\] (5.41)

**Proof.** We proceed analogously to the proof of [32, Theorem 3.1] and directly apply
Lemma 5.2.2 to readily derive the following estimates:

\[ \| y - y_h(u) \|_{2,h} + \| p - p_h(u) \|_{0,\Omega} \leq C h \left( \| y \|_{2,\Omega} + \| p \|_{1,\Omega} \right), \]
\[ \| w - w_h(y) \|_{2,h} + \| r - r_h(y) \|_{0,\Omega} \leq C h \left( \| w \|_{2,\Omega} + \| r \|_{1,\Omega} \right). \]

Next, the derivation of \( L^2 \)-estimates for \( y - y_h(u) \) and \( w - w_h(y) \) follows an Aubin-Nitsche duality argument. Let us consider the dual problem: find \( (z, \rho) \in H^1_0(\Omega) \times L^2_0(\Omega) \) such that

\[ K^{-1}(x) - \text{div}(\mu(x)\varepsilon(z) - \rho I) = y - y_h(u) \quad \text{in} \ \Omega, \]
\[ \text{div} z = 0 \quad \text{in} \ \Omega, \]
\[ z = 0 \quad \text{on} \ \partial \Omega, \]

which is uniquely solvable and moreover the following \( H^2(\Omega) \times H^1(\Omega) \)-regularity property is satisfied:

\[ \| z \|_{2,\Omega} + \| \rho \|_{1,\Omega} \leq \| y - y_h(u) \|_{0,\Omega}. \]

Let us denote by \( z_I \in V_h \) the usual continuous piecewise linear interpolant of \( z \), satisfying the following approximation properties:

\[ \| z - z_I \|_{2,h} \leq C h \| z \|_{2,\Omega} \quad \text{and} \quad \| z - z_I \|_{0,\Omega} \leq C h^2 \| z \|_{2,\Omega}. \]

Also, let \( \Pi_1 \) denote the \( L^2 \)-projection from \( L^2_0(\Omega) \) to \( Q_h \), satisfying

\[ \| \rho - \Pi_1 \rho \|_{0,\Omega} \leq C h \| \rho \|_{1,\Omega}. \]

Multiplying (5.42) by \( y - y_h(u) \), integrating by parts, and using the fact that \( [\varepsilon(z)n]_e = 0 \) and \( [\rho]_e = 0 \), we can obtain

\[ \| y - y_h(u) \|_{0,\Omega}^2 = A_h^x(y - y_h(u), z) + c_h^x(y - y_h(u), z) - b_h^x(y - y_h(u), \rho), \]
where the auxiliary bilinear forms adopt the following expressions

\[ A_h^*(w_h, v_h) := (K^{-1}(x)w_h, v_h)_{0, \Omega}; \]
\[ b_h^*(v_h, q_h) := b(v_h, q_h) + \sum_{e \in \mathcal{E}_h} \int_e [q_h n]_e \cdot [v_h]_e \, ds, \]
\[ c_h^*(w_h, v_h) := c(w_h, v_h) - \sum_{e \in \mathcal{E}_h} \int_e \{ \mu(x) \varepsilon(v_h) n \}_e \cdot [v_h]_e \, ds \]
\[- \sum_{e \in \mathcal{E}_h} \int_e \{ \mu(x) \varepsilon(v_h) n \}_e \cdot [w_h]_e \, ds + \sum_{e \in \mathcal{E}_h} \int_e \frac{\alpha}{h_e^2} [w_h]_e \cdot [v_h]_e \, ds. \]

Since \( z_I \in \mathbf{V}_h \) is a continuous interpolant of \( z \), we note that the pair \((y - y_h(u), p - p_h(u))\) will be a solution of the following problem

\[ \bar{A}_h(y - y_h(u), z_I) + c_h(y - y_h(u), z_I) + C_h(z_I, p - p_h(u)) = 0, \quad (5.47) \]
\[ B_h(y - y_h(u), \Pi_1 \rho) = 0. \quad (5.48) \]

Using the definition of \( c_h(\cdot, \cdot) \) and \( C_h(\cdot, \cdot) \) we can assert that

\[ C_h(z_I, p - p_h(u)) = -(\text{div } z_I, p - p_h(u))_{T_h} - (\nabla p, z_I - \gamma z_I)_{T_h}, \quad (5.49) \]

where the inner product over the primal mesh is understood as the sum of the inner products over each element in \( T_h \). On subtracting equation (5.47) from the sum of equations (5.46) and (5.48), and using (5.49), it follows that

\[ \|y - y_h(u)\|_{0, \Omega}^2 \]
\[ = \left[ A_h^*(y - y_h(u), z) - \bar{A}_h(y - y_h(u), z_I) \right]_{R_1} + c_h^*(y - y_h(u), z - z_I)_{R_2} \]
\[ + \left[ c_h^*(y - y_h(u), z_I) - c_h(y - y_h(u), z_I) + \sum_{T \in T_h} \int_T (z_I - \gamma z_I) \cdot \nabla p \, dx \right]_{R_3} \]
\[ + (p - p_h(u), \text{div } z_I)_{0, \Omega} - b_h^*(y - y_h(u), \rho)_{R_4} + B_h(y - y_h(u), \Pi_1 \rho). \quad (5.50) \]

The estimation of \( R_1 \) relies on (5.3), (5.45), the self-adjointness and approximation
properties of $\gamma$, and (5.44). This gives

$$R_1 \leq |(y - y_h(u), K^{-1}(x)z)_{0,\Omega} - (K^{-1}(x)(y - y_h(u)), \gamma z_t)_{0,\Omega}|$$

$$\leq k_2 |(y - y_h(u), z - z_t)_{0,\Omega} + (y - y_h(u)) - \gamma(y - y_h(u), z_t)_{0,\Omega}|$$

$$\leq C(h^2 \|y - y_h(u)\|_{0,\Omega} \|z\|_{2,\Omega} + h \|y - y_h(u)\|_{2,h} \|z_t\|_{0,\Omega})$$

$$\leq Ch^2(\|y - y_h(u)\|^2_{0,\Omega} + \|y\|_{2,\Omega} \|y - y_h(u)\|_{0,\Omega}),$$

where the last inequality follows from (5.39). For the second term we employ the definition of $c_h(\cdot, \cdot)$, and relations (5.45),(5.44) to verify that

$$R_2 \leq Ch^2 \|y\|_{2,\Omega} \|z\|_{2,\Omega} \leq Ch^2 \|y\|_{2,\Omega} \|y - y_h(u)\|_{0,\Omega}.$$ 

Bounds for the remaining terms can be obtained following the proof of [52, Theorem 3.4] and [32, Theorem 3.2], as follows

$$R_3 \leq Ch^2[\|y\|_{2,\Omega} + \|u\|_{1,\Omega} + \|f\|_{1,\Omega}] \|y - y_h(u)\|_{0,\Omega},$$

$$R_4 \leq |(p - p_h(u), \text{div}(z - z_t))_{0,\Omega}| \leq Ch^2 \|p\|_{1,\Omega} \|y - y_h(u)\|_{0,\Omega},$$

$$R_5 \leq Ch^2 \|y\|_{2,\Omega} \|y - y_h(u)\|_{0,\Omega}.$$ 

Combining the five estimates above with (5.50), we straightforwardly obtain

$$\|y - y_h(u)\|_{0,\Omega} \leq Ch^2 \left[\|y\|_{2,\Omega} + \|p\|_{1,\Omega} + \|u\|_{1,\Omega} + \|f\|_{1,\Omega}\right],$$

and very much in the same way, one arrives at

$$\|w - w_h(y)\|_{0,\Omega} \leq Ch^2 \left[\|w\|_{2,\Omega} + \|r\|_{1,\Omega} + \|y\|_{1,\Omega} + \|y_d\|_{1,\Omega}\right].$$

Now, for a given control $u$, let $(w_h(u), r_h(u))$ be the solution of

$$\hat{A}_h(w_h(u), z_h) + c_h(w_h(u), z_h) + B_h(z_h, r_h(u)) = (y_h(u) - y_d, \gamma z_h)_{0,\Omega} \quad \forall z_h \in V_h,$$

$$B_h(w_h(u), \psi_h) = 0 \quad \forall \psi_h \in Q_h,$$

and notice that similar arguments as those appearing in the proof of Lemma 5.3.2 and
in the derivation of the estimate \( \| y - y_h(u) \|_{0,\Omega} = O(h^2) \), will readily lead to

\[
\| w - w_h(u) \|_{0,\Omega} = O(h^2). 
\] (5.51)

The following result plays a vital role in deriving error estimates of the control, state and co-state variables. Its proof is similar to that in [59, Theorem 4.1].

**Lemma 5.3.3.** Under the assumptions \( \mu \in W^{2,\infty}(\Omega) \) and \( u, f, y_d \in H^1(\Omega) \), we have the following estimate

\[
(w - w_h, u_h - u)_{0,\Omega} \leq C h^2 \| u_h - u \|_{0,\Omega} + C h \| u_h - u \|_{0,\Omega}^2. 
\] (5.52)

**Proof.** We split \( (w - w_h, u_h - u)_{0,\Omega} \) as

\[
(w - w_h, u_h - u)_{0,\Omega} = (w - w_h(y), u_h - u)_{0,\Omega} + (w_h(y) - w_h - \gamma(w_h(y) - w_h), u_h - u)_{0,\Omega}. 
\] (5.53)

Then, using the approximation property of \( \gamma \) together with Lemmas 5.3.1 and 5.3.2 implies

\[
(w - w_h(y), u_h - u)_{0,\Omega} + (w_h(y) - w_h - \gamma(w_h(y) - w_h), u_h - u)_{0,\Omega}
\]

\[
\leq \| w - w_h(y) \|_{0,\Omega} \| u_h - u \|_{0,\Omega} + C h \| y - y_h \|_{0,\Omega} \| u_h - u \|_{0,\Omega}
\]

\[
\leq C h^2 \| u_h - u \|_{0,\Omega} + C h (\| y - y_h(u) \|_{0,\Omega} + \| u_h - u \|_{0,\Omega}) \| u_h - u \|_{0,\Omega}
\]

\[
\leq C h^2 \| u_h - u \|_{0,\Omega} + C h \| u_h - u \|_{0,\Omega}^2. 
\] (5.54)

Now we subtract (5.28) and (5.29) from (5.23) and (5.24), respectively and test the result against \( v_h = w_h(y) - w_h \) and \( \phi_h = r_h(y) - r_h \) to obtain the relation

\[
(\gamma(w_h(y) - w_h), u_h - u)_{0,\Omega}
\]

\[
= A_h(y_h - y_h(u), w_h(y) - w_h) + c_h(y_h - y_h(u), w_h(y) - w_h)
\]

\[
- B_h(w_h(y) - w_h, p_h - p_h(u)) + B(y_h - y_h(u), r_h(y) - r_h). 
\] (5.55)

Similarly, on subtracting equations (5.25) and (5.26) from (5.30) and (5.31), respec-
tively, and taking \( z_h = y_h - y_h(u) \) and \( \psi_h = p_h - p_h(u) \), we can assert that

\[
\hat{A}_h (w_h(y) - w_h, y_h - y_h(u)) + c_h (w_h(y) - w_h, y_h - y_h(u)) \\
= (y - y_h, \gamma (y_h - y_h(u)))_{0, \Omega} - B_h (y_h - y_h(u), r_h(y) - r_h) \\
+ B (w_h(y) - w_h, p_h - p_h(u)).
\] (5.56)

Adding equations (5.55) and (5.56) and using the fact that \((y_h - y_h(u), \gamma (y_h - y_h(u)))_{0, \Omega} \geq 0\), we arrive at

\[
(\gamma (w_h(y) - w_h), u_h - u)_{0, \Omega} \\
\leq [\hat{A}_h (y_h - y_h(u), w_h(y) - w_h) - \hat{A}_h (w_h(y) - w_h, y_h - y_h(u))] \\
+ [c_h (y_h - y_h(u), w_h(y) - w_h) - c_h (w_h(y) - w_h, y_h - y_h(u))] \\
+ (y - y_h(u), \gamma (y_h - y_h(u)))_{0, \Omega} \\
\leq Ch \| y_h - y_h(u) \|_{2,h} \| w_h(y) - w_h \|_{2,h} + \| y - y_h(u) \|_{0, \Omega} \| y_h - y_h(u) \|_{2,h},
\]

where we have used relations (5.9), (5.12), (5.16) and (5.19). An application of Lemmas 5.3.1 and 5.3.2 in the above inequality implies

\[
(\gamma (w_h(y) - w_h), u_h - u)_{0, \Omega} \leq Ch \| u_h - u \|_{0, \Omega}^2 + Ch^2 \| u_h - u \|_{0, \Omega}.
\] (5.57)

Inserting the estimates of (5.54) and (5.57) in (5.53), we get the required result. \( \square \)

### 5.3.1 Error estimates under variational discretization

**Theorem 5.3.4.** Let \((y_h, w_h)\) be DFV approximations of \((y, w)\) and let \(u_h\) denote a variational discretization of \(u\). Then

\[
\| u - u_h \|_{0, \Omega} = \mathcal{O}(h^2),
\] (5.58)

\[
\| y - y_h \|_{0, \Omega} = \mathcal{O}(h^2),
\] (5.59)

\[
\| w - w_h \|_{0, \Omega} = \mathcal{O}(h^2).
\] (5.60)

**Proof.** We recall the continuous variational inequality

\[
(w + \lambda \tilde{u}, \tilde{u} - u)_{0, \Omega} \geq 0 \quad \forall \tilde{u} \in U_{ad},
\] (5.61)
and the discrete variational inequality under variational discretization

\[
(w_h + \lambda u_h, \tilde{u}_h - u_h)_{0, \Omega} \geq 0 \quad \forall \tilde{u}_h \in U_{ad}. \tag{5.62}
\]

Choosing \( \tilde{u} = u_h \) and \( \tilde{u}_h = u \) in (5.61) and (5.62), respectively, and adding, yields

\[
(w + \lambda u, u_h - u)_{0, \Omega} + (w_h + \lambda u_h, u - u_h)_{0, \Omega} \geq 0,
\]

and rearranging terms, we get

\[
\lambda \| u - u_h \|_{0, \Omega}^2 \leq (w - w_h, u_h - u)_{0, \Omega}. \tag{5.63}
\]

An application of (5.52) in (5.63) yields the required result (5.58). Using (5.32) and the triangle inequality together with Lemmas 5.3.1 and 5.3.2, and result (5.58), the remaining estimates (5.59)-(5.60) follow in a straightforward manner. \( \square \)

### 5.3.2 L²-error estimates for fully discretized controls

A discrete admissible control \( \tilde{u}_h = (\tilde{u}_{h,j})_{j=1}^d \in U_{h,ad} \) is defined componentwise, and on an arbitrary \( T \in \mathcal{T}_h \), as

\[
\tilde{u}_{h,j} = \begin{cases} 
    u_{a_j} & \text{if } \min_{x \in T} u_j(x) = u_{a_j}, \\
    u_{b_j} & \text{if } \max_{x \in T} u_j(x) = u_{b_j}, \\
    I_h u_j & \text{otherwise},
\end{cases}
\tag{5.64}
\]

where \( I_h u_j \) denotes the Lagrange interpolate of \( u_j \). To avoid ambiguity, we choose \( h \) sufficiently small so that \( \min_{x \in T} u_j(x) = u_{a_j} \) and \( \max_{x \in T} u_j(x) = u_{b_j} \) do not occur simultaneously within the same element \( T \). Next, we proceed to group the elements in the primal mesh into three categories: \( \mathcal{T}_h = \mathcal{T}_{h,1} \cup \mathcal{T}_{h,2} \cup \mathcal{T}_{h,3} \) with \( \mathcal{T}_{h,m} \cap \mathcal{T}_{h,n} = \emptyset \) for \( m \neq n \) according to the local value of \( u_j(x) \) on \( T \). These sets are defined as

\[
\mathcal{T}_{h,1} = \{ T \in \mathcal{T}_h : u_j(x) = u_{a_j} \ \text{or} \ u_j(x) = u_{b_j} \ \forall x \in T \},
\]

\[
\mathcal{T}_{h,2} = \{ T \in \mathcal{T}_h : u_{a_j} < u_j(x) < u_{b_j} \ \forall x \in T \},
\]

\[
\mathcal{T}_{h,3} = \mathcal{T}_h \setminus (\mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}).
\]
Definition (5.64) implies that for any \( \tilde{u}_h \in U_{h,\text{ad}} \), the following relation holds (cf. [17, Lemma 2.1]):

\[
(w + \lambda u, \tilde{u} - \tilde{u}_h)_0,\Omega \geq 0 \quad \forall \tilde{u} \in U_{\text{ad}}.
\]

(5.65)

On the other hand, we have the following technical result, to be instrumental in the subsequent analysis: there exists a positive constant \( C \) independent of \( h \) such that

\[
\sum_{j=1}^{d} \sum_{T \in T^j_{h,3}} |T| \leq Ch.
\]

(5.66)

We will first focus on error bounds for the control field under piecewise linear discretization. Before proceeding we state an auxiliary result, whose proof can be found in [62].

**Lemma 5.3.5.** Assume that (5.66) holds and that \( w \in W^{1,\infty}(\Omega) \). Then, there exists \( C > 0 \) independent of \( h \) such that

\[
| (w + \lambda u, \tilde{u}_h - u)_0,\Omega | \leq \frac{C}{\lambda} h^3 \| \nabla w \|_{\infty,\Omega}^2,
\]

for any \( \tilde{u}_h \in U_{h,\text{ad}} \).

The main result in this Section is stated as follows.

**Theorem 5.3.6.** Let \( u \in U_{\text{ad}} \) be the solution of (5.1)-(5.2) and \( u_h \in U_{h,\text{ad}} \) be the solution of (5.23)-(5.27), under piecewise linear control discretization. Then

\[
\| u - u_h \|_{0,\Omega} = \mathcal{O}(h^{3/2}).
\]

*Proof.* Testing the continuous and discrete variational inequalities against \( u_h \in U_{h,\text{ad}} \subset U_{\text{ad}} \) and \( \tilde{u}_h \in U_{h,\text{ad}} \), respectively, and adding them, leads to

\[
(w + \lambda u, u_h - u)_0,\Omega + (w_h + \lambda u_h, \tilde{u}_h - u_h)_0,\Omega \geq 0.
\]

Addition and subtraction of \( \tilde{u}_h \) in the first term above yields

\[
\lambda(u - u_h, u_h - \tilde{u}_h)_0,\Omega + (w - w_h, u_h - \tilde{u}_h)_0,\Omega + (w + \lambda u, \tilde{u}_h - u)_0,\Omega \geq 0.
\]
and after rearranging terms we obtain

\[ \lambda \| \mathbf{u} - \mathbf{u}_h \|_{0,\Omega}^2 \leq \lambda (\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \tilde{\mathbf{u}}_h)_{0,\Omega} + (\mathbf{w} - \mathbf{w}_h, \mathbf{u}_h - \mathbf{u})_{0,\Omega} \\
+ (\mathbf{w} - \mathbf{w}_h, \mathbf{u} - \tilde{\mathbf{u}}_h)_{0,\Omega} + (\mathbf{w} + \lambda \mathbf{u}, \mathbf{u}_h - \mathbf{u})_{0,\Omega}. \]  

(5.67)

Now, in order to estimate \( \| \mathbf{u} - \tilde{\mathbf{u}}_h \|_{0,\Omega} \), we rewrite it as

\[ \| \mathbf{u} - \tilde{\mathbf{u}}_h \|_{0,\Omega}^2 = \sum_{j=1}^{d} \sum_{T \in \mathcal{T}_h} \| u_j - \tilde{u}_{j,h} \|_{0,T}^2 \\
= \sum_{j=1}^{d} \sum_{T \in \mathcal{T}_{h,2}^j} \| u_j - \tilde{u}_{j,h} \|_{0,T}^2 + \sum_{j=1}^{d} \sum_{T \in \mathcal{T}_{h,3}^j} \| u_j - \tilde{u}_{j,h} \|_{0,T}^2 \]  

(5.68)

where we have used the fact that \( \tilde{u}_{j,h} = u_j \) on \( \mathcal{T}_{h,1}^j \), and hence \( \sum_{T \in \mathcal{T}_{h,1}^j} \| u_j - \tilde{u}_{j,h} \|_{0,T}^2 = 0 \), for \( j = 1, \ldots, d \). In order to bound \( T_1 \) we use the relation \( u_j = \frac{1}{\lambda} w_j \) on all triangles \( T \in \mathcal{T}_{h,2}^j \), to obtain

\[ \sum_{j=1}^{d} \sum_{T \in \mathcal{T}_{h,2}^j} \| u_j - \tilde{u}_{j,h} \|_{0,T}^2 \leq C h^4 \sum_{j=1}^{d} \sum_{T \in \mathcal{T}_{h,2}^j} \| \nabla^2 u_j \|_{0,T}^2 \\
\leq \frac{C}{\lambda^2} h^4 \sum_{j=1}^{d} \| \nabla^2 w_j \|_{0,T}^2, \]

whereas for \( T_2 \), we employ the projection property (5.6) together with Assumption 5.66 to get

\[ \sum_{j=1}^{d} \sum_{T \in \mathcal{T}_{h,3}^j} \| u_j - \tilde{u}_{j,h} \|_{0,T}^2 \leq C \sum_{j=1}^{d} \sum_{T \in \mathcal{T}_{h,3}^j} |T| \| u_j - \tilde{u}_{j,h} \|_{L^\infty(T)}^2 \\
\leq C h^3 \sum_{j=1}^{d} \| \nabla u_j \|_{L^\infty(\Omega)}^2 \leq \frac{C}{\lambda^2} h^3 \sum_{j=1}^{d} \| \nabla w_j \|_{L^\infty(\Omega)}^2. \]

Inserting the bounds of \( T_1 \) and \( T_2 \) in (5.68) we arrive at

\[ \| \mathbf{u} - \tilde{\mathbf{u}}_h \|_{0,\Omega} = \mathcal{O}(h^{3/2}). \]  

(5.69)

Finally, applying Cauchy-Schwarz and Young’s inequalities, the estimates (5.52), (5.69)
and Lemmas 5.3.2 and 3.3.8 in (2.36), we readily obtain the required result

\[ \| u - u_h \|_{0,\Omega} = O(h^{3/2}). \]

We now turn to the \( L^2 \)–error analysis for the control field under element-wise constant discretization. The main idea follows from [17], where the \( L^2 \)–projection operator \( \Pi_0 : L^2(\Omega) \to U_{h,0} \) is introduced, having the following property: there exists a positive constant \( C \) independent of \( h \) such that

\[ \| u - \Pi_0 u \|_{0,\Omega} \leq Ch \| u \|_{1,\Omega}. \] (5.70)

The error estimate reads as follows.

**Theorem 5.3.7.** Let \( u \) be the unique solution of problem (5.1)-(5.2) and \( u_h \) be the unique control, solution of discrete problem (5.23)-(5.27) under element-wise constant discretization. Then we can obtain the following result.

\[ \| u - u_h \|_{0,\Omega} = O(h). \]

**Proof.** Since \( \Pi_0 U_{ad} \subset U_{h,ad} \), the continuous and discrete optimality conditions readily imply

\[ (w + \lambda u, u_h - u)_{0,\Omega} + (w_h + \lambda u_h, \Pi_0 u - u_h)_{0,\Omega} \geq 0. \]

Adding and subtracting \( u \) and rearranging terms we obtain

\[ \lambda \| u - u_h \|_{0,\Omega}^2 \leq (w - w_h, u_h - u)_{0,\Omega} + (w_h + \lambda u_h, \Pi_0 u - u)_{0,\Omega}, \]

and since \( \Pi_0 \) is an orthogonal projection and \( u_h \in U_{h,ad} \), then the term \( \lambda(u_h, \Pi_0 u - u)_{0,\Omega} \) vanishes to give

\[ \lambda \| u - u_h \|_{0,\Omega}^2 \leq (w - w_h, u_h - u)_{0,\Omega} + (w_h, \Pi_0 u - u)_{0,\Omega} =: I_1 + I_2. \] (5.71)
For the first term, we use (5.52)

\[ I_1 \leq Ch^2 \| u - u_h \|_{0, \Omega} + Ch \| u - u_h \|_{0, \Omega}^2, \]

whereas a bound for \( I_2 \) follows from the orthogonality of \( \Pi_0 \):

\[ I_2 = (w_h - \Pi_0 w_h, \Pi_0 u - u)_{0, \Omega} \leq \| w_h - \Pi_0 w_h \|_{0, \Omega} \| \Pi_0 u - u \|_{0, \Omega} \]
\[ \leq Ch \| w_h \|_{2, h} \| \Pi_0 u - u \|_{0, \Omega}. \]

It is left to show that \( w_h \) is uniformly bounded, which can be readily derived using the coercivity of the forms \( \hat{A}_h(\cdot, \cdot) \) and \( c_h(\cdot, \cdot) \) and the uniform boundedness of \( U_{h, \text{ad}} \):

\[ \| w_h \|_{2, h} \leq C \left( \| u_h \|_{0, \Omega} + \| f \|_{0, \Omega} + \| y_d \|_{0, \Omega} \right) \leq C. \]

Therefore, substituting the bounds for \( I_1 \) and \( I_2 \) in (5.71), and using (2.30) the desired result follows.

5.3.3 \( L^2 \)-error estimates for velocity under full discretization of control

The main result in this Section is given as follows (see similar ideas, based on duality arguments also applied in [61, 67]).

**Theorem 5.3.8.** Let \((y, w)\) be the state and co-state velocities, solutions of (5.1)-(5.2), and let \((y_h, w_h)\) be their DFV approximations under piecewise linear (or piecewise constant) discretization of the control field. Then

\[ \| y - y_h \|_{0, \Omega} = O(h^2), \quad \text{and} \quad \| w - w_h \|_{0, \Omega} = O(h^2). \]

**Proof.** We start by splitting the total error and applying triangle inequality as:

\[ \| y - y_h \|_{0, \Omega} \leq \| y - y_h(u) \|_{0, \Omega} + \| y_h(u) - y_h(\Pi_h u) \|_{0, \Omega} + \| y_h(\Pi_h u) - y_h \|_{0, \Omega}, \]

(5.72)

where \( \Pi_h \) represents the \( L^2 \)-projection operator onto the discrete control space \( U_h \).

Next, let \((\tilde{w}_h, \tilde{r}_h) \in V_h \times Q_h\) be the unique solution of the auxiliary discrete dual
Brinkman problem for all \( \tilde{z}_h \in V_h \) and \( \tilde{\psi}_h \in Q_h \)

\[
\begin{align*}
\hat{A}_h(\tilde{w}_h, \tilde{z}_h) + c_h(\tilde{w}_h, \tilde{z}_h) - B_h(\tilde{z}_h, \tilde{r}_h) &= (\gamma \tilde{z}_h, y_h(u) - y_h(\Pi_h u))_{0, \Omega}, \quad (5.73) \\
B_h(\tilde{w}_h, \tilde{\psi}_h) &= 0. \quad (5.74)
\end{align*}
\]

We then choose \( \tilde{z}_h = \hat{w}_h \) and \( \tilde{\psi}_h = \hat{r}_h \) in (5.73) and (5.74), respectively, next we add the result, and we use the coercivity properties (5.15) and (5.18), to derive that

\[
\| \hat{w}_h \|_{2,h} \leq C \| y_h(u) - y_h(\Pi_h u) \|_{0, \Omega}. \quad (5.75)
\]

After testing (5.73)-(5.74) against \( \hat{z}_h = y_h(u) - y_h(\Pi_h u) \) and \( \hat{\psi}_h = p_h(u) - p_h(\Pi_h u) \), respectively, and adding the result, we obtain

\[
\begin{align*}
\hat{A}_h(\hat{w}_h, y_h(u) - y_h(\Pi_h u)) + c_h(\hat{w}_h, y_h(u) - y_h(\Pi_h u)) - B_h(y_h(u) - y_h(\Pi_h u), \hat{r}_h) \\
- B_h(\hat{w}_h, p_h(u) - p_h(\Pi_h u)) &= (\gamma y_h(u) - y_h(\Pi_h u), y_h(u) - y_h(\Pi_h u))_{0, \Omega}. \quad (5.76)
\end{align*}
\]

In addition, employing the discrete state equation for \( y_h(u) \) and \( y_h(\Pi_h u) \), we obtain

\[
\begin{align*}
\hat{A}_h(y_h(u) - y_h(\Pi_h u), \hat{w}_h) + c_h(y_h(u) - y_h(\Pi_h u), \hat{w}_h) - B_h(\hat{w}_h, p_h(u) - p_h(\Pi_h u)) \\
- B_h(y_h(u) - y_h(\Pi_h u), \hat{r}_h) &= (u - \Pi_h u, \gamma \hat{w}_h)_{0, \Omega}. \quad (5.77)
\end{align*}
\]

We then proceed to subtract (5.77) from (5.76) and to rearrange terms, to arrive at

\[
\begin{align*}
(\gamma y_h(u) - y_h(\Pi_h u), y_h(u) - y_h(\Pi_h u))_{0, \Omega} \\
= \hat{A}_h(\hat{w}_h, y_h(u) - y_h(\Pi_h u)) - \hat{A}_h(y_h(u) - y_h(\Pi_h u), \hat{w}_h) \\
+ c_h(\hat{w}_h, y_h(u) - y_h(\Pi_h u)) - c_h(y_h(u) - y_h(\Pi_h u), \hat{w}_h) \\
+ (u - \Pi_h u, \gamma \hat{w}_h)_{0, \Omega}.
\end{align*}
\]

Using the definition of the norm \( \| \cdot \|_{0,h} \) and its equivalence with the norm \( \| \cdot \|_{0,\Omega} \) we
find that

\[
\| \mathbf{y}_h(u) - y_h(\Pi_h u) \|^2_{0,\Omega} \\
\leq (u - \Pi_h u, \gamma \bar{w}_h)_{0,\Omega} + |\hat{A}_h(\bar{w}_h, y_h(u) - y_h(\Pi_h u)) - \hat{A}_h(y_h(u) - y_h(\Pi_h u), \bar{w}_h)| \\
+ |c_h(\bar{w}_h, y_h(u) - y_h(\Pi_h u)) - c_h(y_h(u) - y_h(\Pi_h u), \bar{w}_h)|. 
\]

By virtue of properties of \( \Pi_h \) applied in the above inequality, we can assert that

\[
\| \mathbf{y}_h(u) - y_h(\Pi_h u) \|^2_{0,\Omega} \\
\leq (u - \Pi_h u, \gamma \bar{w}_h - \bar{w}_h)_{0,\Omega} + (u - \Pi_h u, \bar{w}_h - \Pi_h \bar{w}_h)_{0,\Omega} \\
+ |\hat{A}_h(\bar{w}_h, y_h(u) - y_h(\Pi_h u)) - \hat{A}_h(y_h(u) - y_h(\Pi_h u), \bar{w}_h)| \\
+ |c_h(\bar{w}_h, y_h(u) - y_h(\Pi_h u)) - c_h(y_h(u) - y_h(\Pi_h u), \bar{w}_h)| \\
=: S_1 + S_2 + S_3 + S_4. 
\] (5.78)

Approximation properties of \( \gamma \) and the \( L^2 \)-projection readily yield appropriate bounds for \( S_1 \) and \( S_2 \), respectively:

\[
S_1 \leq Ch \| u - \Pi_h u \|_{0,\Omega} \| \bar{w}_h \|_{2,h}, \\
S_2 \leq Ch \| u - \Pi_h u \|_{0,\Omega} \| \bar{w}_h \|_{2,h}. 
\]

Then, a direct application of (5.75) yields

\[
S_1 + S_2 \leq Ch \| u - \Pi_h u \|_{0,\Omega} \| \mathbf{y}_h(u) - y_h(\Pi_h u) \|_{0,\Omega}. 
\]

We next use relations (5.16), (5.19) and (5.75) to obtain

\[
S_3 + S_4 \leq Ch \| \mathbf{y}_h(u) - y_h(\Pi_h u) \|_{2,h} \| \bar{w}_h \|_{2,h} \\
\leq Ch \| u - \Pi_h u \|_{0,\Omega} \| \mathbf{y}_h(u) - y_h(\Pi_h u) \|_{0,\Omega}. 
\]

Finally, substituting the estimates for \( S_1, S_2, S_3 \) and \( S_4 \) in (5.78), one straightforwardly arrives at

\[
\| \mathbf{y}_h(u) - y_h(\Pi_h u) \|_{0,\Omega} \leq Ch \| u - \Pi_h u \|_{0,\Omega}. 
\] (5.79)

For the third term in (2.43) we exploit (5.12) and proceed similarly as in the proof of
Lemma 5.3.1 to obtain

\[ \| y_h(\Pi_h u) - y_h \|_{0,\Omega} \leq \| y_h(\Pi_h u) - y_h \|_{2,h} \leq C \| \Pi_h u - u_h \|_{0,\Omega}. \]  

(5.80)

Using the discrete variational inequality along with the projection property of \( \Pi_h \) and (5.65), we have the following relation

\[
\begin{align*}
\lambda \| \Pi_h u - u_h \|_{0,\Omega}^2 &= \lambda (u - u_h, \Pi_h u - u_h)_{0,\Omega} \\
&\leq (w - w_h, u_h - \Pi_h u)_{0,\Omega} \\
&= (w - w_h(u), u_h - \Pi_h u)_{0,\Omega} + (w_h(u) - w_h(\Pi_h u), u_h - \Pi_h u)_{0,\Omega} \\
&\quad + (w_h(\Pi_h u) - w_h, u_h - \Pi_h u)_{0,\Omega} \\
&\quad + (\gamma(w_h(\Pi_h u) - w_h - \gamma(w_h(\Pi_h u) - w_h), u_h - \Pi_h u)_{0,\Omega} \\
&= J_1 + J_2 + J_3 + J_4. 
\end{align*}
\]

(5.81)

Next, we use Cauchy-Schwarz inequality and (5.51) to bound the first term:

\[ J_1 \leq \| w - w_h(u) \|_{0,\Omega} \| u_h - \Pi_h u \|_{0,\Omega} \leq Ch^2 \| u_h - \Pi_h u \|_{0,\Omega}. \]

For \( J_2 \), an application of Lemma 5.3.1 and (5.79) suffices to get

\[ J_2 \leq \| y_h(u) - y_h(\Pi_h u) \|_{0,\Omega} \| u_h - \Pi_h u \|_{0,\Omega} \leq Ch \| u - \Pi_h u \|_{0,\Omega} \| u_h - \Pi_h u \|_{0,\Omega}. \]

To bound the third term we use the approximation property of \( \gamma \) and Lemma 5.3.1

\[ J_3 \leq Ch \| w_h(y_h(\Pi_h u)) - w_h \|_{2,h} \| u_h - \Pi_h u \|_{0,\Omega} \leq Ch \| u_h - \Pi_h u \|_{0,\Omega}^2. \]

Proceeding analogously to the proof of Lemma 5.3.3, using (5.16) and (5.19), the last
term of the expression (5.81) can be estimated as

\[ J_4 \leq \hat{A}_h(y_h - y_h(\Pi_h u), w_h(\Pi_h u) - w_h) - \hat{A}_h(w_h(\Pi_h u) - w_h, y_h - y_h(\Pi_h u)) \\
+ c_h(\hat{y}_h - y_h(\Pi_h u), w_h(\Pi_h u) - w_h) - c_h(w_h(\Pi_h u) - w_h, y_h - y_h(\Pi_h u)) \]

\[ \leq C h \| \hat{y}_h - y_h(\Pi_h u) \|_{2,h} \| w_h(\Pi_h u) - w_h \|_{2,h} \leq C h \| u_h - \Pi_h u \|_{0,\Omega}^2. \]

Plugging the bounds for \( J_1, J_2, J_3 \) and \( J_4 \) in (5.81), putting (2.49) and (5.80) into (2.43), and using interpolation estimates, along with Lemma 5.3.2; we obtain an optimal estimate for the state velocity error

\[ \| y - y_h \|_{0,\Omega} = \mathcal{O}(h^2). \]  

Finally, splitting the co-state velocity error as \( w - w_h = w - w_h(y) + w_h(y) - w_h \), using triangle inequality and Lemmas 5.3.1, 5.3.2, and relation (5.82), we get the second desired estimate

\[ \| w - w_h \|_{0,\Omega} \leq \| w - w_h(y) \|_{0,\Omega} + \| w_h(y) - w_h \|_{0,\Omega} \]

\[ \leq \| w - w_h(y) \|_{0,\Omega} + \| y - y_h \|_{0,\Omega} = \mathcal{O}(h^2). \]

\[ \square \]

5.3.4 Error bounds in the energy norm

**Theorem 5.3.9.** Let \((y, w, p, r)\) be the state and co-state velocities, and pressures, solutions of (5.1)-(5.2), and let \((y_h, w_h, p_h, r_h)\) be their DFV approximations. Then

\[ \| y - y_h \|_{2,h} + \| p - p_h \|_{0,\Omega} = \mathcal{O}(h) \quad \text{and} \quad \| w - w_h \|_{2,h} + \| r - r_h \|_{0,\Omega} = \mathcal{O}(h). \]

**Proof.** Using (5.32) and (5.33), applying triangle inequality and Lemma 5.3.1, we obtain

\[ \| y - y_h \|_{2,h} + \| p - p_h \|_{0,\Omega} \leq \| y - y_h(u) \|_{2,h} + \| y_h(u) - y_h \|_{2,h} \\
+ \| p - p_h(u) \|_{0,\Omega} + \| p_h(u) - p_h \|_{0,\Omega} \]

\[ \leq \| y - y_h(u) \|_{2,h} + \| p - p_h(u) \|_{0,\Omega} + C \| u - u_h \|_{0,\Omega}, \]

149
Therefore, the proof is complete after combining the estimates of Lemma 5.3.2 and the estimates of \( \| u - u_h \|_{0,\Omega} \) and \( \| y - y_h \|_{0,\Omega} \).

\section{5.4 Numerical experiments}

In this Section, we present a set of numerical examples to illustrate the theoretical results previously described. For the sake of completeness, before jumping into the tests we provide some details about the implementation and algorithms for the efficient computation of the DFV method applied to the optimal control of Brinkman equations.

\subsection*{Implementation aspects}

We will use the well-known active set strategy (proposed in [8]) involving primal and dual variables (see also [33, 68] for its application in Stokes flow). The principle is to approximate the constrained optimal control problem by a sequence of unconstrained problems, using active sets as summarized in Algorithm 2 below. By \( u^n_h, w^n_h \) we will denote the optimal control and adjoint velocity, solutions to the discrete problem (5.23)-(5.27) at the current iteration. Also, the control constraints are \( u_a = (u_{a_1}, ..., u_{a_d})^T \) and \( u_b = (u_{b_1}, ..., u_{b_d})^T \).

Let \( \{ \phi_i^\ast \}_{i=1}^N, \{ \xi_i \}_{i=1}^L, \{ \bar{\psi}_i \}_{i=1}^M \) be the basis functions for \( V_h, Q_h \), and \( U_h \), respectively, whereas the space \( V_h^\ast \) is spanned by \( \{ \phi_i^\ast \}_{i=1}^N \), with (explicit here for \( d = 3 \))

\[ \phi_i^\ast (x) = \{ \chi_{T_i^\ast} (1, 0, 0), \chi_{T_i^\ast} (0, 1, 0), \chi_{T_i^\ast} (0, 0, 1) \}, \]

where \( \chi_{T_i^\ast} \) is the characteristic function assuming the value 1 on \( T_i^\ast \in T_h^\ast \) and zero elsewhere.
Table 5.1: Example 1: convergence history and optimization iteration count for the approximations of the optimal control of the Brinkman problem.

We next proceed to define the discrete active and inactive sets, based on the degrees of freedom of $U_h$, as follows

$$A_{u_a}^{n+1} = \{ k \in \{ 1, \ldots, M \} : -w_{j,h}^{n,k} / \lambda < u_{a,j} \}, \quad \text{for any } j \in \{ 1, \ldots, d \},$$

$$A_{u_b}^{n+1} = \{ k \in \{ 1, \ldots, M \} : -w_{j,h}^{n,k} / \lambda > u_{b,j} \}, \quad \text{for any } j \in \{ 1, \ldots, d \},$$

$$I_{n+1} = \{ 1, \ldots, M \} \setminus (A_{u_a}^{n+1} \cup A_{u_b}^{n+1}),$$

where, in general, $s_{j,h}^{n,k}$ stands for the discrete value associated to the degree of freedom at position $k$, related to the spatial component $j$ of the vector field $s$, at the step $n$ of Algorithm 2. By the definition of the optimal control problem, we have that

$$u_h^n = \begin{cases} u_a & \text{in } A_{u_a}^{n+1}, \\ -\lambda^{-1}w_h^n & \text{in } I_{n+1}, \\ u_b & \text{in } A_{u_b}^{n+1}, \end{cases}$$
and if we further introduce the following characteristic sets

\[
\chi_{A_{a}^{n+1}(k,k)} = \begin{cases} 
1 & \text{if } k \in A_{a}^{n+1}, \\
0 & \text{else}, 
\end{cases} \\
\chi_{A_{b}^{n+1}(k,k)} = \begin{cases} 
1 & \text{if } k \in A_{b}^{n+1}, \\
0 & \text{else}, 
\end{cases}
\]

then we get

\[
\lambda^{-1} w_{h}^{n} (1 - \chi_{A_{a}^{n+1}} - \chi_{A_{b}^{n+1}}) + u_{h}^{n} = u_{a} \chi_{A_{a}^{n+1}} + u_{b} \chi_{A_{b}^{n+1}}.
\] (5.85)

Finally, we define the matrix blocks

\[
A := [A_{h}(\phi_{i}, \phi_{j})]_{1 \leq i,j \leq N}, \quad C := [c_{h}(\phi_{i}, \phi_{j})]_{1 \leq i,j \leq N}, \\
B := [B_{h}(\xi_{i}, \phi_{j})]_{1 \leq i,j \leq N}, \\
M := [(\phi_{i}, \phi_{j})_{0,0}]_{1 \leq i,j \leq N}, \\
G := [(\phi_{i}, \psi_{j})_{0,0}]_{1 \leq i,j \leq N}, \\
D := [(\psi_{i}, \psi_{j})_{0,0}]_{1 \leq i,j \leq M}, \\
\hat{E} := \lambda^{-1}(1 - \chi_{A_{a}^{n+1}} - \chi_{A_{b}^{n+1}})
\]

along with the vectors

\[
F := [(f, \phi_{i})_{0,0}]_{1 \leq i \leq N}, \quad Y_d := [(y_{d}, \phi_{i})_{0,0}]_{1 \leq i \leq N}, \\
\hat{S} := [(a \chi_{A_{a}^{n+1}} + b \chi_{A_{b}^{n+1}}, \psi_{i})_{0,0}]_{1 \leq i \leq M},
\]

so that after testing (5.85) against \{\psi_{i}^{*}\}_{i=1}^{M} we end up with the following matrix form of the discrete optimal control problem (5.23)-(5.27):

\[
\begin{pmatrix} 
A + C & -B^{T} & 0 & 0 & -G \\
B & 0 & 0 & 0 & 0 \\
-M & 0 & A + C & B^{T} & 0 \\
0 & 0 & -B & 0 & 0 \\
0 & 0 & \hat{E}G^{T} & 0 & D
\end{pmatrix} \begin{pmatrix} 
Y \\
P \\
W \\
R \\
U
\end{pmatrix} = \begin{pmatrix} 
F \\
0 \\
-Y_d \\
0 \\
\hat{S}
\end{pmatrix},
\] (5.86)

where \(Y, P, W, R, U\) are the coefficients in the expansion of \(y_{h}^{n+1}, p_{h}^{n+1}, w_{h}^{n+1}, r_{h}^{n+1}\) and \(u_{h}^{n+1}\), respectively, and the hats indicate quantities associated with the previous iteration.

**Example 1.** We start by assessing the experimental convergence of the proposed scheme applied to the optimal control problem (5.1)-(5.2) defined on the unit square \(\Omega = (0,1)^{2}\). Viscosity, permeability and the weight for the control cost assume the following constant values \(\mu = 1, K = I, \lambda = 1\), respectively. The set of admissible controls is
characterized by the constants $u_{a_1} = u_{a_2} = -\frac{1}{10}$, $u_{b_1} = u_{b_2} = \frac{1}{4}$, and manufactured solutions are explicitly given by

$$ y = w = \begin{pmatrix} \sin^2(\pi x_1) \sin(\pi x_2) \cos(\pi x_2) \\ -\sin^2(\pi x_2) \sin(\pi x_1) \cos(\pi x_1) \end{pmatrix}, \quad p = -r = \sin(2\pi x_1) \sin(2\pi x_2), $$

$$ u = P_{[u_a, u_b]} \left( \frac{-1}{\lambda} w \right), $$

(see e.g. [78]) which satisfy the homogeneous Dirichlet boundary conditions under which the analysis was performed. Source term and desired velocity field of the problem
are constructed according to these exact solutions, that is, respectively

\[ f = K^{-1}y - \text{div}(\mu \varepsilon(y) - pI) - u, \quad y_d = y - K^{-1}w + \text{div}(\mu \varepsilon(w) + rI). \]

A family of nested primal and dual triangulations of \( \Omega \) is generated, on which we compute errors in the \( L^2 \)-and mesh-dependent norm \( \| \cdot \|_{1,h} \) for the state and co-state velocity, in the \( L^2 \)-norm for pressures, and in the \( L^2 \)-norm for the control approximation. Table 5.1 displays the error history for this first test, where we observe optimal convergence rates for velocity and pressure (only those of the state equation are shown) in their natural norms, along with an \( O(h) \) convergence for the control when approximated by piecewise constant elements, which improves to roughly a \( O(h^{3/2}) \) rate under piecewise linear approximations. We can also confirm that a maximum of three iterations are needed to reach the stopping criterion that the active sets are equal to those in the previous optimization step. This indicates a mesh independence of the method in the sense that the number of iterations needed to achieve the stopping criterion is independent of the resolution. In addition we portray in Figure 5.2 the obtained approximate solutions at the finest resolution level, where we highlight the active sets with a
contour plot on top of the control and state velocities. In all examples herein we employ a BiCGSTAB method with AMG preconditioning to solve the linear systems involved at each step of Algorithm 2. Moreover, the zero-mean pressure condition is applied for both pressure and adjoint pressure using a real Lagrange multiplier approach.

At this point we also present a basic comparison with other classical methods in terms of accuracy. For instance, we have performed the same test as above but employing discontinuous coefficients. Both fluid viscosity and medium permeability have now a jump of five orders of magnitude at \( x_1 = 0.5 \). The tested methods are: a conforming stable \( P_2 - P_0 \) and MINI-element pairs for velocity and pressure approximation, a classical interior penalty DG method using the same stabilization parameters as in (5.10)-(5.11), and the proposed DFV formulation. In all cases we consider a piecewise linear approximation of the control variable.

The results are collected in Figure 5.3, where convergence histories (errors for velocity and pressure vs. the number of degrees of freedom \( \text{DoF} = 2(N + L) + M \)) associated to the studied discretizations are shown. For all fields, the DFV approximation exhibits a slightly better accuracy than its pure-DG counterpart. This may be explained by the smaller elements used in the dual mesh (but being associated to the same number of DoF). On the other hand, for coarse meshes the conforming approximation \( P_2 - P_0 \) outperforms all other methods, but for finer meshes the discontinuous coefficients of the problem imply a badly conditioned system matrix requiring more (inner) iterations of the linear solver and eventually the conforming methods lose their optimal convergence. For a fixed number of DoF, the proposed DFV scheme produces smaller errors for the pressure approximation than the other methods. We stress that some recent theoretical comparison results are available for forward Stokes problems (see e.g. [15]), but only in the case of smooth solutions and constant coefficients.

**Example 2.** Our second test focuses on the optimal control problem applied to the well-known lid driven cavity problem. The objective function still corresponds to (5.1), but no analytic exact solution is available. Again the domain consists of the unit square, and the data of the problem are given by a traction boundary condition on the top of the
Figure 5.4: Example 2: DFV approximation of state velocity components and magnitude along with state pressure (top panels), adjoint velocity and pressure (center row), components and magnitude of the control variable under piecewise constant approximation, and state velocity streamlines (bottom row). Contours of the active sets associated to \( u_{a1} = u_{a2} \) (in white curves) and \( u_{b1} = u_{b2} \) (red curves) are displayed on each plot.

lid, the applied body force, and an observed velocity field defined by:

\[
y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ on the top and zero elsewhere}, \quad f = y_d = 0 \text{ in } \Omega.
\]

The adjoint problem is subject to homogeneous Dirichlet data. The viscosity is set to \( \mu = 0.1 \), the control weight is now \( \lambda = 0.2 \), the admissible control space is characterized by \( u_{a1} = u_{a2} = -0.15 \), \( u_{b1} = u_{b2} = 0.15 \), and the permeability exhibits a discontinuity on the line \( x_2 = 0.4 \): \( K = \kappa I \), with \( \kappa = \begin{cases} 10000 & \text{if } x_2 \geq 0.4; \\ 10 & \text{elsewhere} \end{cases} \) in \( \Omega \). The domain is discretized into 20000 primal triangular elements, and Figure 5.4
Table 5.2: Example 2: iteration count vs. the regularization parameter for the DFV approximations of the optimal control of the Brinkman problem.

<table>
<thead>
<tr>
<th>λ</th>
<th>1.0</th>
<th>0.2</th>
<th>0.04</th>
<th>0.008</th>
<th>0.0016</th>
<th>0.00032</th>
<th>0.000064</th>
</tr>
</thead>
<tbody>
<tr>
<td>it</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>12</td>
<td>19</td>
<td>34</td>
</tr>
</tbody>
</table>

portrays all fields obtained with our DFV scheme, where the stabilization parameter is \( \alpha_d = 10 \). From Figure 5.4 we observe that the controlled velocity approaches to the desired velocity, that is, it goes to zero and the movement of the fluid concentrates in the upper section of the cavity. In addition, we study the influence of the Tikhonov regularization in the iteration count of the active set algorithm applied to a coarse solve of this test. As in [33], we immediately observe that a larger number of iterations are required for smaller values of \( \lambda \) (see Table 5.2).

**Example 3.** Next we turn to the numerical solution of a three-dimensional optimal control problem. The domain now consists of a cylinder with height 4 and radius 1, aligned with the \( x_2 \) axis. The permeability field is now anisotropic \( \mathbf{K} = \text{diag}(0.1, 10^{-6}\chi_B + 0.1\chi_{B^c}, 0.1) \), where \( B \) is a ball of radius 1/4 located at the center of the domain. As boundary condition for the state velocity, a Poiseuille inflow profile is imposed at the bottom of the cylinder (i.e. on \( x_2 = 0 \)): \( \mathbf{y} = (0, 10(1 - x_2^2 - (x_3 - 1/2)^2), 0)^T \), a zero-pressure is considered on \( x_2 = 4 \), whereas homogeneous Dirichlet data are enforced on the remainder of \( \partial\Omega \). The viscosity is constant \( \mu = 0.01 \), the Tikhonov regularization is \( \lambda = 1/2 \), the desired velocity is zero \( \mathbf{y}_d = 0 \), the bounds for the control are \( u_{a_j} = u_a = -0.1 \) and \( u_{b_j} = u_b = 0.2 \), and a smooth body force is set as in [2]:

\[
\mathbf{f} = \mathbf{K}^{-1}(\exp(-x_2x_3) + x_1 \exp(-x_2^2)\cos(\pi x_1)\cos(\pi x_3) - x_2 \exp(-x_2^2) - x_1x_2x_3 - x_3 \exp(-x_3^2))^T.
\]

The primal meshes has 76766 internal tetrahedral elements and 13663 vertices. For this test we observe that five iterations are required to reach the stopping criterion (5.83). Snapshots of the resulting approximate fields are collected in Figure 5.5.
Figure 5.5: Example 3: streamlines of the DFV approximate state and co-state velocities, along with control field (top row), iso-surfaces of approximate state and co-state pressures (middle), and iso-surfaces of the control components associated to $a = u_{a1} = u_{a2} = u_{a3}$ (in red) and $b = u_{b1} = u_{b2} = u_{b3}$ (blue) (bottom panels).
Algorithm 2 Active set implementation and overall solution strategy.

1: choose and store arbitrary initial guess

\[ u_h^0 = u_b, \quad w_h^0 = -\lambda u_h^0 \]

2: initialize active and inactive sets

\[ A_{0}^{u_a} = A_{0}^{u_b} = \emptyset, \quad I_0 = \{1, \ldots, M\} \]

3: for \( n = 0, 1, \ldots, \) do

4: For known \( u_h^n \) and \( w_h^n \), construct the new finite active sets \( A_{n+1}^{u_a}, A_{n+1}^{u_b} \) as well as the finite inactive set \( I_{n+1} \) from (5.84)

5: if

\[ n \geq 1, \text{ and } A_{n+1}^{u_a} = A_n^{u_a}, \text{ and } A_{n+1}^{u_b} = A_n^{u_b}, \] (5.83)

then

6: stop

7: else

8: find \((y_h^{n+1}, p_h^{n+1}, w_h^{n+1}, r_h^{n+1}, u_h^{n+1})\), solution to the coupled system \((5.86)\)

9: end if

10: reinitialize active and inactive sets and control variable

\[ A_{n+1}^{u_a} \leftarrow A_{n}^{u_a}, \quad A_{n+1}^{u_b} \leftarrow A_{n}^{u_b}, \quad I_{n+1} \leftarrow I_{n+1}, \quad u_h^n \leftarrow u_h^{n+1} \]

11: end for