Chapter 6

STABILITY OF FUZZY RECURRENT NEURAL NETWORKS WITH MARKOVIAN JUMPING PARAMETERS

6.1 Introduction

A recurrent neural network, which naturally involves dynamic elements in the form of feedback connections used as internal memories. Unlike the feed forward neural network whose output is a function of its current inputs only and is limited to static mapping, recurrent neural network perform dynamic mapping. Recurrent networks are needed for problems where there exist at least one system state variable which cannot be observed. Most of the existing recurrent neural networks are obtained by adding trainable temporal elements to feed forward neural networks (like multilayer perceptron networks and radial basis function networks [6]) to make the output history to be sensitive. Like feed forward
neural networks, these network functions as block boxes and the meaning of each weight in these nodes are not known. They play an important role in applications such as classification of patterns, associate memories and optimization (see [6] and the references therein). Thus, research on properties of stability problem and relaxed stability problem for recurrent neural networks, has become a very active area in the past few years [14, 32]. When the neural network incorporates abrupt changes in its structure, the Markovian jump linear system is very appropriate to describe its dynamics. Recently, the problem of stochastic robust stability for uncertain delayed neural networks with Markovian jumping parameters is investigated via LMI technique in [51].

Inspired by the aforementioned works, we generalize the ordinary T–S fuzzy models to express a class stochastic Markovian jumping recurrent neural networks with discrete and distributed time-varying delays, uncertain Markovian jumping recurrent neural networks with time-varying delays. By using the Lyapunov functional technique, global stability conditions for stochastic fuzzy Markovian jumping recurrent neural networks (TSSFMJRNNS) with discrete and distributed time-varying delays, uncertain fuzzy Markovian jumping recurrent neural networks (TSUFMJRNNS) with time-varying delays are given in terms of LMIs, which can be easily calculated by MATLAB LMI toolbox [22]. Numerical examples are provided to demonstrate the effectiveness and applicability of the proposed stability results.
6.2 Stability of stochastic fuzzy Markovian jumping recurrent neural networks with discrete and distributed time-varying delays

6.2.1 System description and Preliminaries

Consider the following Markovian jumping recurrent neural networks with discrete and distributed time-varying delays described by,

\[
\dot{u}_i(t) = -a_i(\varrho_t)u_i(t) + \sum_{j=1}^{n} w_{ij}(\varrho_t)F_j(u_j(t)) + \sum_{j=1}^{n} h_{ij}(\varrho_t)f_j(u_j(t - \tau_j(t))) \\
+ \sum_{j=1}^{n} c_{ij}(\varrho_t) \int_{t-\rho_j(t)}^{t} F_j(u_j(s))ds + I_i \quad i = 1, 2, \ldots, n, \tag{6.1}
\]

in which \( u_i(t) \) is the activation of the \( i \)th neuron. Positive constant \( a_i(\varrho_t) \) denotes the rates with which the cell \( i \) reset their potential to the resting state when isolated from the other cells and inputs. \( w_{ij}(\varrho_t), h_{ij}(\varrho_t) \) and \( c_{ij}(\varrho_t) \) are the connection weights at the time \( t \), \( I_i \) denote the external input and \( F_j(\cdot) \) is the neuron activation function of \( j \)th neuron. \( \tau_j(t) \) is the bounded time varying delay in the state and satisfy \( 0 \leq \tau_j(t) \leq \bar{\tau}, \; 0 \leq \bar{\tau}_j(t) \leq d_1 < \infty, \; j = 1, 2, \ldots, n. \) \( \rho_j(t) \) is the distributed time varying delay in the state and satisfy \( 0 \leq \rho_j(t) \leq \bar{\rho}, \; 0 \leq \bar{\rho}_j(t) \leq d_2 < \infty, \; j = 1, 2, \ldots, n. \)

Assume that \( u^* = (u^*_1, u^*_2, \ldots, u^*_n)^T \) is the equilibrium point of the system (6.1), then we shift the equilibrium points to the origin by the transformation, \( x(\cdot) = u(\cdot) - u^*, \; f_j(x_j(t)) = F_j(x_j(t) + u^*) - F_j(u^*) \). Then transformed system is given by,

\[
\dot{x}_i(t) = -a_i(\varrho_t)x_i(t) + \sum_{j=1}^{n} w_{ij}(\varrho_t)f_j(x_j(t)) + \sum_{j=1}^{n} h_{ij}(\varrho_t)f_j(x_j(t - \tau_j(t))) \\
+ \sum_{j=1}^{n} c_{ij}(\varrho_t) \int_{t-\rho_j(t)}^{t} f_j(x_j(s))ds, \; i = 1, 2, \ldots, n. \tag{6.2}
\]

Conveniently, we can write (6.2) in the form

\[
\dot{x}(t) = -A(\varrho_t)x(t) + W(\varrho_t)f(x(t)) + H(\varrho_t)f(x(t - \tau(t))) + C(\varrho_t) \int_{t-\rho(t)}^{t} f(x(s))ds,
\]

\[
\tag{6.3}
\]
where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $A(\varrho_i) = \text{diag}\{a_1(\varrho_i), a_2(\varrho_i), \ldots, a_n(\varrho_i)\}$, $W(\varrho_i) = [(w_{ij}(\varrho_i))_{n \times n}]^T$, $H(\varrho_i) = [(h_{ij}(\varrho_i))_{n \times n}]^T$, $C(\varrho_i) = [(c_{ij}(\varrho_i))_{n \times n}]^T$, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T$, $\tau(t) = (\tau_1(t), \tau_2(t), \ldots, \tau_n(t))^T$, and $\rho(t) = (\rho_1(t), \rho_2(t), \ldots, \rho_n(t))^T$.

For convenience, each possible value of $\varrho_i$ is denoted by $i$, $i \in S$ in the sequel. Then we have $A_i = A(\varrho_i)$, $W_i = W(\varrho_i)$, $H_i = H(\varrho_i)$, $C_i = C(\varrho_i)$, where $A_i$, $W_i$, $C_i$ and $H_i$, for any $i \in S$, are known constant matrices of appropriate dimensions.

The system (6.3) can be written as

$$\dot{x}(t) = -A_i x(t) + W_i f(x(t)) + H_i f(x(t - \tau(t))) + C_i \int_{t-\rho(t)}^{t} f(x(s))ds,$$

Based on discussions in the introduction section, we generalize the ordinary T-S fuzzy models to express a complex system whose consequent parts are a set of stochastic Markovian jumping recurrent neural networks with discrete and distributed time varying delays.

The T-S fuzzy model has $r$ rules, where the $k^{th}$ rule of this T-S fuzzy model is of the following form

**Plant Rule $k$:**

**IF** \{\theta_1(t) is $\eta_1^k$\} and \ldots and \{\theta_n(t) is $\eta_n^k$\} **THEN**

$$\dot{x}(t) = \left[-A_{ik}x(t) + W_{ik} f(x(t)) + H_{ik} f(x(t - \tau(t))) + C_{ik} \int_{t-\rho(t)}^{t} f(x(s))ds \right]dt$$

$$+ \sigma_{ik}(t, x(t), x(t - \tau(t)), \int_{t-\rho(t)}^{t} f(x(s))ds) dw(t),$$

Let $\delta_k(\varrho(t))$ be the normalized membership function of the inferred fuzzy set $\omega_k(\varrho(t))$.

The state equation is defined as follows:

$$\dot{x}(t) = \sum_{k=1}^{r} \delta_k(\varrho(t)) \left\{ \left[-A_{ik}x(t) + W_{ik} f(x(t)) + H_{ik} f(x(t - \tau(t))) \right]dt$$

$$+ C_{ik} \int_{t-\rho(t)}^{t} f(x(s))ds \right \} dt + \sigma_{ik}(t, x(t), x(t - \tau(t)), \int_{t-\rho(t)}^{t} f(x(s))ds) dw(t) \right \},$$

(6.4)

Throughout this section we assume that,
(A1) The neuron activation functions \( f_j(\cdot) \) in (6.2) are bounded and satisfies the following Lipschitz condition,

\[
|f_j(x) - f_j(y)| \leq l_j|x - y|,
\]

for all \( x, y \in R, j = 1, 2, \ldots, n \) where \( l_j \) are positive numbers. Then we have

\[
f^T(x(t))f(x(t)) \leq x^T(t)L^TLx(t),
\]

where \( L = \text{diag}\{l_1, l_2, \ldots, l_n\} \).

(A2) There exist matrices \( X_{1i} \geq 0, X_{2i} \geq 0, X_{3i} \geq 0 \) such that

\[
\begin{align*}
\text{trace}\left[ &\sigma^T_k(t, x(t - \tau(t)))\int_{t-\rho(t)}^{t} f(x(s))ds \right] \\
&+ \int_{t-\rho(t)}^{t} f(x(s))ds \right] X_{1i}\left( \int_{t-\rho(t)}^{t} f(x(s))ds \right) \\
&\leq x^T(t)X_{1i}x(t) + x^T(t - \tau(t))X_{2i}x(t - \tau(t)) + \left( \int_{t-\rho(t)}^{t} f(x(s))ds \right)^T X_{3i}\left( \int_{t-\rho(t)}^{t} f(x(s))ds \right).
\end{align*}
\]

Defining the following state variables for the TSSFMJRN as,

\[
y(t) = \sum_{k=1}^{r} \delta_k(\theta(t)) \left[ - A_{ik}x(t) + W_{ik}f(x(t)) + H_{ik}f(x(t - \tau(t))) + C_{ik} \int_{t-\rho(t)}^{t} f(x(s))ds \right],
\]

\[
g(t) = \sum_{k=1}^{r} \delta_k(\theta(t)) \left[ \sigma_{ik}(t, x(t), x(t - \tau(t))), \int_{t-\rho(t)}^{t} f(x(s))ds \right].
\]

Then the TSSFMJRN can be represented as

\[
dx(t) = y(t)dt + g(t)d\omega(t). \tag{6.5}
\]

The following Definition is essential for the proof of main results.

**Definition 6.2.1** [60] An \( \hat{R}^n \) -valued stochastic process \( \{x(t)\}_{t_0 \leq t \leq T} \) is called a solution of equation (6.5) if it has the following properties:

1. \( \{x(t)\}_{t_0 \leq t \leq T} \) is continuous and \( \mathcal{F}_t \) adapted;

2. \( \{y(t)\}_{t_0 \leq t \leq T} \in \mathcal{L}^1([t_0, T]; R^n) \) while \( \{g(t)\}_{t_0 \leq t \leq T} \in \mathcal{L}^3([t_0, T]; R^{n \times m}) \);

3. for any \( t \in [t_0, T] \) equation

\[
x(t) = x(t_0) + \int_{t_0}^{t} y(s)ds + \int_{t_0}^{t} g(s)d\omega(s) \tag{6.6}
\]

holds with probability 1.
6.2.2 Global stability results

In this section, some sufficient conditions of global stability for system (6.4) is obtained.

**Theorem 6.2.2** Suppose that the assumption (A1) – (A2) holds, then for given $\bar{\tau} > 0$, $\bar{\rho} > 0$, $d_1 > 0$ and $d_2 > 0$ the system (6.4) is globally asymptotically stable if there exist symmetric positive definite matrices $P_i > 0, Q > 0, R > 0, S > 0, T > 0, U > 0, W > 0$ symmetric positive definite matrices $N_i, M_i, O_l(l = 1, 2, \ldots, 7)$ and scalars $a > 0, b > 0$ such that feasible solution exist for LMIs

$$
\Omega_k = \begin{bmatrix}
\Omega_{k11} & \Omega_{k12} & \Omega_{k13} & \Omega_{k14} & \Omega_{k15} & \Omega_{k16} & \Omega_{k17} & \sqrt{\bar{\tau}}N_1 & \sqrt{\bar{\rho}}M_1 \\
* & \Omega_{k22} & \Omega_{k23} & \Omega_{k24} & \Omega_{k25} & \Omega_{k26} & \Omega_{k27} & \sqrt{\bar{\tau}}N_2 & \sqrt{\bar{\rho}}M_2 \\
* & * & \Omega_{k33} & \Omega_{k34} & \Omega_{k35} & \Omega_{k36} & \Omega_{k37} & \sqrt{\bar{\tau}}N_3 & \sqrt{\bar{\rho}}M_3 \\
* & * & * & \Omega_{k44} & \Omega_{k45} & \Omega_{k46} & \Omega_{k47} & \sqrt{\bar{\tau}}N_4 & \sqrt{\bar{\rho}}M_4 \\
* & * & * & * & \Omega_{k55} & \Omega_{k56} & \Omega_{k57} & \sqrt{\bar{\tau}}N_5 & \sqrt{\bar{\rho}}M_5 \\
* & * & * & * & * & \Omega_{k66} & \Omega_{k67} & \sqrt{\bar{\tau}}N_6 & \sqrt{\bar{\rho}}M_6 \\
* & * & * & * & * & * & \Omega_{k77} & \sqrt{\bar{\tau}}N_7 & \sqrt{\bar{\rho}}M_7 \\
* & * & * & * & * & * & * & -U & 0 \\
* & * & * & * & * & * & * & * & -W
\end{bmatrix}, \tag{6.7}
$$

where

$$
\Omega_{k11} = -A_{ik}^T P_i - P_i^T A_{ik} + Q + R + X_{11} + \sum_{j=1}^{8} \pi_{ij} P_j + N_1^T + N_1 + M_1^T + M_1 - O_1 A_{ik}
$$

$$
- A_{ik}^T O_1^T + b L T L, \quad \Omega_{k12} = N_2^T - N_1 + M_2^T - A_{ik}^T O_2^T, \quad \Omega_{k13} = N_3^T - M_1 + M_3^T
$$

$$
- A_{ik}^T O_3^T, \quad \Omega_{k14} = P_i W_{ik} + N_4^T + M_4^T + O_1 W_{ik} - A_{ik}^T O_4^T, \quad \Omega_{k15} = P_i H_{ik} + N_5^T + M_5^T - A_{ik}^T O_5^T + O_1 H_{ik}, \quad \Omega_{k16} = P_i C_{ik} + N_6^T + M_6^T + O_1 C_{ik} - A_{ik}^T O_6^T,
$$

$$
\Omega_{k17} = N_7^T + M_7^T - O_1 - A_{ik}^T O_7^T, \quad \Omega_{k22} = -(1 - d_1) R + X_{22} + a L T L - N_2^T - N_2,
$$

$$
\Omega_{k23} = -N_3^T - M_2, \quad \Omega_{k24} = -N_4^T + O_2 W_{ik}, \quad \Omega_{k25} = -N_5^T + O_2 H_{ik}, \quad \Omega_{k26} = -N_6^T + O_2 C_{ik}, \quad \Omega_{k27} = -N_7^T + O_2 - O_2,
$$

$$
- O_3 C_{ik}, \quad \Omega_{k37} = -N_7^T - O_3, \quad \Omega_{k34} = -M_4^T + O_3 W_{ik}, \quad \Omega_{k35} = -M_5^T + O_3 H_{ik}, \quad \Omega_{k36} = -M_6^T + O_3 C_{ik}, \quad \Omega_{k37} = -M_7^T - O_3,
$$

$$
\Omega_{k44} = \beta T + S - b I + O_4 W_{ik} + W_{ik}^T O_4^T,
$$

155
\[ \Omega_{k45} = O_4 H_{ik} + W_{ik}^T O_5^T, \quad \Omega_{k46} = O_4 C_{ik} + W_{ik}^T O_6^T, \quad \Omega_{k47} = W_{ik}^T O_7^T - O_4, \]
\[ \Omega_{k55} = -(1 - a_1)S - aI + O_5 H_{ik} + H_{ik}^T O_6^T, \quad \Omega_{k56} = H_{ik}^T O_6^T + O_5 C_{ik}, \]
\[ \Omega_{k57} = H_{ik}^T O_7^T - O_5, \quad \Omega_{k66} = X_{2t} - \tau^{-1}T + O_6 C_{ik} + C_{ik}^T O_6^T, \quad \Omega_{k67} = C_{ik}^T O_7^T - O_6, \]
\[ \Omega_{k77} = \tau U + \bar{p} W - O_7^T - O_7. \]

**Proof:** We consider the following Lyapunov functional to derive the stability result,

\[ V(t, x(t)) = \sum_{i=1}^{6} V_i(t, x(t)), \]

where

\[ V_1(t, x(t)) = x^T(t) P_1 x(t), \]
\[ V_2(t, x(t)) = \int_{t-\rho(t)}^{t} x^T(s) Q x(s) ds, \]
\[ V_3(t, x(t)) = \int_{t-\tau(t)}^{t} [x^T(s) R x(s) ds + f^T(x(s)) S f(x(s))] ds, \]
\[ V_4(t, x(t)) = \int_{-\tau}^{0} \int_{t+s}^{t} f^T(x(\theta)) T f(x(\theta)) d\theta ds, \]
\[ V_5(t, x(t)) = \int_{-\tau}^{0} \int_{t+s}^{t} y^T(\theta) U y(\theta) d\theta ds, \]
\[ V_6(t, x(t)) = \int_{-\tau}^{0} \int_{t+s}^{t} y^T(\theta) W y(\theta) d\theta ds. \]

According to the Itô's rule [56], the infinitesimal operator \( \mathcal{L}(\cdot) \) of the process \( \{x(t), t \geq 0, \omega_t\} \) for system (6.4) at the point \( \{x(t), t, i\} \) is given by

\[ \mathcal{L}V = \frac{\partial V}{\partial t} + \dot{x}(t) \frac{\partial V}{\partial x} |_{\omega_t = i} + \sum_{j=1}^{6} \pi_{ij} V(t, x(t)). \]

Then, we obtain

\[ \mathcal{L}V(t, x(t)) = \sum_{i=1}^{6} \mathcal{L}V_i(t, x(t)), \quad (6.8) \]

where

\[ \mathcal{L}V_1(t, x(t)) = \sum_{k=1}^{\tau} \delta_k(\theta(t)) \{2 x^T(t) P_1 [-A_{ik}x(t) + W_{ik} f(x(t)) + H_{ik} f(x(t) - \tau(t))] \}

156
\[ \mathcal{L}V_2(t, x(t)) = x^T(t)Qx(t) - (1 - d_2)x^T(t - \rho(t))Qx(t - \rho(t)), \]

\[ \mathcal{L}V_3(t, x(t)) = x^T(t)Rx(t) - (1 - d_1)x^T(t - \tau(t))Rx(t - \tau(t)) + f^T(x(t))Sf(x(t)) - (1 - d_1)f^T(x(t - \tau(t)))Sf(x(t - \tau(t))), \]

\[ \mathcal{L}V_4(t, x(t)) = \tilde{\rho}f^T(x(t))Tf(x(t)) - \int_{t-\tilde{\beta}}^t f^T(x(s))Tf(x(s))ds, \]

\[ \mathcal{L}V_5(t, x(t)) = \bar{\tau}y^T(t)Uy(t) - \int_{t-\bar{\tau}}^t y^T(s)Uy(s)ds, \]

\[ \mathcal{L}V_6(t, x(t)) = \rho y^T(t)Wy(t) - \int_{t-\rho}^t y^T(s)Wy(s)ds. \]

Thus by using Lemma 1.5.9 we obtain,

\[ \mathcal{L}V \leq \sum_{k=1}^r \delta_k(\theta(t)) \left\{ x^T(t) \left[ - A_{ik}^T P_i - P_i^T A_{ik} + Q + R + X_{2k} + \sum_{j=1}^s \pi_{ij} P_j \right] x(t) \right. \]

\[ + 2x^T(t)P_iW_{ik}f(x(t)) + 2x^T(t)P_iH_{ik}f(x(t - \tau(t))) + 2a^T(t)P_iC_{ik} \int_{t-\rho(t)}^t f(x(s))ds \]

\[ - \int_{t-\tilde{\tau}}^t y^T(s)Uy(s)ds + x^T(t - \tau(t)) [X_{2k} - (1 - d_1)R] x(t - \tau(t)) \]

\[ - x^T(t - \rho(t))(1 - d_2)Qx(t - \rho(t)) + f^T(x(t))[\bar{\tau} T + S]f(x(t)) \]

\[ - (1 - d_1)f^T(x(t - \tau(t)))Sf(x(t - \tau(t))) + y^T(t)[\bar{\tau}U + \rho W]y(t) \]

\[ + \left( \int_{t-\tilde{\beta}}^t f(x(s))ds \right)^T (X_{3k} - \bar{\tau}^{-1}T) \left( \int_{t-\tilde{\beta}}^t f(x(s))ds - \int_{t-\tilde{\beta}}^t y^T(s)Wy(s)ds \right), \]

It follows from Assumption (A1) that for any \( a > 0, b > 0 \) the following equations hold,

\[ ax^T(t - \tau(t))L^T Lx(t - \tau(t)) - af^T(x(t - \tau(t)))f(x(t - \tau(t))) \geq 0 \]

\[ bx^T(t)L^T Lx(t) - bf^T(x(t))f(x(t)) \geq 0. \]
From (6.6) and the definition of \( y(t) \) for any matrices \( N_i, M_i, O_i, \quad (i = 1, 2, \ldots, 7) \) the following equations hold

\[
2 \xi^T(t) N \left[ x(t) - x(t - \tau(t)) \right] - \int_{t-\tau(t)}^{t} y(s) ds - \int_{t-\rho(t)}^{t} dw(s) = 0,
\]

\[
2 \xi^T(t) M \left[ x(t) - x(t - \rho(t)) \right] - \int_{t-\rho(t)}^{t} y(s) ds - \int_{t-\rho(t)}^{t} dw(s) = 0,
\]

\[
2 \xi^T(t) O \left[ -y(t) + \sum_{k=1}^{r} \delta_k(\theta(t)) \left[ -A_{ik} x(t) + W_{ik} f(x(t)) + H_{ik} f(x(t - \tau(t))) \right] + C_{ik} \int_{t-\rho(t)}^{t} f(x(s)) ds \right] = 0.
\]

where

\[
\xi^T(t) = [x^T(t) \ x^T(t - \tau(t)) \ x^T(t - \rho(t)) \ f^T(x(t)) \ f^T(x(t - \tau(t))) \ f^T(x(t - \rho(t)) \ f^T(x(s)) ds \ y^T(t)],
\]

\[
M^T = [M_1^T \ M_2^T \ M_3^T \ M_4^T \ M_5^T \ M_6^T \ M_7^T], \quad N^T = [N_1^T \ N_2^T \ N_3^T \ N_4^T \ N_5^T \ N_6^T \ N_7^T],
\]

\[
O^T = [O_1^T \ O_2^T \ O_3^T \ O_4^T \ O_5^T \ O_6^T \ O_7^T].
\]

Thus, from (6.10) - (6.15) we obtain,

\[
\mathcal{L}V \leq \sum_{k=1}^{r} \delta_k(\theta(t)) \left\{ \xi^T(t) \tilde{\Omega}_k \xi(t) + 2 \xi^T(t) N \int_{t-\tau(t)}^{t} y(s) ds - 2 \xi^T(t) M \int_{t-\rho(t)}^{t} y(s) ds - 2 \xi^T(t) M \int_{t-\rho(t)}^{t} dw(s) \right\},
\]

\[
\leq \sum_{k=1}^{r} \delta_k(\theta(t)) \left\{ \xi^T(t) \left( \tilde{\Xi}_k + \tilde{\Xi} + \tilde{\Xi} Z \right) \xi(t) - \int_{t-\tau(t)}^{t} \left[ \xi^T(t) y^T(s) \right] \begin{bmatrix} Y & N \\ * & U \end{bmatrix} \left[ \xi(t) y(s) \right] ds - \int_{t-\rho(t)}^{t} \left[ \xi^T(t) y^T(s) \right] \begin{bmatrix} * & M \\ Z & W \end{bmatrix} \left[ \xi(t) y(s) \right] ds - 2 \xi^T(t) N \int_{t-\tau(t)}^{t} dw(s) - 2 \xi^T(t) M \int_{t-\rho(t)}^{t} dw(s) \right\},
\]

158
where
\[
\tilde{\Omega}_k = \begin{bmatrix}
\Omega_{k11} & \Omega_{k12} & \Omega_{k13} & \Omega_{k14} & \Omega_{k15} & \Omega_{k16} & \Omega_{k17} \\
\ast & \Omega_{k22} & \Omega_{k23} & \Omega_{k24} & \Omega_{k25} & \Omega_{k26} & \Omega_{k27} \\
\ast & \ast & \Omega_{k33} & \Omega_{k34} & \Omega_{k35} & \Omega_{k36} & \Omega_{k37} \\
\ast & \ast & \ast & \Omega_{k44} & \Omega_{k45} & \Omega_{k46} & \Omega_{k47} \\
\ast & \ast & \ast & \ast & \Omega_{k55} & \Omega_{k56} & \Omega_{k57} \\
\ast & \ast & \ast & \ast & \ast & \Omega_{k66} & \Omega_{k67} \\
\ast & \ast & \ast & \ast & \ast & \ast & \Omega_{k77}
\end{bmatrix},
\]

Similar to manipulations in [29] we choose \( Y = N^{-1}U^T \) and \( Z = M^{-1}W^T \). It is true that
\[
\begin{bmatrix}
X & N \\
* & U
\end{bmatrix} > 0, \quad \begin{bmatrix}
Z & M \\
* & W
\end{bmatrix} > 0
\]
and
\[
\mathcal{L}V \leq \sum_{k=1}^{r} \delta_k(\theta(t)) \left\{ \xi^T(t)(\tilde{\Omega}_k + \bar{\rho}NU^{-1}U^T + \bar{\rho}MW^{-1}W^T)\xi(t) \right\},
\]
Using the Schur complement Lemma 1.5.2 we have,
\[
\mathcal{L}V \leq \sum_{k=1}^{r} \delta_k(\theta(t)) \left\{ \xi^T(t)(\Omega_k)\xi(t) \right\},
\]
Thus from the Lyapunov stability theory [27], dynamics of the fuzzy neural network (6.4) is globally asymptotically stable, which completes the proof.

In what follows, we will show that our result can be specialized to several cases including those studied extensively in the literature. The following stated corollary is consequences of Theorem 6.2.2 and thus the proof is omitted.

**Case 1:** When \( r = 1 \), system (6.4) is simplified to the general stochastic neural networks with discrete and distributed time varying delays as follows,
\[
dx(t) = [-A_ix(t) + W_if(x(t)) + H_i f(x(t - \tau(t))) + C_i \int_{t-\rho(t)}^{t} f(x(s))ds]dt \\
+ \sigma_i(t,x(t), x(t - \tau(t)), \int_{t-\rho(t)}^{t} f(x(s))ds).
\]
(6.16)
For convenience, we deleted the subscript ”11”. Thus our result makes another novel criterion on stochastic neural networks with discrete and distributed time varying delays.
This result is shown in the following corollary.
Corollary 6.2.3 Suppose that the assumption (A1) – (A2) holds, then for any given $\tau > 0$, $\bar{p} > 0$, $d_1 > 0$, $d_2 > 0$ the system (6.16) is globally asymptotically stable if there exist symmetric positive definite matrices $P_i > 0, Q > 0, R > 0, S > 0, T > 0, U > 0, W > 0$, symmetric matrices $N_i, M_i, O_i, (i = 1, 2, \ldots, 7)$ and scalars $a > 0$, $b > 0$ such that feasible solution exist for LMI's

$$
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} & \sqrt{\bar{p}} N_1 & \sqrt{\bar{p}} M_1 \\
* & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26} & \Omega_{27} & \sqrt{\bar{p}} N_2 & \sqrt{\bar{p}} M_2 \\
* * & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} & \Omega_{37} & \sqrt{\bar{p}} N_3 & \sqrt{\bar{p}} M_3 \\
* * * & \Omega_{44} & \Omega_{45} & \Omega_{46} & \Omega_{47} & \sqrt{\bar{p}} N_4 & \sqrt{\bar{p}} M_4 \\
* * * * & \Omega_{55} & \Omega_{56} & \Omega_{57} & \sqrt{\bar{p}} N_5 & \sqrt{\bar{p}} M_5 \\
* * * * * & \Omega_{66} & \Omega_{67} & \sqrt{\bar{p}} N_6 & \sqrt{\bar{p}} M_6 \\
* * * * * * & \Omega_{77} & \sqrt{\bar{p}} N_7 & \sqrt{\bar{p}} M_7 \\
* * * * * * * & -U & 0 \\
* * * * * * * * & -W 
\end{bmatrix} \quad (6.17)
$$

where

$$
\begin{align*}
\Omega_{11} &= -A_i^T P_i - P_i^T A_i + Q + R + X_{4i} + N_i^T + N_i + M_i^T + M_i - O_i A_i - A_i^T O_i^T + b L^T L, \\
\Omega_{12} &= N_2^T - N_1 + M_2^T - A_i^T O_i^T, \quad \Omega_{13} = N_3^T - M_1 + M_3^T - A_i^T O_i^T, \\
\Omega_{14} &= P_i W_i + N_i^T + M_i^T + O_i W_i - A_i^T O_i^T, \quad \Omega_{15} = P_i H_i + N_5^T + M_6^T - A_i^T O_5^T + O_i H_i, \\
\Omega_{16} &= P_i C_i + N_4^T + M_4^T + O_i C_i - A_i^T O_4^T, \quad \Omega_{17} = N_2^T + M_7^T - O_i - A_i^T O_7^T, \\
\Omega_{22} &= -(1 - d) R + X_{2i} + a L^T L - N_2^T - N_2, \quad \Omega_{23} = -N_3^T - M_2, \quad \Omega_{24} = -N_4^T + O_2 W_i, \\
\Omega_{25} &= -N_5^T + O_2 H_i, \quad \Omega_{26} = -N_6^T + O_2 C_i, \quad \Omega_{27} = -N_7^T - O_2, \\
\Omega_{33} &= -(1 - d_2) R - M_3^T - M_3, \quad \Omega_{34} = -M_8^T + O_3 W_i, \quad \Omega_{35} = -M_9^T + O_3 H_i, \\
\Omega_{36} &= -M_7^T + O_3 C_i, \quad \Omega_{37} = -M_6^T - O_3, \quad \Omega_{44} = \rho T + S - b I + O_4 W_i + W_i^T O_i^T, \\
\Omega_{45} &= O_4 H_i + W_i^T O_i^T, \quad \Omega_{46} = O_4 C_i + W_i^T O_i^T, \quad \Omega_{47} = W_i^T O_i^T - O_4, \\
\Omega_{55} &= -(1 - d_1) S - a I + O_5 H_i + H_i^T O_5^T, \quad \Omega_{56} = H_i^T O_6^T + O_5 C_i, \quad \Omega_{57} = H_i^T O_7^T - O_5, \\
\Omega_{66} &= X_{3i} - \tau^{-1} T + O_6 C_i + C_i^T O_6^T, \quad \Omega_{67} = C_i^T O_7^T - O_6, \quad \Omega_{77} = \tau U + \bar{p} W_i - O_7. 
\end{align*}
$$
6.2.3 Numerical Examples

Example 6.2.1. Consider the TSSMJFRNN with two modes (s=2). The T-S fuzzy model of this system is of the following form:

Plant Rules:

Rule 1: IF \( \{ \theta_1(t) \text{ is } \eta^1 \} \), THEN

\[
\begin{align*}
\dot{x}(t) &= \left[ -A_{11}x(t) + W_{11}f(x(t)) + H_{11}f(x(t - \tau(t))) + C_{11} \int_{t-\rho(t)}^{t} f(x(s))ds \right] dt + \sigma_{11}(t, x(t), x(t - \tau(t)), \int_{t-\rho(t)}^{t} f(x(s))ds) dw(t),
\end{align*}
\]

Rule 2: IF \( \{ \theta_2(t) \text{ is } \eta^2 \} \), THEN

\[
\begin{align*}
\dot{x}(t) &= \left[ -A_{12}x(t) + W_{12}f(x(t)) + H_{12}f(x(t - \tau(t))) + C_{12} \int_{t-\rho(t)}^{t} f(x(s))ds \right] dt + \sigma_{12}(t, x(t), x(t - \tau(t)), \int_{t-\rho(t)}^{t} f(x(s))ds) dw(t),
\end{align*}
\]

with the following parameters,

\[
\begin{align*}
A_{11} &= \begin{bmatrix} 1.5 & 0 \\ 0 & 1.2 \end{bmatrix}, & W_{11} &= \begin{bmatrix} 0.4 & -0.3 \\ -0.1 & 0.3 \end{bmatrix}, & H_{11} &= \begin{bmatrix} -0.6 & -0.2 \\ 0.4 & 0.7 \end{bmatrix}, \\
C_{11} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, & A_{12} &= \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, & W_{12} &= \begin{bmatrix} 0.5 & 1 \\ 0.4 & -0.3 \end{bmatrix}, & H_{12} &= \begin{bmatrix} 0.2 & -0.4 \\ -0.3 & 0.6 \end{bmatrix}, \\
C_{12} &= \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}, & A_{21} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, & W_{21} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, & H_{21} &= \begin{bmatrix} 0.5 & -0.01 \\ -0.1 & 0.3 \end{bmatrix}, \\
C_{21} &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}, & A_{22} &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, & W_{22} &= \begin{bmatrix} -0.2 & 0.3 \\ -0.5 & 0.6 \end{bmatrix}, & H_{22} &= \begin{bmatrix} 0.4 & -0.3 \\ 0.6 & 0.2 \end{bmatrix}, \\
C_{22} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix}, & II &= \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}, & X_{11} &= X_{21} &= X_{31} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
& & & & 
\end{align*}
\]

By using the Matlab LMI toolbox, we solve the LMI (6.7) with \( \bar{\tau} = \bar{\rho} = 0.2824 \), \( d_1 = d_2 = 1.5 \), \( L = I \), the feasible solutions are,

\[
\begin{align*}
P_1 &= \begin{bmatrix} 22.5520 & 14.8089 \\ 14.8089 & 25.4039 \end{bmatrix}, & P_2 &= \begin{bmatrix} 26.4364 & -4.1253 \\ -4.1253 & 18.7611 \end{bmatrix}, & Q &= \begin{bmatrix} 0.0060 & -0.0046 \\ -0.0046 & 0.0056 \end{bmatrix}, \\
R &= \begin{bmatrix} 0.0018 & -0.0012 \\ -0.0012 & 0.0015 \end{bmatrix}, & S &= \begin{bmatrix} 0.0009 & 0.0007 \\ 0.0007 & 0.0013 \end{bmatrix}, & T &= \begin{bmatrix} 6.8387 & 4.3144 \\ 4.3144 & 6.9702 \end{bmatrix}, \\
U &= \begin{bmatrix} 4.9231 & 0.4418 \\ 0.4418 & 5.1874 \end{bmatrix}, & W &= \begin{bmatrix} 1.9267 & -0.2521 \\ -0.2521 & 1.8462 \end{bmatrix}.
\end{align*}
\]
Therefore, the concerned fuzzy neural network with time-varying delays is asymptotically stable. To provide relatively complete information we calculated the upper bounds of $\bar{\tau}$ and $\bar{\rho}$ for different $d_1,d_2$ and listed in Table 6.1.

<table>
<thead>
<tr>
<th>$d_1,d_2$</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 6.2.1</td>
<td>0.8371</td>
<td>0.3595</td>
<td>0.2887</td>
<td>0.2824</td>
<td>0.2824</td>
<td>0.2824</td>
</tr>
</tbody>
</table>

**Table 6.1: Maximum allowable upper bounds of $\bar{\tau}$ and $\bar{\rho}$ with different $d_1,d_2$**

**Example 6.2.2.** Consider a stochastic neural networks with mixed time-varying delays [76] with the following parameters,

$$
A_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 2.5 \end{bmatrix}, \quad W_{11} = \begin{bmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{bmatrix}, \quad H_{11} = \begin{bmatrix} 0.3 & 0 \\ 0.1 & 0.2 \end{bmatrix},
$$

$$
C_{11} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad W_{12} = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix},
$$

$$
H_{12} = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},
$$

We select $X_{1t} = 0.1I$, $X_{2t} = 0.2I$, $X_{3t} = 0.2I$, $\bar{\tau} = \bar{\rho} = 0.9$ and $d_1 = d_2 = 1.5$. By using the Matlab LMI toolbox, we solve the LMIs (6.17) the feasible solutions are

$$
P_1 = \begin{bmatrix} 84.3256 & -53.4954 \\ -53.4954 & 157.9951 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 72.6960 & -27.7430 \\ -27.7430 & 72.8334 \end{bmatrix},
$$

$$
Q = \begin{bmatrix} 0.3576 & -0.3479 \\ -0.3479 & 0.3979 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0956 & -0.0936 \\ -0.0936 & 0.0920 \end{bmatrix},
$$

$$
S = \begin{bmatrix} 0.0194 & -0.0182 \\ -0.0182 & 0.0173 \end{bmatrix}, \quad T = \begin{bmatrix} 3.6010 & -0.7356 \\ -0.7356 & 3.6670 \end{bmatrix},
$$

$$
U = \begin{bmatrix} 8.5272 & -0.0435 \\ -0.0435 & 8.5246 \end{bmatrix}, \quad W = \begin{bmatrix} 4.3906 & -0.2292 \\ -0.2292 & 4.3750 \end{bmatrix}.
$$

Therefore, the concerned Markovian jumping neural networks with discrete distributed time-varying delays is globally asymptotically stable. In [76] Theorem 1 provided delay-independent stability criterion for stochastic Markovian jumping neural networks with discrete distributed time-varying delays. It is noted that the results [76] are not able to conclude stability for $d > 1$. According to Corollary 6.2.3, we can demonstrate that this system is stable for $d_1 = d_2 > 1$. Thus, our results are less conservative than the results in [76].

162
6.3 Stability of uncertain fuzzy Markovian jumping recurrent neural networks with time-varying delays

6.3.1 System description and Preliminaries

Consider the following uncertain Markovian jumping recurrent neural networks with time-varying delays described by,

\[ \dot{x}(t) = -(A(\eta_t) + \Delta A(\eta_t))x(t) + (W(\eta_t) + \Delta W(\eta_t))f(x(t)) + (H(\eta_t) + \Delta H(\eta_t))f(x(t - \tau(t))), \]  \hspace{1cm} (6.18)

where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \), \( A(\eta_t) = \text{diag}\{a_1(\eta_t), a_2(\eta_t), \ldots, a_n(\eta_t)\} \),
\( W(\eta_t) = [(w_{ij}(\eta_t))_{n\times n}]^T, \ H = [(h_{ij}(\eta_t))_{n\times n}]^T, f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T \) and \( \tau(t) = (\tau_1(t), \tau_2(t), \ldots, \tau_n(t))^T \).

Then we have

\[ f^T(x(t))f(x(t)) \leq x^T(t)L^TLx(t), \]

where \( L = \text{diag}\{l_1, l_2, \ldots, l_n\} \).

For convenience, each possible value of \( \eta_t \) is denoted by \( i, i \in S \) in the sequel. Then we have

\[ A_i = A(\eta_t), \ W_i = W(\eta_t), \ H_i = H(\eta_t), \]
\[ \Delta A_i = \Delta A(\eta_t), \ \Delta W_i = \Delta W(\eta_t), \ \Delta H_i = \Delta H(\eta_t), \]

where \( A_i, W_i \) and \( H_i, \) for any \( i \in S \), are known constant matrices of appropriate dimensions. \( \Delta A_i, \Delta W_i \) and \( \Delta H_i \) for \( i \in S \) are unknown matrices that represent the time-varying parameter uncertainties and are assumed to be of the form:

\[ [\Delta A_i \ \Delta W_i \ \Delta H_i] = N_iG_i(t)[E_{3i} \ E_{2i} \ E_{1i}], \]  \hspace{1cm} (6.19)

where \( N_i, E_{3i}, E_{2i} \) and \( E_{1i} \) are known real constant matrices and \( G_i(t), \ \forall i \in S \), are known time-varying matrix functions satisfying

\[ G_i^T(t)G_i(t) \leq I, \ \forall i \in S. \]  \hspace{1cm} (6.20)
It is assumed that all elements $G_i(t)$ are Lebesgue measurable, $\Delta A_i$, $\Delta W_i$ and $\Delta H_i$ are said to be admissible if both (6.19) and (6.20) hold. The system (6.18) can be written as

$$\dot{x}(t) = -A_i(t)x(t) + W_i(t)f(x(t) + H_i(t)f(x(t - \tau(t))),$$

where

$$A_i(t) = A_i + \Delta A_i, \quad W_i(t) = W_i + \Delta W_i, \quad H_i(t) = H_i + \Delta H_i.$$

Now we generalize the ordinary T–S fuzzy models to express a complex system whose consequent parts are a set of uncertain Markovian jumping recurrent neural networks with time varying delays.

The T–S fuzzy model has $r$ rules, where the $k^{th}$ rule of this T–S fuzzy model is of the following form,

**Plant Rule $k$:**

IF \{$\theta_1(t)$ is $\eta^k_1$\} and \ldots and \{$\theta_r(t)$ is $\eta^k_r$\}

THEN

$$\dot{x}(t) = -A_{ik}(t)x(t) + W_{ik}(t)f(x(t)) + H_{ik}(t)f(x(t - \tau(t))),$$ \hspace{1cm} (6.21)

Let $\delta_k(\theta(t))$ be the normalized membership function of the inferred fuzzy set $\omega_k(\theta(t))$. The state equation is defined as follows:

$$\dot{x}(t) = \sum_{k=1}^{r} \delta_k(\theta(t))\{ -A_{ik}(t)x(t) + W_{ik}(t)f(x(t)) + H_{ik}(t)f(x(t - \tau(t))) \},$$ \hspace{1cm} (6.22)

### 6.3.2 Global stability results

In this section, first some sufficient conditions of global stability for system (6.22) with $\Delta A_{ik} = \Delta W_{ik} = \Delta H_{ik} = 0$ are obtained.

**Theorem 6.3.1** Suppose that the assumption (A1) holds, then the system (6.22) with $\Delta A_{ik} = \Delta W_{ik} = \Delta H_{ik} = 0$ is globally asymptotically stable if there exist symmetric
positive definite matrices $P > 0, Q > 0, R > 0, S > 0$, positive scalars $\epsilon_j > 0, (j = 1, 2)$ and positive diagonal matrix $M = \text{diag}\{m_1, m_2, \ldots, m_n\} > 0$ such that feasible solution exist for LMIs

$$
\Omega_k = \begin{bmatrix}
\Omega_{k11} & \Omega_{k12} & 0 & \epsilon_1 P_i W_{ik} & \epsilon_2 P_i H_{ik} \\
\ast & \Omega_{k22} & 0 & 0 & 0 \\
\ast & \ast & -\tau^{-1} S & 0 & 0 \\
\ast & \ast & \ast & -\epsilon_1 I & 0 \\
\ast & \ast & \ast & \ast & -\epsilon_2 I
\end{bmatrix} < 0, \quad (6.23)
$$

where

$$
\Omega_{k11} = -A_i^T P_i - P_i^T A_i k + L^T R L + \tau L^T S L + \epsilon_1^{-1} L^T L + Q - 2LM A_i k + 2L^T M W_{ik} L \\
+ \sum_{j=1}^{s} \pi_{ij} P_j, \quad \Omega_{k12} = L^T M H_{ik} L, \quad \Omega_{k22} = -(1 - d) Q - (1 - d) L^T R L + \epsilon_2^{-1} L^T L.
$$

**Proof:** We consider the following Lyapunov functional to derive the stability result,

$$
V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t)) + V_4(t, x(t)),
$$

where

$$
V_1(t, x(t)) = x^T(t) P_i x(t),
$$

$$
V_2(t, x(t)) = 2 \int_{0}^{x_i} f_i(s) ds,
$$

$$
V_3(t, x(t)) = \int_{t-\tau(t)}^{t} [x^T(s) Q x(s) ds + f^T(x(s)) R f(x(s))] ds,
$$

$$
V_4(t, x(t)) = \int_{t-\tau(t)}^{t} (\theta - t + \tau) f^T(x(\theta)) S f(x(\theta)) d\theta.
$$

It follows from Lemma 1.5.4 that,

$$
2 x^T(t) P_i W_{ik} f(x(t)) \leq \epsilon_1 x^T(t) P_i W_{ik} W_{ik}^T P_i x(t) + \epsilon_1^{-1} x^T(t) L^T L x(t),
$$

$$
2 x^T(t) P_i H_{ik} f(x(t - \tau(t))) \leq \epsilon_2 x^T(t) P_i H_{ik} H_{ik}^T P_i x(t) + \epsilon_2^{-1} x^T(t - \tau(t)) L^T L x(t - \tau(t)),
$$

$$
2 f^T(x(t)) M A_i k x(t) \leq 2 x^T(t) L M A_i k x(t),
$$

$$
2 f^T(x(t)) M W_{ik} f(x(t)) \leq 2 x^T(t) L^T M W_{ik} L x(t),
$$

165
$$2f^T(x(t))MH_{ik}f(x(\tau(t))) \leq 2x^T(t)L^TMH_{ik}Lx(t-\tau(t)).$$

According to the Itô’s rule [56], using Lemma 1.5.10 we obtain,

$$LV \leq \sum_{k=1}^r \delta_k(\theta(t)) \left\{ x^T(t) \left[ -A^{T}_{ik}P_i - P_iA_{ik} + \epsilon_1 P_i W_{ik} W^T_{ik} P_i + \epsilon_2^{-1}L^T L - 2LM A_{ik} ight] \\
+ 2L^T MW_{ik} L + \epsilon_2 P_i H_{ik} H^T_{ik} P_i + Q + L^T RL + \bar{\tau} L^T S L + \sum_{j=1}^r \pi_{ij} P_j \right\} x(t) \\
+ 2x^T(t)L^T M H_{ik} L x(t-\tau(t)) - \left( \int_{t-\tau}^t f(x(s))ds \right)^T \bar{\tau}^{-1} S \left( \int_{t-\tau}^t f(x(s))ds \right) \\
+ x^T(t-\tau(t))[\epsilon_2^{-1}L^T L - (1-d)Q - (1-d)L^T RL]x(t-\tau(t)) \right\}$$

$$\leq \sum_{k=1}^r \delta_k(\theta(t)) \left\{ \psi^T(t) \Pi_k \psi(t) \right\},$$

where $\psi^T(t) = [x^T(t) x^T(t-\tau(t)) \left( \int_{t-\tau}^t f(x(s))ds \right)^T]$ and

$$\Pi_k = \begin{bmatrix} \Pi_{k11} & \Pi_{k12} & 0 \\ * & \Pi_{k22} & 0 \\ * & * & -\bar{\tau}^{-1} S \end{bmatrix},$$

with $\Pi_{k11} = \Omega_{k11} + \epsilon_1 P_i W_{ik} W^T_{ik} P_i + \epsilon_2 P_i H_{ik} H^T_{ik} P_i$, $\Pi_{k12} = \Omega_{k12}$ and $\Pi_{k22} = \Omega_{k22}$.

By using the Schur complement Lemma 1.5.2, $\Pi_k$ can be written as $\Omega_k$, which indicates from the Lyapunov stability theory [27], that the dynamics of the fuzzy neural network (6.22) with $\Delta A_{ik} = \Delta W_{ik} = \Delta H_{ik} = 0$ is globally asymptotically stable, which completes the proof.

Now, we will provide the global robust stability condition for the uncertain MJFRNNs (6.22) with norm-bounded uncertainties satisfying (6.19) and (6.20).

**Theorem 6.3.2** Suppose that the assumption (A1) holds, then the system (6.22) is globally asymptotically stable if there exist symmetric positive definite matrices $P > 0, Q > 0, R > 0, S > 0$, positive scalars $\epsilon_j > 0, (j = 1, 2), \mu_j > 0, (j = 1, 2, 3, 4)$ and positive diagonal matrix $M = \text{diag}\{m_1, m_2, \ldots, m_n\} > 0$ such that the feasible solution exist for
\[ \Xi_k < 0, \quad \text{where} \]
\[
\Xi_k = \begin{bmatrix}
\Xi_{k1} & \Xi_{k2} & 0 & \Xi_{k3} & \Xi_{k4} & \Xi_{k5} & \Xi_{k6} & \Xi_{k7} & \Xi_{k8} & \Xi_{k9} & 0 & 0 & 0 \\
+ & \Xi_{k10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + & + & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + & + & + & + & + & -I & 0 & 0 & 0 & 0 & 0 \\
+ & + & + & + & + & + & + & + & -I & 0 & 0 & 0 \\
+ & + & + & + & + & + & + & + & + & + & + & -I \\
\end{bmatrix} < 0,
\]
\[(6.24)\]

\[
\Xi_{k1} = \Omega_{k11} + (\mu_1 + \mu_2) E_{11k}^T E_{11k} - \mu_2 E_{11k}^T E_{11k} L - \mu_2 L^T E_{11k}^T E_{11k},
\]
\[
\Xi_{k2} = \Omega_{k12} + \mu_2 L^T E_{12k}^T E_{12k} L - \mu_2 E_{12k}^T E_{12k} L, \quad \Xi_{k3} = c_1 P_l W_k,
\]
\[
\Xi_{k4} = e_2 P_l H_k, \quad \Xi_{k5} = P N_{ik}, \quad \Xi_{k6} = L^T M N_{ik}, \quad \Xi_{k7} = \mu_2 L^T E_{21k}, \quad \Xi_{k8} = c_1 N_{ik}^T P_i,
\]
\[
\Xi_{k9} = e_2 N_{ik}^T P_i, \quad \Xi_{k10} = \Omega_{k22}, \quad \Xi_{k11} = \mu_2 L^T E_{31k}, \quad \Xi_{k12} = \mu_4 E_{31k}, \quad \Xi_{k13} = \mu_4 E_{31k}^T.
\]

**Proof:** By Theorem 6.3.1, the system \((6.22)\) is robustly globally asymptotically stable if the following inequality holds:

\[
\Omega_k + [N_{ik}^T P_i 0 0 0 0]^T G_{ik}(t)[-E_{11k} 0 0 0 0]
+ [-E_{11k} 0 0 0 0]^T G_{ik}(t)[N_{ik}^T P_i 0 0 0 0]
+ [N_{ik}^T M L 0 0 0 0]^T G_{ik}(t)[-E_{11k} + E_{21k} L E_{31k} L 0 0 0]
+ [-E_{11k} + E_{21k} L E_{31k} L 0 0 0]^T G_{ik}(t)[N_{ik}^T M L 0 0 0 0]
+ [\epsilon_1 N_{ik}^T P_i 0 0 0 0]^T G_{ik}(t)[0 0 0 0 E_{21k} 0]
+ [0 0 0 E_{21k} 0]^T G_{ik}(t)[\epsilon_1 N_{ik}^T P_i 0 0 0 0]
+ [\epsilon_2 N_{ik}^T P_i 0 0 0 0]^T G_{ik}(t)[0 0 0 0 E_{31k}]
+ [0 0 0 0 E_{31k} 0]^T G_{ik}(t)[\epsilon_2 N_{ik}^T P_i 0 0 0 0] < 0.
\]

By Lemma 1.5.4, \((6.25)\) can be written as,

\[
\Omega_k + \mu_1^{-1} [N_{ik}^T P_i 0 0 0 0]^T [N_{ik}^T P_i 0 0 0 0]
\]

\[
< 0.
\]

167
\[ + \mu_1 [-E_{1ik} 0 0 0 0]^T [-E_{1ik} 0 0 0 0] \\
+ \mu_2^{-1} [N_{ik}^T M^T L 0 0 0 0]^T [N_{ik}^T M^T L 0 0 0 0] \\
+ \mu_2 [-E_{1ik} + E_{2ik} L E_{2ik} L 0 0 0]^T [-E_{1ik} + E_{2ik} L E_{2ik} L 0 0 0] \\
+ \mu_3^{-1} [\epsilon_1 N_{ik}^T P_1 0 0 0 0]^T [\epsilon_1 N_{ik}^T P_1 0 0 0 0] \\
+ \mu_3 [0 0 0 E_{2ik} 0]^T [0 0 0 E_{2ik} 0] \\
+ \mu_4^{-1} [\epsilon_2 N_{ik}^T P_1 0 0 0 0]^T [\epsilon_2 N_{ik}^T P_1 0 0 0 0] \\
+ \mu_4 [0 0 0 E_{2ik} 0]^T [0 0 0 E_{2ik} 0]. \] (6.26)

Then by using the Schur complement Lemma 1.5.2, (6.26) is equivalent to (6.24). Thus, if the LMI (6.24) holds system (6.22) is robustly globally asymptotically stable. This completes the proof.

**Remark 6.3.3** When \( r = 1, s = 1 \) with \( \Delta A_{ik} = \Delta W_{ik} = \Delta H_{ik} = 0 \) system (6.23) is simplified to the general cellular neural networks with time varying delays as follows,

\[ \dot{x}(t) = -Ax(t) + Wf(x(t)) + Hf(x(t - \tau(t))). \] (6.27)

For convenience, we deleted the subscript "11". Thus our result makes another novel criterion on cellular neural networks. This result is shown in the following corollary. The proof of the above theorem is similar to that of Theorem 6.3.1 by choosing the functionals \( V_1, V_3 \) and \( V_4 \) to be as described in the proof of Theorem 6.3.1 and it is thus omitted.

**Corollary 6.3.4** Suppose that the assumption \( (A1) \) holds, then the system (6.27) is asymptotically stable if there exist symmetric positive definite matrices \( P > 0, Q > 0, R > 0, S > 0 \) and positive scalars \( \epsilon_j > 0, (j = 1, 2) \) such that feasible solution exist for LMI,

\[ \Gamma = \begin{bmatrix}
\Gamma_{11} & 0 & 0 & \epsilon_1 PW & \epsilon_2 PH \\
* & \Gamma_{22} & 0 & 0 & 0 \\
* & * & -\tau^{-1}S & 0 & 0 \\
* & * & * & -\epsilon_1 I & 0 \\
* & * & * & * & -\epsilon_2 I 
\end{bmatrix} < 0, \] (6.28)

where \( \Gamma_{11} = -A^T P - P^T A + L^T RL + \tau L^T S L + \epsilon_1^{-1} L^T L + Q, \) \( \Gamma_{22} = -(1 - d)Q - (1 - d) L^T RL + \epsilon_2^{-1} L^T L. \)
6.3.3 Numerical Examples

Example 6.3.1. Consider the MJFRNNs with uncertainties and two modes \((s=2)\). The T–S fuzzy model of this system is of the following form:

**Plant Rules:**

**Rule 1:** IF \(\{\theta_1(t) is \eta^1\}\), THEN

\[
\dot{x}(t) = -(A_{11} + \Delta A_{11})x(t) + (W_{11} + \Delta W_{11})f(x(t)) + (H_{11} + \Delta H_{11})f(x(t - \tau(t))),
\]

**Rule 2:** IF \(\{\theta_2(t) is \eta^2\}\), THEN

\[
\dot{x}(t) = -(A_{12} + \Delta A_{12})x(t) + (W_{12} + \Delta W_{12})f(x(t)) + (H_{12} + \Delta H_{12})f(x(t - \tau(t))),
\]

with the following parameters,

\[
A_{11} = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.2 \end{bmatrix}, \quad W_{11} = \begin{bmatrix} 0.01 & -0.02 \\ -0.1 & 0.01 \end{bmatrix}, \quad H_{11} = \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.02 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2.5 & 0 \\ 0 & 2 \end{bmatrix},
\]

\[
W_{12} = \begin{bmatrix} -0.5 & 0.1 \\ 0.2 & -0.1 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad W_{21} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},
\]

\[
H_{21} = \begin{bmatrix} 0.5 & -0.01 \\ -0.1 & 0.3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.2 \end{bmatrix}, \quad W_{22} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},
\]

\[
H_{22} = \begin{bmatrix} 0.2 & -0.5 \\ 0.1 & 0.6 \end{bmatrix}, \quad H = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}, \quad G_{ik}(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{bmatrix},
\]

\(N_{ik} = E_{ijk} = E_{2ik} = E_{3ik} = 0.2I, i, l = 1, 2\). The membership functions are \(\eta^1 = \frac{1}{\varepsilon_i^0}, \eta^2 = 1 - \eta^1\). By using the Matlab LMI toolbox, we solve the LMI (6.26) for \(\varepsilon_i > 0, (i = 1, 2)\), with \(\delta = 0.5, d = 0.5, L = 0.5I\), the feasible solutions are,

\[
P_1 = \begin{bmatrix} 1.4073 & -0.1704 \\ -0.1704 & 2.3542 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.2676 & 0.3810 \\ 0.3810 & 0.9423 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.5798 & -0.0686 \\ -0.0686 & 0.4107 \end{bmatrix},
\]

\[
R = \begin{bmatrix} 2.1067 & -0.2560 \\ -0.2560 & 1.5293 \end{bmatrix}, \quad S = \begin{bmatrix} 6.1560 & -0.8629 \\ -0.8629 & 4.3265 \end{bmatrix}, \quad M = \begin{bmatrix} 0.0969 & 0 \\ 0 & 0.0721 \end{bmatrix},
\]

Therefore, the concerned fuzzy neural network with time-varying delays is robustly asymptotically stable.

**Example 6.3.2.** Consider the Markovian Jumping Hopfield neural network [52] with two modes \((s=2)\).

The network parameters are given as follows,
\[
A_{11} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, \quad H_{11} = \begin{bmatrix} 0.8 & -0.7 \\ 0.4 & 0.6 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad H_{21} = \begin{bmatrix} 0.6 & 0.4 \\ -0.5 & 0.4 \end{bmatrix},
\]
\[
H = \begin{bmatrix} -0.4 & 0.4 \\ 0.3 & -0.3 \end{bmatrix}.
\]

By using the Matlab LMI toolbox, we solve the LMI (6.25) with \( k = 1 \) for \( \epsilon > 0 \), \( (i = 1, 2) \), \( \bar{\tau} = 0.5 \), \( d = 0.5 \), \( L = I \) the feasible solutions are,

\[
P_1 = \begin{bmatrix} 1.5450 & 0.0614 \\ 0.0614 & 1.1236 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.7327 & 0.2311 \\ 0.2311 & 1.4297 \end{bmatrix}, \quad Q = \begin{bmatrix} 11.8360 & 0.7426 \\ 0.7426 & 18.1513 \end{bmatrix},
\]
\[
R = \begin{bmatrix} 11.8360 & 0.7426 \\ 0.7426 & 18.1513 \end{bmatrix}, \quad S = \begin{bmatrix} 9.6644 & 0.0280 \\ 0.0280 & 11.2458 \end{bmatrix}, \quad M = \begin{bmatrix} 9.1274 & 0 \\ 0 & 10.1315 \end{bmatrix}.
\]

Therefore, the concerned neural network with time varying delays is asymptotically stable. It was reported in [52], the delay-dependent stability condition in [52] is satisfied for \( \bar{\tau} = 0.6 \). However, by using LMI (6.24) in Theorem 6.3.1 for \( d = 0 \), it is found that the system described by Example 6.3.2 is stable for any constant delay \( \bar{\tau} \).

**Example 6.3.3.** Consider the cellular neural networks with time varying delays (6.27) with the parameters,

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},
\]

We select \( \bar{\tau} = 0.5 \), \( d = 0.5 \), \( L = diag\{1,1\} \) and solve the LMI (6.28) by using the Matlab LMI toolbox the feasible solutions are

\[
P = \begin{bmatrix} 2.6184 & -0.8142 \\ -0.8142 & 2.6184 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.5800 & -0.2157 \\ -0.2157 & 0.5800 \end{bmatrix},
\]
\[
R = \begin{bmatrix} 0.5800 & -0.2157 \\ -0.2157 & 0.5800 \end{bmatrix}, \quad S = \begin{bmatrix} 1.4378 & -0.4129 \\ -0.4129 & 1.4378 \end{bmatrix}.
\]

Therefore, the concerned neural network with time-varying delays is globally asymptotically stable. It can be observed that,

\[
-(W + H) = \begin{bmatrix} 0 & -0.3 \\ -0.3 & -0.2 \end{bmatrix}, \quad -(W + H + rI) = \begin{bmatrix} -r & -0.3 \\ -0.3 & -0.2 - r \end{bmatrix}, \quad r > 0
\]

It is easy to see that the matrix \(-(W + H)\) is not positive definite and the matrix \(-(W + H + rI)\) cannot be positive definite for any \( r \geq 0 \). Hence the conditions in [2, 8] are not satisfied. Thus, our results are less conservative than the results presented in [2, 8].

170
Figure 6.1: State trajectory of the system in Example 6.2.1 when r=1, s=1.

Figure 6.2: State trajectory of the system in Example 6.2.1 when r=1, s=2.
Figure 6.3: State trajectory of the system in Example 6.2.1 when $r=2, s=1$.

Figure 6.4: State trajectory of the system in Example 6.2.1 when $r=2, s=2$.  

172