Chapter 3

Stability Analysis for Switched Discrete-Time Systems with Actuator Failures

3.1 Introduction

Fault detection is considered as a noteworthy task, because for a human operator it is not easy to surveil constantly a fault occurrence on a complex dynamical system from a reliability point of view [94, 119]. Fault-tolerance means a system possesses the ability to maintain its stability and performance in spite of unknown faults within components of the system. Over the past few decades, it is essential to ensure safety and reliability of real-time dynamic systems due to the growing demand for reliability, maintainability and survivability in industrial processes [17, 69]. Moreover, faults may result in poor system performance, lead to instability and even catastrophic accidents. The dissipative analysis problem of fault tolerant method for discrete-time systems has been interrogated in [79] with actuator failures.

Besides for the aero-engine fault tolerant control system, it is required that not only the stable operation of the aero-engine but also the robustness to external disturbance should be guaranteed when failure occurs. Meanwhile, the response time and control cost should be
minimized. While considering both robust performance and cost performance as constraints, it is fairly necessary to study the fault tolerant control method. Hence in recent years, the fault tolerant control technique has been attained considerable attention among researchers [11, 79, 118]. Generally speaking, fault tolerance can be achieved either passively by using feedback control laws that are robust to possible system faults, or actively by means of faults diagnosis and accommodation architecture [108].

Moreover, switched systems are capable of describing systems with multiple dynamics with one of them governing the system at a time. In addition, they can be used to model systems of a single process being controlled by multiple switching controllers [102]. In practice, it can be applied to various modeling and control problems present in robotics, automotive systems, process control, power systems and air traffic control. The interaction of continuous and discrete dynamics in a switched system will lead to a rich dynamical behavior that is not come across in purely continuous or discrete-time systems. Therefore, study on switched systems has both theoretical significance and practical value [57, 95]. Recently, the average dwell-time method becomes an important method to analyze a suitable switching signal to guarantee the stability of switching systems. However, the fault tolerant control is not included in [57] to study the stability analysis of discrete-time switched systems.

On the other hand, it is always unavoidable that the possibility of impulsive effects [49, 116] and external disturbances [33, 107] in the systems due to sudden changes and environment, respectively. That is, some kinds of unexpected discrepancies in structures or parameters are often exist that are caused by component failures or needing for repairs, changes in the interconnections of subsystems and environmental disturbance. These phenomena cannot be exactly described by most deterministic DSs whereas the stochastic equations are used to model such diverse phenomena [58]. To this end, some researchers devoted many attempts to examine the various control problem of discrete-time systems with external disturbances and impulses. Even though the perfect consensus cannot be achieved under the effects of actuator faults/uncertainties, it is always desirable to define a cost function which helps to ensure that the system is not only stable but also guarantees an adequate level of performance. Also, there are only few results have been reported for
switched discrete-time impulsive systems with actuator failures and guaranteed cost control. Furthermore, the goal of this chapter is to design the fault tolerant control and appropriate cost function for the class of discrete-time switched impulsive systems.

Inspired by the above, Section 3.2 investigates the exponential stability of the guaranteed cost analysis for discrete-time switched impulsive systems with actuator failures and external disturbances. The effectiveness of the derived results are verified through the numerical examples.

3.2 Guaranteed Cost Fault Tolerant Control for Discrete-Time Switched Impulsive Systems with External Disturbances

3.2.1 System Formulation

Consider a class of impulsive discrete-time switched systems as follows

\[
\begin{align*}
    x(k+1) &= A_{\sigma(k)}(k)x(k) + A_{\sigma(k)}(k)x(k - \tau(k)) + B_{\sigma(k)}u^f(k) \\
    &\quad + C_{\sigma(k)}(k)w(k), \quad k \neq k_v, \\
    \Delta x(k) &= (H_{\sigma(k)} - I)x(k), \quad k = k_v, \quad v \in \mathbb{Z}^+, \\
    x(s) &= \phi(s), \quad \forall s = -\tau_M, -\tau_M + 1, \ldots, 0
\end{align*}
\]  

(3.1)

where \( x(k) \in \mathbb{R}^n \) is the state vector, \( u^f(k) \in \mathbb{R}^m \) is the fault control input vector and \( w(k) \) is the external disturbance belongs to \( l_2[0, \infty] \). \( \sigma(k) : [0, +\infty) \to \mathbb{S} = \{1, 2, \ldots, N\} \) is the switching sequence independent of the state and let \( \sigma(k) = I \). The term \( \tau(k) \) describes the time-varying delay that satisfies

\[
0 < \tau_m \leq \tau(k) \leq \tau_M
\]  

(3.2)

with \( \tau_m \) and \( \tau_M \) are known positive integers representing respectively the lower and upper bounds of the time-delay. The matrices \( A_l(k) = A_l + \Delta A_l(k), \quad A_{dl}(k) = A_{dl} + \Delta A_{dl}(k) \)
and $C_l(k) = C_l + \Delta C_l(k)$ represent the bounded matrices containing parameter uncertainties $\Delta A_l(k), \Delta A_d(k)$, and $\Delta C_l(k)$, respectively and also satisfy the following condition

$$[\Delta A_l(k) \Delta A_d(k) \Delta C_l(k)] = G_l F(k) [E_{al} E_{ad} E_{el}],$$  \hspace{1cm} (3.3)$$

where $A_l, B_l, A_d, C_l$ are known constant matrices with appropriate dimensions and $F(k)$ is the time-varying nonlinear function satisfying $F^T(k)F(k) \leq I$. $H_{el}$ is a constant matrix representing the impulse effect on the system at switching time and $\phi(s)$ is the initial condition.

Now, the failure model reliable controller can be chosen as follows

$$u_f(k) = F u(k)$$

with $u(k)$ is taken as delayed state feedback controller, $u(k) = K_i x(k - \tau(k))$. Therefore, the switched feedback controller can be designed as

$$u_f(k) = F K_i x(k - \tau(k)).$$  \hspace{1cm} (3.4)$$

The unknown matrix $F = diag\{f_1, f_2, \ldots, f_n\}$ is the actuator fault matrix, where each $f_i$, $i = 1, 2, \ldots, n$ is a fixed value or varying within the known lower and upper bound as

$$0 \leq \underline{f}_i \leq f_i \leq \overline{f}_i \leq 1,$$

where $\underline{f}_i$ and $\overline{f}_i$ are known constants.

**Remark 3.1.** The unknown actuator fault matrix can be described in three forms: the actuator is in normal case if $f_i = 1$, the actuator is in completely failed case if $f_i = 0$ and the partial fail case if $0 < f_i < 1$, i.e., the actuator is degraded partially.

Denoting the matrices $\hat{F} = diag\{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\}, D = diag\{d_1, d_2, \ldots, d_n\}$ and $|L| = diag\{|l_1|, |l_2|, \ldots, |l_n|\}$ with

$$\hat{f}_i = \frac{\underline{f}_i + \overline{f}_i}{2}, \quad d_i = \frac{\underline{f}_i - \overline{f}_i}{2\hat{f}_i}, \quad l_i = \frac{\hat{f}_i + \overline{f}_i}{d_i}.$$
Then, \( F \) can be expressed as \( F = \hat{F}(I + L) \) with \(|L| \leq D \leq I\) for some known matrix \( \hat{F} \).

From system (3.1) and control (3.4), it follows that
\[
\begin{align*}
    x(k + 1) &= A_{t}(k)x(k) + (A_{dl}(k) + B_{t}FK_{t})x(k - \tau(k)) + C_{t}w(k), \quad k \neq k_{v}, \\
    \Delta x(k) &= (H_{el} - I)x(k), \quad k = k_{v}, \quad v \in Z^{+}.
\end{align*}
\tag{3.5}
\]

From the fact (3.3), system (3.5) can be equivalently rewritten as
\[
\begin{align*}
    x(k + 1) &= A_{l}x(k) + (A_{dl} + B_{l}FK_{l})x(k - \tau(k)) + C_{l}w(k) + G_{l}q_{l}(k), \\
    q_{l}(k) &= F_{l}q_{l}(k), \\
    q_{1}(k) &= E_{ad}x(k) + E_{adl}x(k - \tau(k)), \quad k \neq k_{v}, \\
    \Delta x(k) &= (H_{el} - I)x(k), \quad k = k_{v}, \quad v \in Z^{+}.
\end{align*}
\tag{3.6}
\]

For given positive-definite matrices \( U_{l} \) and \( W_{l} \), let define the cost function associated with the system (3.6) as
\[
J = \sum_{k=0}^{\infty} [x^{T}(k)U_{l}x(k) + (w^{f})^{T}(k)W_{l}w^{f}(k)].
\tag{3.7}
\]

### 3.2.2 Stability Results

In this section, the following theorem is given to derive the sufficient conditions for the robust exponential stabilization of considered discrete-time systems with time-varying delays involving the probabilistic effects. The following notations are used in the sequel.

\[
\begin{align*}
    \bar{e}_{i} &= [0_{n \times (i-1)n}, I_{n}, 0_{n \times (11-i)n}], \\
    \bar{e}^{T}_{1} &= [\bar{e}_{3} - \bar{e}_{4}; \sqrt{3}(\bar{e}_{3} + \bar{e}_{4} - \bar{e}_{9})], \\
    \bar{e}^{T}_{2} &= [\bar{e}_{2} - \bar{e}_{3}; \sqrt{3}(\bar{e}_{2} + \bar{e}_{3} - \bar{e}_{10})], \\
    \bar{e}_{3} &= [\bar{v}_{1}; \bar{v}_{2}]^{T}, \quad \bar{\tau} = \tau_{M} - \tau_{m}, \\
    \Gamma_{l} &= \begin{bmatrix} \bar{T}_{l} & \bar{Y}_{l} \\ 0 & \bar{T}_{l} \end{bmatrix}, \quad \bar{T}_{l} = \begin{bmatrix} T_{l} & 0 \\ 0 & T_{l} \end{bmatrix}, \quad \bar{Y}_{l} = \begin{bmatrix} Y_{l} & 0 \\ 0 & Y_{l} \end{bmatrix}, \\
    \xi^{T}(k) &= [x^{T}(k) \ x^{T}(t - \tau_{m}) \ x^{T}(t - \tau(k)) \ x^{T}(t - \tau_{M}) \ \eta^{T}(k) \ \eta^{T}(t - \tau_{M}) \ \mu_{1}^{T}(k) \ \mu_{2}^{T}(k) \ \varphi_{l}^{T}(k)], \\
    \mu_{1}^{T}(k) &= \Lambda(k, \tau(k), \tau_{M}), \quad \mu_{2}^{T}(k) = \Lambda(k, \tau_{m}, \tau(k)).
\end{align*}
\]

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Further according to actuator failures, the stability results are derived through two cases.

**Case i:** For known actuator failures.

**Theorem 3.1.** For given scalars $0 < \alpha < 1$, $\mu > 1$, known actuator failure matrix $F$, if there exists positive-definite symmetric matrices $P_i, Q_r(r = 1, 2, 3)$, $R_i, S_i, T_i, V_i$ and $\tilde{W}_i$, any matrices $X_i, Y_i$ with appropriate dimensions and positive scalars $\epsilon_i$ such that the following LMIs hold for all $l = 1, 2, \ldots, N$

\[
\Omega_{1l} = \begin{bmatrix}
\Omega_{11} & \epsilon_i \psi_i^T & \tilde{W}_l \\
* & \epsilon_i I & 0 \\
* & * & -\tilde{W}_l
\end{bmatrix} < 0, \quad (3.8)
\]

\[
\Omega_{2l} = \begin{bmatrix}
-(1 - \alpha)P_i & H_{nl}P_l \\
* & -P_l
\end{bmatrix} < 0, \quad (3.9)
\]

\[P_i < \mu P_j, \quad Q_r i < \mu Q_j (r = 1, 2, 3), \quad R_i < \mu R_j, \quad S_i < \mu S_j \quad \text{and} \quad T_i < \mu T_j,
\]

for all $i, j = 1, 2, \ldots, N$ and $i \neq j$,

where

\[
\Omega_{11} = \alpha P_i + Q_{11i} + Q_{22i} + (\tau + 1) R_i - \alpha \tau_m S_i + U_i - 2V_i + V_i A_i,
\]

\[
\Omega_{12} = -\alpha_m \tau_m S_i, \quad \Omega_{1,3} = V_i A_i + X_i F B_i,
\]

\[
\Omega_{1,5} = P_i + Q_{12i} + Q_{22i} + (\tau + 1) R_2 + A_i^T V_i - 2V_i, \quad \Omega_{1,11} = V_i G_i,
\]

\[
\Omega_{2,2} = -\alpha_m Q_{32i} - \alpha_m Q_{11i} - 2\alpha M - \alpha_m \tau_m S_i, \quad \Omega_{2,5} = -\alpha \tau_m Y_i - 2\alpha M T_i,
\]

\[
\Omega_{2,4} = -2\alpha M T_i - 2\alpha M Y_i, \quad \Omega_{2,6} = -\alpha_m Q_{32i} - \alpha_m Q_{12i}, \quad \Omega_{2,9} = 3\alpha M Y_i,
\]

\[
\Omega_{2,10} = 3\alpha M T_i, \quad \Omega_{3,3} = -\alpha M R_{1i} - 8\alpha M T_i - 4\alpha M Y_i, \quad \Omega_{3,4} = -2\alpha M T_i - 4\alpha M Y_i,
\]

\[
\Omega_{3,5} = A_i^T V_i + B_i^T F X_i^T, \quad \Omega_{3,7} = -\alpha M R_{2i}, \quad \Omega_{3,9} = 3\alpha M T_i + 3\alpha M Y_i,
\]

\[
\Omega_{3,10} = 3\alpha M Y_i + 3\alpha M T_i, \quad \Omega_{4,4} = -\alpha_m Q_{22i} - \alpha_m Q_{33i} - 4\alpha M T_i,
\]

\[
\Omega_{4,8} = -\alpha M Q_{22i} - \alpha M Q_{33i}, \quad \Omega_{4,9} = 3\alpha M T_i, \quad \Omega_{4,10} = 3\alpha M Y_i,
\]

\[
\tilde{\Omega}_{5,5} = P_i + Q_{13i} + Q_{23i} + (\tau + 1) R_2 + (\tau_m)^2 S_i + U_i - V_i - V_i^T,
\]

\[
\tilde{\Omega}_{5,11} = V_i G_i, \quad \tilde{\Omega}_{6,6} = \alpha_m Q_{33i} - \alpha_m Q_{13i} + \alpha_m (\tau)^2 T_i, \quad \tilde{\Omega}_{7,7} = -\alpha M R_{3i},
\]

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\[ \Omega_{8,8} = -\alpha_M Q_{23M} - \alpha_M Q_{233}, \quad \Omega_{9,9} = -3\alpha_M T_l, \quad \Omega_{9,10} = 3\alpha_M y_l, \]
\[ \Omega_{10,10} = -3\alpha_M T_l, \quad \Omega_{11,11} = -c_{\text{d}I}, \]
\[ \psi_i^T = [E_{\text{ad}}, 0, E_{\text{add}}, 0_{8n \times 8n}]; \quad \tilde{W}_l = [0, 0, B_l^T F^T X_l^T, 0_{8n \times 8n}]. \]

Then, the system (3.6) is said to be exponentially stable for an arbitrary switched signal \( \sigma(k) \) with average dwell-time \( T_a \geq T^*_a = \frac{\ln(\mu)}{\ln(1-\alpha)}. \) The control gain matrices can be obtained through \( K_l = V_l^{-1} X_l. \)

**Proof.** To prove the sufficient conditions for exponential stability of switched discrete-time system (3.6), divide the proof into two parts.

**Part 1:** For \( k \neq k_\mu. \)

Let us construct the LKF as

\[ V_l(k) = V_{II}(k) + V_{II}(k) + V_{II}(k) + V_{II}(k) \quad (3.10) \]

with

\[ V_{II}(k) = x^T(k) P_l x(k), \]
\[ V_{II}(k) = \sum_{s=k-r_\alpha}^{k-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right]^T Q_{11}(1-\alpha)^{k-s-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right] \]
\[ + \sum_{s=k-r_\alpha}^{k-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right]^T Q_{22}(1-\alpha)^{k-s-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right], \]
\[ + \sum_{s=k-r_\mu}^{k-r_\alpha-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right]^T Q_{33}(1-\alpha)^{k-s-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right], \]
\[ V_{II}(k) = \sum_{s=k-r(k)}^{k-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right]^T R_l(1-\alpha)^{k-s-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right] \]
\[ + \sum_{j=-r_\mu+1}^{k-1} \sum_{s=k+j}^{k-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right]^T R_l(1-\alpha)^{k-s-1} \left[ \begin{array}{c} x(s) \\ \eta(s) \end{array} \right], \]

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\[
V_{3t}(k) = \tau_m \sum_{j=-\tau_m}^{k-1} \sum_{s=k+j}^{\tau_m-1} \eta^T(s) S_t(1 - \alpha)^{k-s-1} \eta(s) \\
+ \tau \sum_{j=-\tau_M}^{\tau_M-1} \sum_{s=k+j}^{\tau_M-1} \eta^T(s) T_t(1 - \alpha)^{k-s-1} \eta(s),
\]

where \( Q_{il} = \begin{bmatrix} Q_{il} & Q_{il2} \\ * & Q_{il3} \end{bmatrix} \) (i=1,2,3), \( R_t = \begin{bmatrix} R_{1t} & R_{2t} \\ * & R_{3t} \end{bmatrix} \). Also, let \( \alpha_m = (1 - \alpha)^{\tau_m} \) and \( \alpha_M = (1 - \alpha)^{\tau_M} \). Taking the difference and expectation on (3.10), one can get

\[
\mathbb{E}\{\Delta V_{1l}(k) + \alpha V_{1l}(k)\} = \mathbb{E}\{x^T(k + 1) P_t x(k + 1) - (1 - \alpha)x^T(k) P_t x(k)\}
\]

\[
= \mathbb{E}\{[\eta^T(k) + x^T(k)] P_t [\eta(k) + x(k)]
\]

\[-(1 - \alpha)x^T(k) P_t x(k)\}.
\]

(3.11)

\[
\mathbb{E}\{\Delta V_{2l}(k) + \alpha V_{2l}(k)\} = \mathbb{E}\left\{ \sum_{s=k-\tau_m}^{k-1} \sum_{s=k+1}^{\tau_m+1} x(s) \eta(s) \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix}^T Q_{1l} (1 - \alpha)^{k-s-1} \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix} \\
- \sum_{s=k-\tau_m}^{k-2} \sum_{s=k+1}^{\tau_m+1} x(s) \eta(s) \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix}^T Q_{2l} (1 - \alpha)^{k-s-1} \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix} \\
+ \sum_{s=k-\tau_M}^{k-1} \sum_{s=k+1}^{\tau_M+1} x(s) \eta(s) \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix}^T Q_{3l} (1 - \alpha)^{k-s-1} \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix} \\
- \sum_{s=k-\tau_M}^{k-2} \sum_{s=k+1}^{\tau_M+1} x(s) \eta(s) \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix}^T Q_{2l} (1 - \alpha)^{k-s-1} \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix} \\
+ \sum_{s=k-\tau_M}^{k-1} \sum_{s=k+1}^{\tau_M+1} x(s) \eta(s) \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix}^T Q_{3l} (1 - \alpha)^{k-s-1} \begin{bmatrix} x(s) \\ \eta(s) \end{bmatrix} \right\}
\]

\[
= \mathbb{E}\left\{ \begin{bmatrix} x(k) \\ \eta(k) \end{bmatrix}^T (Q_{1l} + Q_{2l}) \begin{bmatrix} x(k) \\ \eta(k) \end{bmatrix} \right\}.
\]

(3.12)
\begin{align*}
&+\alpha_m \begin{bmatrix}
  x(k-\tau_m) \\
  \eta(k-\tau_m)
\end{bmatrix}^T (Q_{2l} - Q_{2l}) \begin{bmatrix}
  x(k-\tau_m) \\
  \eta(k-\tau_m)
\end{bmatrix} \\
&-\alpha_M \begin{bmatrix}
  x(k-\tau_M) \\
  \eta(k-\tau_M)
\end{bmatrix}^T (Q_{2l} + Q_{3l}) \begin{bmatrix}
  x(k-\tau_M) \\
  \eta(k-\tau_M)
\end{bmatrix}.
\end{align*}
\tag{3.13}

Similarly,
\begin{align*}
\mathbb{E}\{\Delta V_{3l}(k) + \alpha V_{3l}(k)\} &\leq \mathbb{E}\left\{ (\tau + 1) \begin{bmatrix}
  x(k) \\
  \eta(k)
\end{bmatrix}^T R_t \begin{bmatrix}
  x(k) \\
  \eta(k)
\end{bmatrix} \\
&-\alpha_M \begin{bmatrix}
  x(k-\tau(k)) \\
  \eta(k-\tau(k))
\end{bmatrix}^T R_t \begin{bmatrix}
  x(k-\tau(k)) \\
  \eta(k-\tau(k))
\end{bmatrix} \right\}. \tag{3.14}
\end{align*}

\begin{align*}
\mathbb{E}\{\Delta V_{4l}(k) + \alpha V_{4l}(k)\} &\leq \mathbb{E}\left\{ \tau_m^2 \eta^T(k) S_l \eta(k) - \tau_m \alpha_m \sum_{s=k-\tau_m}^{k-1} \eta^T(s) \right. \\
&\times S_l \eta(s) + \alpha_m \bar{\tau}^2 \eta^T(k-\tau_m) T_l \eta(k-\tau_m) \\
&\left. -\alpha_M \bar{\tau}^2 \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j) R_{2l} \eta(j) \right\}.
\end{align*}

By utilizing Lemma 1.7, it is seen that
\begin{align*}
\mathbb{E}\{\Delta V_{4l}(k) + \alpha V_{4l}(k)\} &< \mathbb{E}\left\{ \tau_m^2 \eta^T(k) S_l \eta(k) - \alpha_m \tau_m [x(k) - x(k-\tau_m)]^T \\
&\times S_l [x(k) - x(k-\tau_m)] + \alpha_m \bar{\tau}^2 \eta^T(k-\tau_m) \\
&\times T_l \eta(k-\tau_m) - \alpha_M \bar{\tau} \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j) T_l \eta(j) \right\}. \tag{3.15}
\end{align*}

Then, it follows that
\begin{align*}
-\bar{\tau} \sum_{j=k-\tau_M}^{k-\tau(k)-1} \eta^T(j) T_l \eta(j) = -\bar{\tau} \sum_{j=k-\tau_M}^{k-\tau_M-1} \eta^T(j) T_l \eta(j) - \bar{\tau} \sum_{j=k-\tau(k)}^{k-\tau_M-1} \eta^T(j) T_l \eta(j).
\end{align*}
From Lemma 1.6, one can obtain that

\[
\tau \sum_{j=k-\tau_M}^{k-\tau(k)-1} \eta^T(j)T_i \eta(j) \leq \frac{-\bar{\tau}}{\tau_M - \tau(k)} \begin{bmatrix} x(k - \tau(k)) - x(k - \tau_M) \\ x(k - \tau(k)) + x(k - \tau_M) - \mu_1(k) \end{bmatrix}^T \times \begin{bmatrix} T_i & 0 \\ 0 & 3T_i \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) - x(k - \tau_M) \\ x(k - \tau(k)) + x(k - \tau_M) - \mu_1(k) \end{bmatrix} \\
\leq \frac{-\bar{\tau}}{\tau_M - \tau(k)} \xi^T(k) \begin{bmatrix} \bar{e}_3 - \bar{e}_4 \\ \sqrt{3} \bar{e}_3 + \bar{e}_4 - \bar{e}_9 \end{bmatrix}^T \times \begin{bmatrix} T_i & 0 \\ 0 & T_i \end{bmatrix} \begin{bmatrix} \bar{e}_3 - \bar{e}_4 \\ \sqrt{3} \bar{e}_3 + \bar{e}_4 - \bar{e}_9 \end{bmatrix} \xi(k) = \frac{-\bar{\tau}}{\tau_M - \tau(k)} \xi^T(k) \tilde{\nu}_1^T \tilde{T}_i \tilde{\nu}_2 \xi(k). \tag{3.16}
\]

Likewise,

\[
-\bar{\tau} \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j)T_i \eta(j) \leq \frac{-\bar{\tau}}{\tau(k) - \tau_m} \xi^T(k) \begin{bmatrix} \bar{e}_2 - \bar{e}_3 \\ \sqrt{3} (\bar{e}_2 + \bar{e}_3 - \bar{e}_{10}) \end{bmatrix}^T \times \begin{bmatrix} T_i & 0 \\ 0 & T_i \end{bmatrix} \begin{bmatrix} \bar{e}_2 - \bar{e}_3 \\ \sqrt{3} (\bar{e}_2 + \bar{e}_3 - \bar{e}_{10}) \end{bmatrix} \xi(k) = \frac{-\bar{\tau}}{\tau(k) - \tau_m} \xi^T(k) \tilde{\nu}_2^T \tilde{T}_i \tilde{\nu}_2 \xi(k). \tag{3.17}
\]

Therefore from (3.16) and (3.17), one can get

\[
-\bar{\tau} \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j)T_i \eta(j) \leq -\xi^T(k) \left\{ \frac{\bar{\tau}}{\tau_M - \tau_m} \tilde{\nu}_1^T \tilde{T}_i \tilde{\nu}_1 + \frac{\bar{\tau}}{\tau(k) - \tau_m} \tilde{\nu}_2^T \tilde{T}_i \tilde{\nu}_2 \right\} \xi(k).
\]

By applying reciprocally convex Lemma 1.3, if there exist \( \tilde{Y}_i \) such that \( \Gamma_i > 0 \) holds. Hence, the above inequality becomes

\[
-\bar{\tau} \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j)T_i \eta(j) \leq -\xi^T(k) \tilde{\nu}_3 \Gamma_i \tilde{\nu}_3^T \xi(k). \tag{3.18}
\]

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Now, the equation (3.15) becomes
\[
\mathbb{E}\{\Delta V_{t}(k) + \alpha V_{t}(k)\} \leq \mathbb{E}\left\{r_{m}^{2}\eta^{T}(k)S_{t}\eta(k) - \alpha_{m}[x(k) - x(k - \tau_{m})^{T}S_{t}T_{t}^{-1}x(k - \tau_{m}) + \alpha_{m}\tau_{m}^{2}\eta^{T}(k - \tau_{m})T_{t}\eta(k - \tau_{m}) - \xi^{T}(k)\tilde{\nu}_{3}\Gamma_{1}\tilde{\nu}_{3}^{T}\xi(k)\right\}.
\]  
(3.19)

For any matrix \(V_{i}\), one can have
\[
2[\eta^{T}(k) + x^{T}(k)]V_{i}[x(k + 1) - x(k) - \eta(k)] = 0
\]
which implies
\[
2[\eta^{T}(k) + x^{T}(k)]V_{i}\left\{A_{i}x(k) + A_{dl}x(k - \tau(k)) + K_{li}F_{i}B_{i}x(k - \tau(k)) + C_{i}\omega(k) + C_{i}\varphi_{i}(k) - x(k) - \eta(k)\right\} = 0
\]
Hence,
\[
\mathbb{E}\{2[\eta^{T}(k) + x^{T}(k)]V_{i}\left\{A_{i}x(k) + A_{dl}x(k - \tau(k)) + K_{li}F_{i}B_{i}x(k - \tau(k)) + C_{i}\omega(k) + C_{i}\varphi_{i}(k) - x(k) - \eta(k)\right\}\} = 0.
\]  
(3.20)

Combining the equations from (3.11)-(3.14),(3.19) and (3.20), one can obtain
\[
\mathbb{E}\{\Delta V_{t}(k) + \alpha V_{t}(k)\} \leq \mathbb{E}\left[\xi^{T}(k)\left\{[\bar{e}_{1}^{T} + \bar{e}_{1}^{T}]P_{i}[\bar{e}_{5} + \bar{e}_{1}] - (1 - \alpha)e_{5}^{T}P_{i}\bar{e}_{1}
+ \bar{e}_{5}\begin{bmatrix} Q_{11} + Q_{21} & \bar{e}_{1} \\ \bar{e}_{1} & \bar{e}_{5} \end{bmatrix} + \bar{e}_{5}\begin{bmatrix} Q_{21} - Q_{11} & \bar{e}_{1} \\ \bar{e}_{1} & \bar{e}_{5} \end{bmatrix}
+ \bar{e}_{5}\begin{bmatrix} Q_{12} + Q_{21} & \bar{e}_{4} \\ \bar{e}_{4} & \bar{e}_{5} \end{bmatrix} + (\gamma_{1} + 1)
\right\}R_{i}\left[\begin{bmatrix} \bar{e}_{3} \\ \bar{e}_{7} \end{bmatrix} - \alpha_{m}\begin{bmatrix} \bar{e}_{3} \\ \bar{e}_{7} \end{bmatrix}\right]R_{i}\left[\begin{bmatrix} \bar{e}_{3} \\ \bar{e}_{7} \end{bmatrix}
+ \tau_{m}^{2}\bar{e}_{5}^{T}S_{i}\bar{e}_{5} + \alpha_{m}\tau_{m}[\bar{e}_{1}^{T} - \bar{e}_{2}^{T}]S_{i}[\bar{e}_{1} - \bar{e}_{2}] + \alpha_{m}\tau_{m}^{2}\bar{e}_{6}^{T}T_{i}\bar{e}_{6}
- \alpha_{M}\bar{e}_{3}\Gamma_{1}\bar{e}_{3}^{T} + 2[\bar{e}_{5}^{T} + \bar{e}_{1}^{T}]V_{i}\left\{A_{i}\bar{e}_{1} + A_{dl}\bar{e}_{3}
+ K_{li}F_{i}\bar{e}_{3} + C_{i}\bar{e}_{i1} - \bar{e}_{1} - \bar{e}_{5}\right\}\right\}\xi(k)\right].
\]  
(3.21)
Also from (3.6), one can have
\[ \xi^T(k) \psi_1(k) \leq \tilde{\psi}_i^T(k) \psi_i(k). \]

Then, there exist positive constants \( \psi \_i \) satisfying the following inequality
\[ \psi \_i (\xi^T(k) \psi_1(k) \psi_i(k) - \tilde{\psi}_i^T(k) \psi_i(k)) \geq 0. \]  

Using S-procedure Lemma 1.8 and including LHS of (3.22) to (3.21), one can achieve
\[ E\{ \Delta V_i(k) + \alpha V_i(k) \} \leq E\left\{ \xi^T(k) \left[ \tilde{\Omega}_{11} + \psi_i^T \psi_i + \tilde{W}_i^T W_i \tilde{W}_i \right] \xi(k) - x^T(k) U_i x(k) - (u^f)^T(k) W_i u^f(k) \right\} \]

where \( \tilde{W}_i = [0, \ 0, \ R_{i}^T F K_{i}^T, \ 0_{m \times n}] \). Pre and post-multiply \( \tilde{\Omega}_{11} \) by \( \text{diag}\{I_{12m \times 12m}, \ V_i\} \) and its transpose, respectively and also let \( V_i W_i^{-1} V_i = \tilde{W}_i \). Now, it affords that
\[ E\{ \Delta V_i(k) + \alpha V_i(k) \} \leq E\left\{ \xi^T(k) \tilde{\Omega}_{11} \xi(k) - x^T(k) U_i x(k) - (u^f)^T(k) W_i u^f(k) \right\}. \]  

**Part 2:** For \( \nu = k \nu \),
\[ V_i(k \nu) = x^T(k \nu) P_i x(k \nu). \]

Then taking the difference, one can obtain that
\[ E\{ \Delta V_i(k \nu) + \alpha V_i(k \nu) \} = E\{ x^T(k \nu + 1) P_i x(k \nu + 1) - (1 - \alpha) x^T(k \nu) P_i x(k \nu) \} \]
\[ \leq E\{ x^T(k \nu) \Omega_{21} x(k \nu) \}. \]

Then from (3.23), (3.24) and using Schur complement Lemma 1.2, it can be easily found that \( \Omega_{11} < 0 \) and \( \Omega_{21} < 0 \) which leads to
\[ E\{ \Delta V_i(k) + \alpha V_i(k) \} \leq 0. \]

That is,
\[ V_i(k + 1) \leq (1 - \alpha) V_i(k) \]
which implies that

\[ V_{\sigma(k)}(k) \leq (1 - \alpha)^{k - k_t} V_{\sigma(k_t)}(k_t). \]

Thus, it can be obtained that

\begin{align*}
V_{\sigma(k)}(k) &\leq (1 - \alpha)^k V_{\sigma(k_t)}(k_t), \\
&\leq \mu(1 - \alpha)^{k - k_t} V_{\sigma(k_t - 1)}(k_{t-1}), \\
&\leq \mu(1 - \alpha)^{k - k_{t-1}} V_{\sigma(k_{t-1})}(k_{t-1}), \\
&\vdots \\
&\leq \mu^{N_{\sigma}(k_0,k)}(1 - \alpha)^{k - k_0} V_{\sigma(k_0)}(k_0). \hspace{1cm} (3.25)
\end{align*}

As we know from the Definition 1.3, \(N_{\sigma}(k_0,k) \leq (k - k_0)/T_a\), then (3.25) becomes

\[ V_{\sigma(k)}(k) \leq ((1 - \alpha)\mu^{1/T_a})^{k - k_0} V_{\sigma(k_0)}(k_0). \hspace{1cm} (3.26)\]

So, it can be verified from (3.10) that

\[ V_{\sigma(k)}(k) \geq \delta_1 \|x(k)\|^2 \text{ and } V_{\sigma(k_0)}(k_0) \leq \delta_2 \|\phi\|^2_L. \]

From (3.25), one can get

\[ \delta_1 \|x(k)\|^2 \leq ((1 - \alpha)\mu^{1/T_a})^{k - k_0} \delta_2 \|\phi\|^2_L, \]

\[ \|x(k)\|^2 \leq \frac{\delta_2}{\delta_1} \lambda^{k - k_0} \|\phi\|^2_L. \hspace{1cm} (3.27)\]

where \( \delta_1 = \min_{\mathbb{S}} \lambda_{\min}(P), \ \delta_2 = \max_{\mathbb{S}} \lambda_{\max}(P) + \max_{\mathbb{S}} \lambda_{\max}(Q_u) + \max_{\mathbb{S}} \lambda_{\max}(R_i) + \tau \max_{\mathbb{S}} \lambda_{\max}(S_i) + \tau^2 \max_{\mathbb{S}} \lambda_{\max}(T_i), \ i = 1, 2, 3. \) Therefore from the Lyapunov functional stability theory Definition 1.2, it is proved that (3.6) is exponentially stable in the mean square with the controller gain matrices \( K_i = V_i^{-1}X_i. \) To find the guaranteed cost value, it is obvious from (3.23) that

\[ x^T(k)U_i x(k) + (u^T)^I(k)W_i u^I(k) \leq E\{(1 - \alpha)V(k) - V(k + 1)\}. \]
Summing both sides from 0 to $n$ and letting $n \to \infty$ implies to note that $V(n) \to 0$, it can be acquired
\[
\sum_{k=0}^{n} [x^T(k)U_l x(k) + (u^T)^f(k)W_l u^f(k)] \leq \delta_2\|x(0)\|^2 = J^*.
\]
This completes the proof. \hfill \Box

The following corollary provides the sufficient stability criteria for the system (3.6) without uncertainties ($G_l = 0$) given as
\[
\begin{align*}
    x(k+1) &= A_l x(k) + (A_{dl} + B_l F K_l)x(k - \tau(k)) + C_l w(k), \quad k \neq k_v, \\
    \Delta x(k) &= (H_{vl} - I)x(k), \quad k = k_v, \quad v \in Z^+.
\end{align*}
\] (3.28)

**Corollary 3.1.** For given scalars $0 < \alpha < 1$, $\mu > 1$, known actuator failure matrix $F$, if there exists positive-definite symmetric matrices $P_l$, $Q_{rl}(r = 1, 2, 3)$, $R_l$, $S_l$, $T_l$, $V_l$ and $\bar{W}_l$, any matrices $X_l$, $V_l$ with appropriate dimensions and positive scalars $\epsilon_l$ such that for all $l = 1, 2, \ldots, N$ the following LMIs hold
\[
\Omega_{2l} = \begin{bmatrix}
    \Omega_{10 \times 10} & \bar{W}_l \\
    \bar{W}_l^T & \bar{W}_l
\end{bmatrix} < 0,
\] (3.29)

\[
\Omega_{2l} < 0, \quad P_l < \mu P_j, \quad Q_{rl} < \mu Q_{rj} (r = 1, 2, 3), \quad R_l < \mu R_j, \quad S_l < \mu S_j
\]

and $T_l < \mu T_j$ for all $i, j = 1, 2, \ldots, N$ and $i \neq j$

with the terms defined as in Theorem 3.1. Then, the switched discrete-time impulsive system (3.28) is said to be exponentially stable and the control gain matrices is given by $K_l = V_l^{-1} X_l$.

**Proof.** The proof remains the same as in the case of Theorem 3.1 by taking $G_l = 0$ with
\[
E\{\Delta V_l(k) + \alpha V_l(k)\} \leq E\left\{\tilde{\xi}^T(k) \left[\Omega_{10 \times 10} + \bar{W}_l^T \bar{W}_l\right] \tilde{\xi}(k) \right.
\]
\[
- x^T(k)U_l x(k) - (u^f)^T(k)W_l u^f(k)\left\}
\]
\[
= E\left\{\tilde{\xi}^T(k)\Omega_{3l} \tilde{\xi}(k) - x^T(k)U_l x(k) - (u^f)^T(k)W_l u^f(k)\right\},
\]

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where \( \xi^T(k) = [x^T(k) \ x^T(t - \tau_m) \ x^T(t - \tau(k)) \ x^T(t - \tau_M) \ \eta^T(k) \ \eta^T(t - \tau_M) \ \eta^T(t - \tau(k)) \ \eta^T(t - \tau_M) \ \mu_1^T(k) \ \mu_2^T(k)] \) and hence it is omitted.

**Case ii:** For unknown actuator failures.

**Theorem 3.2.** For given scalars \( 0 < \alpha < 1, \mu > 1 \), unknown actuator failure matrix \( F \), if there exists positive-definite symmetric matrices \( P_i, Q_r(i = 1, 2, 3), R_i, S_i, T_i, V_i \) and \( \tilde{W}_i, \) any matrices \( X_i, V_i \) with appropriate dimensions and positive scalars \( \epsilon_i \) such that the following LMIs hold for all \( l = 1, 2, \ldots, N \)

\[
\Omega_{2l} = \begin{bmatrix}
\Omega_{2l} & \epsilon_i \tilde{B} & \tilde{X}_i \\
* & -\epsilon_i I & 0 \\
* & * & -\epsilon_i I
\end{bmatrix} < 0, \tag{3.30}
\]

\( \Omega_{2l} < 0, \ P_i < \mu P_j, \ Q_r < \mu Q_j \ (r = 1, 2, 3), \ R_i < \mu R_j, \ S_i < \mu S_j \)

and \( T_i < \mu T_j \) for all \( i, j = 1, 2, \ldots, N \) and \( i \neq j \)

with the control gain matrices can be described through \( K_i = V_i^{-1} X_i \). Then, the system (3.6) with unknown actuator failure is said to be exponentially stable for an arbitrary switched signal \( \sigma(k) = l \).

**Proof.** Since \( F \) is unknown actuator failure, \( \Omega_{2l} \) is incurred by replacing \( F \) by \( \tilde{F} \) in \( \Omega_{2l} \) and also neglecting the occurrence of fault matrix terms. So that, one can get

\[
\Omega_{2l} = \Omega_{2l} + \text{sym}(BLX_i),
\]

where \( \tilde{B} = [B^T \ 0 \ B^T \ 0_{10n \times 10n}]^T, \tilde{X}_i = [\tilde{F} X_i^T \ 0_{12n \times 12n}]^T \), completes the proof. \( \Box \)

The sufficient stability criteria for the system (3.6) without impulses

\[
\begin{align*}
x(k + 1) &= A_{il} x(k) + (A_{il} + B_i F K_i) x(k - \tau(k)) + C_i w(k) + G_i s_i(k), \\
q_l(k) &= F(k) q_l(k), \\
q_1(k) &= E_{a1} x(k) + E_{a1} a(k - \tau(k)). \tag{3.31}
\end{align*}
\]

are described by the following corollary.
Corollary 3.2. For given scalars $0 < \alpha < 1$, $\mu > 1$, known actuator failure matrix $F$, if there exists positive-definite symmetric matrices $P_r, Q_r (r = 1, 2, 3)$, $R_i, S_i, T_i, V_i$ and $\bar{W}_i$, any matrices $X_i, V_i$ with appropriate dimensions and positive scalars $v_i$ such that the LMI (3.8) alone holds for all $l = 1, 2, \ldots, N$ with the terms defined as in Theorem 3.1 and $P_i < \mu P_j$, $Q_i < \mu Q_j (r = 1, 2, 3)$, $R_i < \mu R_j$, $S_i < \mu S_j$ and $T_i < \mu T_j$ for all $i, j = 1, 2, \ldots, N$ and $i \neq j$. Then, the system (3.31) is said to be exponentially stable for an arbitrary switched signal $\sigma(k) = l$ with the control gain matrices $K_l = V_l^{-1} X_l$.

### 3.2.3 Numerical Examples

The following examples show the effectiveness of the obtained theoretical results.

**Example 3.1.** Consider the uncertain switched impulsive system (3.6) with three subsystems ($l = 1, 2, 3$) as follows

\[
A_1 = \begin{bmatrix} 0.56 & 0.25 \\ 0 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.25 & 0.1 \\ 1 & 0.2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.2 & 0.5 \\ 1 & 0.4 \end{bmatrix},
\]

\[
A_{d1} = \begin{bmatrix} 0.8 & 0.5 \\ 0.1 & 0.4 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.5 & 0.12 \\ 0 & -0.2 \end{bmatrix}, \quad A_{d3} = \begin{bmatrix} 0.2 & 0.5 \\ 0.3 & 0.4 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},
\]

\[
H_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix},
\]

\[
G_1 = \begin{bmatrix} 0.4 & 0 \\ 0.1 & 0.3 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.25 & 0.2 \\ 0 & 0.4 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0.68 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
E_{a1} = \begin{bmatrix} 1.2 & 0 \\ 0.5 & 0 \end{bmatrix}, \quad E_{a2} = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad E_{a3} = \begin{bmatrix} 2 & 0.2 \\ 0.3 & 0.1 \end{bmatrix},
\]

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\[ E_{ad1} = \begin{bmatrix} 0.2 & 0.5 \\ 0 & 0.3 \end{bmatrix}, \quad E_{ad2} = \begin{bmatrix} 1 & 0 \\ 0.2 & -0.25 \end{bmatrix}, \quad E_{ad3} = \begin{bmatrix} 0.3 & 0 \\ 0.8 & 0 \end{bmatrix}, \]
\[ E_{c1} = \begin{bmatrix} 0.5 & 0 \\ 0.7 & 0 \end{bmatrix}, \quad E_{c2} = \begin{bmatrix} 1 & 0 \\ 0.2 & 0.25 \end{bmatrix}, \quad E_{c3} = \begin{bmatrix} 0.03 & 0 \\ 0.3 & 0 \end{bmatrix}, \]
\[ F(k) = [\sin(k) 0; 0 \cos(k)]. \]

Time-varying delay \( \tau(k) \) varies over the interval \([1, 14]\) and so set the minimum and maximum bounds of delay to be \( \tau_m = 1 \) and \( \tau_M = 14 \), respectively. Let take \( \alpha = 0.3 \) and \( \mu \), then choose the \( T_a \) such that the average dwell-time can be obtained as \( T_a > T_a^* = 1.1308 \).

The simulation results are given for the following both known and unknown actuators.

**(I) For known actuator \((F = 0.4I)\)**

Solving the matrix inequalities in Theorem 3.1, the feasible solutions can be incurred using LMI toolbox in Matlab as follows

\[ X_1 = \begin{bmatrix} -0.2478 & 0.4638 \\ -0.2815 & -0.7017 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -0.3205 & 0.0625 \\ -0.1473 & 0.2106 \end{bmatrix}, \]
\[ X_3 = \begin{bmatrix} -0.0422 & -0.1375 \\ -0.1015 & -0.0114 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0.0413 & -0.0254 \\ -0.0254 & 0.1120 \end{bmatrix}, \]
\[ V_2 = \begin{bmatrix} 0.0774 & -0.0128 \\ -0.0128 & 0.0893 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0.0180 & -0.0084 \\ -0.0084 & 0.0299 \end{bmatrix}. \]

Then, one can get the control gain matrices as

\[ K_1 = \begin{bmatrix} -8.7839 & 8.5807 \\ -4.5075 & -4.3168 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -4.5236 & 1.2279 \\ -2.2983 & 2.5343 \end{bmatrix}, \]
\[ K_3 = \begin{bmatrix} -4.5144 & -8.9900 \\ -4.6669 & -2.9035 \end{bmatrix}, \quad \text{and } \varepsilon_1 = 0.0265; \ v_2 = 0.0192; \ v_3 = 0.0060. \]

Simulation results plotting state trajectories and switching nature of the subsystems are shown in Figure 3.1 with the initial condition \([-0.1 0.12]\) which illustrates the effectiveness of the proposed theoretical results with the guaranteed cost value, \( J^* = 1.4908 \).
Figure 3.1: State trajectories and switching signal of the subsystems in Example 3.1 for $F = 0.4I$

Figure 3.2: State trajectories and switching signal of the subsystems in Example 3.1 for $F = I$
Figure 3.3: State trajectories and switching signal of the subsystems in Example 3.1 for interval $0.2I < F < 0.8I$

(2) **For unknown actuator**

i) **Actuator fault normal case ($F = I$)**

According to the Theorem 3.2, the feasible solutions and the control gain matrices can be obtained as

$$X_1 = \begin{bmatrix} -0.3402 & 0.6393 \\ -0.4267 & -0.9829 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -0.4402 & 0.0818 \\ -0.2023 & 0.2888 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} -0.1114 & -1.0151 \\ -0.4628 & -0.2114 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0.1462 & -0.0843 \\ -0.0843 & 0.3831 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.2642 & -0.0409 \\ -0.0409 & 0.3047 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0.1771 & -0.0848 \\ -0.0848 & 0.3830 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -3.4007 & 3.3152 \\ -1.8619 & -1.8367 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -1.8063 & 0.4661 \\ -0.9063 & 1.0104 \end{bmatrix},$$

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\[
K_3 = \begin{bmatrix}
-1.3515 & -6.7094 \\
-1.5077 & -2.0382
\end{bmatrix}, \quad \text{and } \epsilon_1 = 0.0907; \epsilon_2 = 0.0668; \epsilon_3 = 0.0786.
\]

\( ii \) Actuator fault between the interval \((0.2I < F < 0.8I)\)

The feasible solutions and the controller gain matrices are

\[
X_1 = \begin{bmatrix}
-0.9116 & 1.7130 \\
-1.1434 & -2.6337
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
-1.1796 & 0.2193 \\
-0.5421 & 0.7740
\end{bmatrix},
\]

\[
X_3 = \begin{bmatrix}
-0.2985 & -2.7201 \\
-1.2401 & -0.5665
\end{bmatrix}, \quad K_1 = \begin{bmatrix}
-9.1125 & 8.8833 \\
-4.9890 & -4.9214
\end{bmatrix},
\]

\[
K_2 = \begin{bmatrix}
-4.8401 & 1.2489 \\
-2.4284 & 2.7074
\end{bmatrix}, \quad K_3 = \begin{bmatrix}
-3.6214 & -17.9783 \\
-4.0401 & -5.4614
\end{bmatrix},
\]

\( \text{and } \epsilon_1 = 0.0907; \epsilon_2 = 0.0668; \epsilon_3 = 0.0786. \)

Figure 3.2 and 3.3 represent the state trajectories and switching of the considered subsystems for \( F = I \) and \( 0.2I < F < 0.8I \), respectively with the guaranteed cost value, \( J^* = 4.8607. \) Thus, the simulations affirm that the exponential stability can be achieved with the proposed controller for all cases.

\( iii \) Actuator fault completely fails \( (F = 0) \)

There will be no desired controller to make system (3.6) to be stable, since \( F = 0 \) implies \( FK_l = 0, (l = 1, 2, 3) \) which leads to \( u^f(k) = 0. \) Hence, the considered system is unstable without the state feedback controller whose state trajectories are depicted in the Figure 3.4.

**Example 3.2.** Let us consider the uncertain switched system (3.31) with two-modes \((l = 1, 2)\) modeled as follows

\[
A_1 = \begin{bmatrix}
0.7 & 0 \\
0.08 & 0.95
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0.7 & 0 \\
0.08 & 0.9
\end{bmatrix}, \quad A_{d1} = \begin{bmatrix}
0.15 & 0 \\
-0.1 & -0.1
\end{bmatrix},
\]

\[
A_{d2} = \begin{bmatrix}
0.14 & 0 \\
-0.1 & -0.05
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0.5 & 0.1 \\
0.3 & 0.1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0.3 & 0.3 \\
0 & 0.5
\end{bmatrix},
\]

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Figure 3.4: State trajectories of the subsystems in Example 3.1 for $F = 0$

Figure 3.5: State trajectories and switching signal of the subsystems in Example 3.2
\[
C_1 = \begin{bmatrix}
0 & 0.1 \\
0.2 & 0.1
\end{bmatrix},
C_2 = \begin{bmatrix}
1 & 0.13 \\
0.05 & 0.3
\end{bmatrix},
G_1 = \begin{bmatrix}
0.4 & 0.1 \\
0.2 & 0.8
\end{bmatrix},
\]
\[
G_2 = \begin{bmatrix}
0.25 & 0.5 \\
0.4 & 0.1
\end{bmatrix},
E_{a1} = \begin{bmatrix}
1.2 & 0.7 \\
0.5 & 0.1
\end{bmatrix},
E_{a2} = \begin{bmatrix}
0.1 & 0.2 \\
0.1 & 0.25
\end{bmatrix},
\]
\[
E_{at1} = \begin{bmatrix}
0.5 & 0.6 \\
0.1 & 0.6
\end{bmatrix},
E_{at2} = \begin{bmatrix}
0.2 & 0.3 \\
0.5 & 0.3
\end{bmatrix},
E_{e1} = \begin{bmatrix}
0.1 & 0.4 \\
0.32 & 0.45
\end{bmatrix},
\]
\[
E_{e2} = \begin{bmatrix}
0.2 & 0.3 \\
0 & 0.13
\end{bmatrix},
F(k) = [\sin(k - 1), 0; 0, \cos(k - 1)] \text{ and } F = I.
\]

The maximum and minimum time-varying delay are assumed to be \(\tau_m = 1\) and \(\tau_M = 14\), respectively. Then by Corollary 3.2, one can get the feasible solutions using the LMI toolbox in Matlab as
\[
X_1 = \begin{bmatrix}
-0.0359 & -0.0076 \\
0.0124 & 0.0277
\end{bmatrix},
X_2 = \begin{bmatrix}
-0.0432 & 0.0367 \\
0.0195 & -0.0114
\end{bmatrix},
\]
\[
V_1 = \begin{bmatrix}
0.0810 & -0.0185 \\
-0.0185 & 0.0239
\end{bmatrix},
V_2 = \begin{bmatrix}
0.0730 & -0.0277 \\
-0.0277 & 0.0959
\end{bmatrix},
\]
\[
U_1 = \begin{bmatrix}
-49.5201 & 51.7456 \\
-106.6290 & -29.8087
\end{bmatrix},
U_2 = \begin{bmatrix}
-26.9634 & 149.6606 \\
-198.0238 & -45.6031
\end{bmatrix},
\]
\[
W_1 = \begin{bmatrix}
0.2858 & -0.0366 \\
-0.0366 & 0.1912
\end{bmatrix},
W_2 = \begin{bmatrix}
0.2576 & -0.0723 \\
-0.0723 & 0.2478
\end{bmatrix},
\]

and \(\epsilon_1 = 0.0141; \epsilon_2 = 0.0234\).

and get the corresponding control gain matrices
\[
K_1 = \begin{bmatrix}
-0.3953 & 0.2089 \\
0.2110 & 1.3235
\end{bmatrix},
K_2 = \begin{bmatrix}
-0.5787 & 0.5135 \\
0.0362 & 0.0289
\end{bmatrix}.
\]
Table 3.1: Maximum upper bound of delays $\tau_M$ for various $\tau_m$.

<table>
<thead>
<tr>
<th>$\tau_m$</th>
<th>2</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>12</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_M$ in [16]</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td>9</td>
<td>10</td>
<td>infeasible</td>
<td>infeasible</td>
</tr>
<tr>
<td>$\tau_M$ in [[114], Theorem 3]</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td>9</td>
<td>10</td>
<td>infeasible</td>
<td>infeasible</td>
</tr>
<tr>
<td>$\tau_M$ in [[51], Theorem 1]</td>
<td>11</td>
<td>11</td>
<td>12</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>infeasible</td>
</tr>
<tr>
<td>$\tau_M$ in Corollary 3.2</td>
<td>17</td>
<td>19</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>22</td>
<td>22</td>
</tr>
</tbody>
</table>

Remark 3.2. From the Table 3.1, it is easy to observe that the maximum upper bound of delay $\tau_M$ obtained through Corollary 3.2 is high compared with the existing methods reported in the literature [16, 51, 114].

Further, the state trajectories and the switching signal of the considered system in Example 3.2 with the initial condition $x(0) = [0.1 \ -0.12]$ are depicted in Figure 3.5. This illustrates that the discrete-time switched system (3.31) is exponentially stable in the mean square sense with the guaranteed cost value $J^* = 1.6514$.

3.3 Conclusions & Future Directions

In this chapter, fault tolerant control against actuator failures with the switching feedback controller has been designed for switched impulsive discrete-time systems with time-varying delays through which the exponential stability has been acquired to the considered system in the mean square sense. An adequate level of performance has been guaranteed by choosing suitable cost function. By defining proper LKFs and using the updated techniques such as Wirtinger-based inequality and reciprocally convex lemma, sufficient criteria have been achieved which are formulated in terms of the LMIs. Numerical simulations have been employed to show the effectiveness of the derived theoretical results. The results obtained in this chapter can be widened to the estimation and filtering analysis of networked control systems which have plenty of applications in this digital world.