Chapter 4

Analysis and Design for Bivariate SSALT Weibull Model

4.1 Introduction

The previous two Chapters are focused on analysis and test design for SSALT model by considering single stress variable with three stress levels under quadratic life-stress relationship. Moreover, as today’s products become extremely reliable due to technological advances, single accelerating variable may not enough to achieve the life-test experiments for highly reliable products under SSALT setup, in such case it may not yield sufficient amount of failure data within specified time. However, it is desirable to add more than one stress variables in many applications and for various reasons, as it will lead to a better understanding of the effect of simultaneously operating stress variables like temperature and humidity.

In this view, only two accelerating variables are considered here in SSALT model, which is known as bivariate SSALT model. In a bivariate SSALT model, the test includes two stress variables and each has two stress levels, and stress levels are changed at different times. This chapter presents the analysis and design for a bivariate SSALT
model for two-parameter Weibull failure time distribution under modified progressive Type-I censored data. The expression of optimum test plan for bivariate SSALT Weibull model under modified progressive Type-I censoring is derived by minimizing asymptotic variance of the maximum likelihood estimators of log of mean life at the design stress and a sensitivity analysis also presented. The expressions for ML estimators of bivariate SSALT Weibull distribution is derived, and the performance of proposed estimators are also examined by providing numerical illustrations based on simulation study. The behaviour of proposed model parameters under bivariate SSALT is also investigated by applying Bayesian approach.

Section 4.2 describes test procedure and model assumptions. Maximum likelihood estimation and Fisher information matrix are presented in section 4.3. Optimization criterion is discussed in section 4.4. Bayesian estimation for the model parameters is presented in section 4.5. A numerical example is provided, followed by comparative study and sensitivity analysis in section 4.6. Conclusion are given in Section 4.7.

4.2 Test Procedure and Assumptions

We consider the SSALT problem with two stress variables, and each has two stress levels. Let \( x_{lm} \) be the \( m \)th stress level for variable \( l \), for \( l = 1, 2 \), and for \( m = 0, 1, 2, 3 \). The \( x_{10}, x_{20} \) denote stress levels at design stress conditions. The step-stress test procedure with two stress variables under modified progressive Type-I censoring is shown as follows: \( n \) identical test units are initially put on low stress level \( (x_{11}, x_{21}) \), and run until pre-specified stress change time \( \tau_1 \), the number of failed units \( n_1 \) are recorded and \( R^*_1 \) surviving units are randomly removed from the test. The test is continued on \( n - n_1 - R^*_1 \) non-removed surviving units are placed on the higher stress level \( (x_{12}, x_{21}) \) and run until time \( \tau_2 \), when the stress is increased to \( (x_{12}, x_{23}) \) and \( R^*_2 \) surviving units are randomly withdrawn from the test. Finally, at time \( \tau_3 \) under stress \( (x_{12}, x_{22}) \), all surviving items \( R^*_3 = n - \sum_{i=1}^{3} n_i - \sum_{i=1}^{2} R^*_i \) are withdrawn from the test, thereby terminating the life test. Note that, when there is no intermediate
censoring (viz., $R_1^* = R_2^* = 0$), this situation corresponds to bivariate step-stress testing under Type-I censoring as a special case. Now, the following assumptions are essential for constructing consequent step-stress model.

**Basic Assumptions**

(i) Testing is done with two stress vectors $(x_{11}, x_{21}), (x_{12}, x_{21}),$ and $(x_{21}, x_{22})$.

(ii) The life of a test unit at each stress level follows Weibull lifetime distribution with distribution function:

$$F_i(t) = 1 - \exp \left( -\frac{t^\delta}{\theta_i^\delta} \right); \quad \theta > 0, \; \delta > 0; \; 0 \leq t < \infty, \; i = 0, 1, 2, 3 \quad (4.1)$$

where $\delta$ and $\theta_i$ are the shape and scale parameters, respectively.

(iii) At the stress level $x_i$, $i = 1, 2, 3$, the scale parameter, $\theta_i$, is a log-linear function of stress, and there is no interaction between the two stress, i.e.

Step 1: $\log (\theta_1) = \beta_0 + \beta_1 x_{11} + \beta_2 x_{21}$

Step 2: $\log (\theta_2) = \beta_0 + \beta_1 x_{12} + \beta_2 x_{21}$

Step 3: $\log (\theta_3) = \beta_0 + \beta_1 x_{12} + \beta_2 x_{22}$  \quad (4.2)

where $\beta_0, \beta_1, \beta_2$ are unknown model parameters, and obtained from test data.

(iv) For all stress level, the shape $\delta$ is common, constant and independent of time.

From basic assumption (i), the cumulative distribution function (CDF) of a test unit under bivariate SSALT follows the K-M model:

$$F(t) = \begin{cases} 
1 - \exp \left( -\frac{t^\delta}{\theta_1^\delta} \right), & 0 \leq t < \tau_1 \\
1 - \exp \left( -\left( \frac{t^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right), & \tau_1 \leq t < \tau_2 \\
1 - \exp \left( -\left( \frac{t^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right), & \tau_2 \leq t < \infty 
\end{cases} \quad (4.3)$$
By taking the derivative of the equation (4.3), the probability density function (p.d.f.) of a random variable $T$ is obtained as follows:

$$f(t) = \begin{cases} 
\frac{\delta^{\delta-1}}{\theta_1^\delta} \exp \left( -\frac{t^\delta}{\theta_1^\delta} \right), & 0 \leq t < \tau_1 \\
\frac{\delta^{\delta-1}}{\theta_2^\delta} \exp \left( -\left( \frac{t^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right), & \tau_1 \leq t < \tau_2 \\
\frac{\delta^{\delta-1}}{\theta_3^\delta} \exp \left( -\left( \frac{t^\delta - \tau_2^\delta}{\theta_3^\delta} + \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right), & \tau_2 \leq t < \infty 
\end{cases} \quad (4.4)$$

### 4.3 Maximum Likelihood Estimation (MLE) and Fisher Information Matrix

The likelihood function of random observations of lifetime $t_{ij}$, $i = 1, 2, 3$; $j = 1, 2, ..., n_i$ under modified progressive Type-I censored units in equation (2.1), is obtained from the CDF in equation (4.3) and the p.d.f. in equation (4.4) as follows:

$$L = L(t_{ij}|\delta_1, \theta_1, \theta_2, \theta_3)$$

$$= \prod_{i=1}^{n_1} \frac{\delta^{\delta-1}}{\theta_1^\delta} \exp \left( -\frac{t_{ij}^\delta}{\theta_1^\delta} \right) \left[ \exp \left( -\frac{\tau_1^\delta}{\theta_1^\delta} \right) \right]^{R_1^i}$$

$$\times \prod_{j=1}^{n_2} \frac{\delta^{\delta-1}}{\theta_2^\delta} \exp \left( -\left( \frac{t_{ij}^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right) \left[ \exp \left( -\left( \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right) \right]^{R_2^i}$$

$$\times \prod_{j=1}^{n_3} \frac{\delta^{\delta-1}}{\theta_3^\delta} \exp \left( -\left( \frac{t_{ij}^\delta - \tau_2^\delta}{\theta_3^\delta} + \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right) \times$$

$$\left[ \exp \left( -\left( \frac{\tau_3^\delta - \tau_1^\delta}{\theta_3^\delta} + \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta} \right) \right) \right]^{R_3^i} \quad (4.5)$$

After simplification, equation (4.5) can be rewritten as

$$L = \prod_{i=1}^{n_1} \prod_{j=1}^{n_i} \frac{\delta^{\delta-1}}{\theta_i^\delta} \exp \left( -\frac{U_i}{\theta_i^\delta} \right) \quad (4.6)$$
where

\[ U_i = \sum_{j=1}^{n_i} \left( t_{ij}^\delta - R_i^* \right) + \frac{R_i^*}{\pi_i^*} \left( R_i^* - t_{i,-1}^\delta \right), \quad i = 1, 2, 3. \]  (4.7)

Note that \( U_i \) is the total time on test statistic for the \( i^{th} \) stage and \( R_i^* \) is defined in equation (2.6), depending on \( \pi_i^* \). Now, using the assumption (iii) and equation (4.6), the log-likelihood of \((\delta, \beta_0, \beta_1, \beta_2)\) can be written as

\[
\log L = \sum_{i=1}^{3} \{ n_i \log(\delta) \} + (\delta - 1) \sum_{j=1}^{n_i} \log(t_{ij}) - n_1\delta(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}) \\
- n_2\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}) - n_3\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}) \\
- U_1 \exp(\delta(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})) - \frac{U_2 \exp(\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))}{U_3 \exp(\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \]  (4.8)

MLEs \( \hat{\delta}, \hat{\beta}_0, \hat{\beta}_1 \) and \( \hat{\beta}_2 \) can then be obtained by taking the first order partial derivatives of the log-likelihood function in equation (4.8) with respect to \( \delta, \beta_0, \beta_1, \) and \( \beta_2 \) respectively and equating to zero, as follows:

\[
\frac{\partial \log L}{\partial \delta} = \sum_{i=1}^{3} \frac{n_i}{\delta} + \sum_{j=1}^{n_i} \log(t_{ij}) - n_1(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}) \\
- n_2(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}) - n_3(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}) \\
+ \frac{(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})U_1 - \frac{U_2}{\delta}}{\exp(\delta(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21})U_2 - \frac{U_3}{\delta}}{\exp(\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}))} \\
+ \frac{(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22})U_3 - \frac{U_2}{\delta}}{\exp(\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} = 0 \]  (4.9)

\[
\frac{\partial \log L}{\partial \beta_0} = -\delta(n_1 + n_2 + n_3) + \frac{U_1}{\exp(\delta(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} \\
+ \frac{U_2}{\exp(\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}))} + \frac{U_3}{\exp(\delta(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} = 0 \]  (4.10)
\[
\frac{\partial \log L}{\partial \beta_1} = -\delta (x_{11} n_1 + x_{12} n_2 + x_{12} n_3) + \frac{x_{11} U_1}{\exp (\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{x_{12} U_2}{\exp (\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}))} + \frac{x_{12} U_3}{\exp (\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} = 0 \quad (4.11)
\]

\[
\frac{\partial \log L}{\partial \beta_2} = -\delta (x_{21} n_1 + x_{21} n_2 + x_{22} n_3) + \frac{x_{21} U_1}{\exp (\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{x_{21} U_2}{\exp (\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}))} + \frac{x_{22} U_3}{\exp (\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} = 0 \quad (4.12)
\]

Clearly, likelihood equations (4.9) to (4.12) don’t have explicit solutions, so numerical method such as Newton–Raphson iterative procedure is used to get the MLEs \(\hat{\delta}, \hat{\beta}_0, \hat{\beta}_1\) and \(\hat{\beta}_2\). In R software the package \textit{maxLik} is used to directly maximize the log-likelihood function.

In order to construct the confidence intervals for \(\delta, \beta_0, \beta_1,\) and \(\beta_2\) the Hessian matrix is used. The Fisher information matrix can be obtained from negative of second and mixed partial derivatives of the log-likelihood function in equation (4.8) with respect to the parameters \(\delta, \beta_0, \beta_1,\) and \(\beta_2\) are as follows:

\[
F(\delta, \beta_0, \beta_1, \beta_2) = \begin{bmatrix}
-\frac{\partial^2 \log L}{\partial \delta^2} & -\frac{\partial^2 \log L}{\partial \delta \beta_0} & -\frac{\partial^2 \log L}{\partial \delta \beta_1} & -\frac{\partial^2 \log L}{\partial \delta \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_0 \delta} & -\frac{\partial^2 \log L}{\partial \beta_0 \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_0 \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_0 \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_1 \delta} & -\frac{\partial^2 \log L}{\partial \beta_1 \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_1 \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_1 \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_2 \delta} & -\frac{\partial^2 \log L}{\partial \beta_2 \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_2 \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_2 \beta_2}
\end{bmatrix} \quad (4.13)
\]

where, the elements of Fisher information matrix in equation (4.13) are given by
\[
\frac{\partial^2 \log L}{\partial \delta^2} = -\frac{(n_1 + n_2 + n_3)}{\delta^2} \frac{U_1 - 2(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} \left[ (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})^2 U_1 - 2(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}) \frac{dU_1}{d\delta} + \frac{d^2 U_1}{d\delta^2} \right] \\
- \frac{(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21})^2 U_2 - 2(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}) \frac{dU_2}{d\delta} + \frac{d^2 U_2}{d\delta^2}}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}))} \\
- \frac{(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22})^2 U_3 - 2(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}) \frac{dU_3}{d\delta} + \frac{d^2 U_3}{d\delta^2}}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))}
\] (4.14)

\[
\frac{\partial^2 \log L}{\partial \delta \partial \beta_0} = -\frac{(n_1 + n_2 + n_3)}{2} + \frac{U_1 - \delta [(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})U_1 - \frac{dU_1}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} \\
+ \frac{U_2 - \delta [(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22})U_2 - \frac{dU_2}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \\
+ \frac{U_3 - \delta [(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22})U_3 - \frac{dU_3}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))}
\] (4.15)

\[
\frac{\partial^2 \log L}{\partial \delta \partial \beta_1} = -\frac{(x_{11} n_1 + x_{12} n_2 + x_{12} n_3) + x_{11}}{2} + \frac{U_1 - \delta [(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})U_1 - \frac{dU_1}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} \\
+ \frac{x_{12}}{2} \frac{U_2 - \delta [(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21})U_2 - \frac{dU_2}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}))} \\
+ \frac{x_{12}}{2} \frac{U_3 - \delta [(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22})U_3 - \frac{dU_3}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))}
\] (4.16)

\[
\frac{\partial^2 \log L}{\partial \delta \partial \beta_2} = -\frac{(x_{21} n_1 + x_{21} n_2 + x_{22} n_3) + x_{21}}{2} + \frac{U_1 - \delta [(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})U_1 - \frac{dU_1}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} \\
+ \frac{x_{21}}{2} \frac{U_2 - \delta [(\beta_0 + \beta_1 x_{12} + \beta_2 x_{21})U_2 - \frac{dU_2}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{21}))} \\
+ \frac{x_{22}}{2} \frac{U_3 - \delta [(\beta_0 + \beta_1 x_{12} + \beta_2 x_{22})U_3 - \frac{dU_3}{d\delta}]}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))}
\] (4.17)
\[
\frac{\partial^2 \log L}{\partial \beta_0^2} = -\delta^2 \left[ \frac{U_1}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{U_2}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \right] + \frac{U_3}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \tag{4.18}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} = -\delta^2 \left[ \frac{x_{11} U_1}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{x_{12} U_2}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \right] + \frac{x_{12} U_3}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \tag{4.19}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} = -\delta^2 \left[ \frac{x_{21} U_1}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{x_{21} U_2}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \right] + \frac{x_{22} U_3}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \tag{4.20}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_1^2} = -\delta^2 \left[ \frac{x_{11}^2 U_1}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{x_{12}^2 U_2}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \right] + \frac{x_{12}^2 U_3}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \tag{4.21}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} = -\delta^2 \left[ \frac{x_{11} x_{21} U_1}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{x_{12} x_{21} U_2}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \right] + \frac{x_{12} x_{22} U_3}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \tag{4.22}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_2^2} = -\delta^2 \left[ \frac{x_{21}^2 U_1}{\exp(\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}))} + \frac{x_{21}^2 U_2}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \right] + \frac{x_{22}^2 U_3}{\exp(\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22}))} \tag{4.23}
\]

Since the MLEs of the model parameters are not in closed-form, it is not possible to derive the exact confidence intervals (CI), so asymptotic CIs instead of exact CIs are derived here. Now, variance-covariance matrix can be obtained as the inverse of Fisher
information matrix in equation (4.13), as follows

\[
\sum = \begin{bmatrix}
    AV(\hat{\delta}) & AC(\hat{\delta}, \hat{\beta}_0) & AC(\hat{\delta}, \hat{\beta}_1) & AC(\hat{\delta}, \hat{\beta}_2) \\
    AC(\hat{\delta}, \hat{\beta}_0) & AV(\hat{\beta}_0) & AC(\hat{\beta}_0, \hat{\beta}_1) & AC(\hat{\beta}_0, \hat{\beta}_2) \\
    AC(\hat{\beta}_1, \hat{\delta}) & AC(\hat{\beta}_1, \hat{\beta}_0) & AV(\hat{\beta}_1) & AC(\hat{\beta}_1, \hat{\beta}_2) \\
    AC(\hat{\beta}_2, \hat{\delta}) & AC(\hat{\beta}_2, \hat{\beta}_0) & AC(\hat{\beta}_2, \hat{\beta}_1) & AV(\hat{\beta}_2) \\
\end{bmatrix} = \hat{F}^{-1} \tag{4.24}
\]

Then the two sided 100(1 − α)% asymptotic CI of the model parameters δ can be obtained from

\[
\hat{\delta} \pm Z_{\frac{\alpha}{2}} \sqrt{AV(\hat{\delta})} \tag{4.25}
\]

where, \(Z_{\frac{\alpha}{2}}\) is the \((1 - \frac{\alpha}{2})\)th quantile of the standard normal distribution. Similarly, the two sided 100(1 − α)% CIs for parameters β₀, β₁ and β₂ can be obtained.

### 4.4 Optimality Criterion

The optimum test design to determine the optimum stress change time is to minimize the AV of the MLEs of the log of mean time-to-failure at usual stress conditions. Therefore, the asymptotic-variance of β₀, β₁ and β₂ can be obtained from the inverse of Fisher information matrix, \(F_3\), as

\[
F_3 = \begin{bmatrix}
    -\frac{\partial^2 \log L}{\partial \beta_0^2} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \\
    -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_1^2} & -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} \\
    -\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_2^2} \\
\end{bmatrix} \tag{4.26}
\]

The expectations of the elements of \(F_3\), can be obtained by using the following properties of the count and order statistics.
Properties:

(1) The random variable \( n_1 \) has a binomial distribution with parameters \((n, F(\tau_1))\). For \( i = 2, 3 \), given \( n_1, ..., n_{i-1} \), the random variable \( n_i \) has a binomial distribution with parameters \((N_i, F_i(\tau))\), where

\[
F_i(\tau) = \frac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})}
\] (4.27)

is the probability that a unit fail in the interval \((\tau_{i-1}, \tau_i]\) with \( \tau_0 = 0 \), and \( F(\tau) \) is as given in (4.3).

(2) For each \( i = 1, 2, 3 \), the random variables \( t_{i,j} \), \( j = 1, 2, ..., n_i \) constitute a random sample from a truncated Weibull distribution on \((\tau_{i-1}, \tau_i]\) where \( \tau_0 = 0 \), with the p.d.f

\[
f_{i,\tau}(z) = \frac{f_i(z)}{F(\tau_i) - F(\tau_{i-1})} \text{ for } \tau_{i-1} \leq z \leq \tau_i.
\]

Using property (1) and the property of conditional expectation, we get \( E(n_i) = E(N_i) F_i(\tau) \). Let us compute the expectation of \( N_i \) and \( R^*_i \), \( i = 1, 2, 3 \). Beginning with \( E(N_1) = n \) and \( N_{i+1} = N_i - n_i - R^*_i \), we obtain, by induction,

\[
E(N_i) = n \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*),
\] (4.28)

\[
E(R^*_i) = E(N_i) [1 - F_i(\tau)] \pi_i^*
\] (4.29)

Using property (2) and p.d.f. given in equation (4.4), the p.d.f. of truncated Weibull distribution can be obtained as:

\[
f_1(t_{1j}) = \frac{f_1(t_{1j})}{F(\tau_1) - F(\tau_0)} = \frac{\delta t_{1j}^{\delta-1} \exp \left( -\frac{t_{1j}^\delta}{\theta_1^\delta} \right)}{1 - \exp \left( -\frac{\tau_1^\delta}{\theta_1^\delta} \right)}; \ 0 \leq t \leq \tau_1,
\] (4.30)

\[
f_2(t_{2j}) = \frac{\delta t_{2j}^{\delta-1} \exp \left( -\frac{t_{2j}^\delta - \tau_1^\delta}{\theta_2^\delta} - \frac{\tau_1^\delta}{\theta_2^\delta} \right)}{\exp \left( -\frac{\tau_1^\delta}{\theta_2^\delta} \right) - \exp \left( -\frac{\tau_1^\delta}{\theta_1^\delta} \right)}; \ \tau_1 < t \leq \tau_2,
\] (4.31)
and

\[ f_3(t_{3j}) = \frac{\delta \, t_{3j}^{\delta-1}}{\theta_3^\delta} \exp \left( -\frac{t_{3j}^\delta - \tau_2^\delta}{\theta_3^\delta} - \frac{\tau_1^\delta - \tau_2^\delta}{\theta_2^\delta} - \frac{\tau_1^\delta}{\theta_1^\delta} \right), \quad \tau_2 < t \leq \tau_3, \quad (4.32) \]

Hence, the expected value of \( U_i \) given in equation (4.7) can be obtained as

\[ E(U_i) = E_{n_i} \left\{ \sum_{j=1}^{n_i} E \left( t_{ij}^\delta - \tau_{i-1}^\delta \right) | n_i \right\} + \frac{E(R_i^\delta)}{\pi_i^*} \left( \tau_i^\delta - \tau_{i-1}^\delta \right), \quad i = 1, 2, 3. (4.33) \]

Thus, using equation (4.30) and equation (4.33), the expected value of \( U_1 \) is obtained as follows:

\[
E(U_1) = E_{n_1} \left[ \sum_{j=1}^{n_1} E \left( t_{1j}^\delta \right) \right]
= E_{n_1} \left[ n_1 \int_0^{\tau_1} t_{1j}^\delta \frac{\delta \, t_{1j}^{\delta-1}}{\theta_1^\delta} \frac{1 - \exp \left( -\frac{t_{1j}^\delta}{\theta_1^\delta} \right)}{1 - \exp \left( -\frac{\tau_1^\delta}{\theta_1^\delta} \right)} dt_{1j} + (n_1 - n_1) \tau_1^\delta \right]
= E_{n_1} \left[ n_1 \theta_1^\delta \left\{ 1 - \exp \left( -\frac{\tau_1^\delta}{\theta_1^\delta} \right) \right\} \exp \left( -\frac{\tau_1^\delta}{\theta_1^\delta} \right) \right]
= E_{n_1} \left[ n_1 \theta_1^\delta \left\{ 1 - \frac{\tau_1^\delta}{\theta_1^\delta} \right\} + (n_1 - n_1) \tau_1^\delta \right]

E(U_1) = E(n_1) \theta_1^\delta - \frac{E(n_1) \tau_1^\delta \exp \left( -\frac{\tau_1^\delta}{\theta_1^\delta} \right)}{F_1(\tau)} + (n_1 - E(n_1)) \tau_1^\delta
= E(N_1) F_1(\tau) \theta_1^\delta - \frac{E(N_1) F_1(\tau) \tau_1^\delta \left[ 1 - F_1(\tau) \right]}{F_1(\tau)} + (n_1 - E(N_1) F_1(\tau)) \tau_1^\delta
= n \theta_1^\delta F_1(\tau) - n \theta_1^\delta \left[ 1 - F_1(\tau) \right] + n \left[ 1 - F_1(\tau) \right] \tau_1^\delta
= n \theta_1^\delta F_1(\tau) = n \exp (\delta (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21})) F_1(\tau) \quad (4.34)
Similarly, \( E(U_2) \) and \( E(U_3) \) by using equation (4.31) and (4.32) can be obtained, respectively, as

\[
E(U_2) = n \exp (\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{21})) F_2(\tau)[1 - F_1(\tau)](1 - \pi_1^*) \tag{4.35}
\]

\[
E(U_3) = n \exp (\delta (\beta_0 + \beta_1 x_{12} + \beta_2 x_{22})) F_3(\tau)^2 \prod_{j=1}^{2} S_j(\tau)(1 - \pi_j^*) \tag{4.36}
\]

Thus, using equation from (4.34) to (4.36), the expected values of the element of Fisher information matrix \( F_3 \), is obtained as

\[
E \left[ -\frac{\partial^2 \log L}{\partial \beta_0^2} \right] = n\delta^2 \sum_{i=1}^{3} F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1 - \pi_j^*) \tag{4.37}
\]

\[
E \left[ -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} \right] = n\delta^2 \sum_{i=1}^{3} (x_{11} + x_{12} + x_{12}) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1 - \pi_j^*) \tag{4.38}
\]

\[
E \left[ -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \right] = n\delta^2 \sum_{i=1}^{3} (x_{21} + x_{21} + x_{22}) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1 - \pi_j^*) \tag{4.39}
\]

\[
E \left[ -\frac{\partial^2 \log L}{\partial \beta_1^2} \right] = n\delta^2 \sum_{i=1}^{3} (x_{11}^2 + x_{12}^2 + x_{12}^2) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1 - \pi_j^*) \tag{4.40}
\]

\[
E \left[ -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} \right] = n\delta^2 \sum_{i=1}^{3} (x_{11}x_{21} + x_{12}x_{21} + x_{12}x_{22}) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1 - \pi_j^*) \tag{4.41}
\]

\[
E \left[ -\frac{\partial^2 \log L}{\partial \beta_2^2} \right] = n\delta^2 \sum_{i=1}^{3} (x_{21}^2 + x_{21}^2 + x_{22}^2) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1 - \pi_j^*) \tag{4.42}
\]
Therefore, the optimal plan under accelerated conditions that minimize the AV of the MLEs of the log of mean time-to-failure at design stress condition can be obtained as follows:

\[
AV \left( \log \hat{\theta} \right) = AV \left( \hat{\beta}_0 + \hat{\beta}_1 x_{10} + \hat{\beta}_2 x_{20} \right) = \left( 1 \ x_{10} \ x_{20} \right) \hat{F}_3^{-1} \left( \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2 \right) \left( 1 \ x_{10} \ x_{20} \right)^T \tag{4.43}
\]

where, \( \hat{F}_3^{-1} \left( \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2 \right) \) is the asymptotic variance-covariance matrix, which is obtained from inverse of Fisher information matrix in equation (4.26) at MLEs \( (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) \).

### 4.5 Bayesian Inference

According to Sinha (1998), the Jeffrey’s rule for choosing the non-informative prior (NIP) density functions for the independent random variables are considered as uniform distribution, as follows

\[
g_1(\beta_0) = k_1; \quad -c_1 \leq \beta_0 \leq c_1, \tag{4.44}
\]

\[
g_2(\beta_1) = k_2; \quad -c_2 \leq \beta_1 \leq c_2, \tag{4.45}
\]

\[
g_3(\beta_2) = k_3; \quad -c_3 \leq \beta_2 \leq c_3, \tag{4.46}
\]

and

\[
g_4(\delta) = k_4; \quad -c_4 < \delta \leq c_4. \tag{4.47}
\]
Therefore, the joint prior density function of the random variables $\delta$, $\beta_0$, $\beta_1$, and $\beta_2$ is obtained as

$$g(\delta, \beta_0, \beta_1, \beta_2) = \prod_{l=1}^{4} k_l; \quad -c_1 \leq \beta_0 \leq c_1, \quad -c_2 \leq \beta_1 \leq c_2,$$

$$-c_3 \leq \beta_2 \leq c_3, \quad -c_4 \leq \delta \leq c_4. \quad (4.48)$$

The joint posterior distribution of $\delta$, $\beta_0$, $\beta_1$ and $\beta_2$ is obtained by combining the likelihood function in equation (4.5) and the joint prior distribution $g(\delta, \beta_0, \beta_1, \beta_2)$ in equation (4.48), as

$$\pi(\delta, \beta_0, \beta_1, \beta_2|t) \propto L(\delta, \beta_0, \beta_1, \beta_2|t) \times g(\delta, \beta_0, \beta_1, \beta_2) \quad (4.49)$$

The marginal posterior density function and estimates of the parameters $\delta$, $\beta_0$, $\beta_1$ and $\beta_2$ can be obtained based on equation (4.49). The marginal posterior densities $\pi^*_1(\delta|t_{ij})$, $\pi^*_2(\beta_0|t_{ij})$, $\pi^*_3(\beta_1|t_{ij})$ and $\pi^*_4(\beta_2|t_{ij})$ have complicated integrations and it not easy to solve them analytically and sometimes even a numerical integration cannot be directly obtained. However, Markov Chain Monte Carlo (MCMC) simulation, which is particularly useful in high dimensional problems, is the easiest alternative way to get reliable results. In the case, we use WinBUGS software. The 100(1 - $\alpha$)% credible interval for the model parameter $\delta$ is $(L_\delta, U_\delta)$, satisfying

$$P(L_\delta \leq \delta \leq U_\delta) = 1 - \alpha = \int_{L_\delta}^{U_\delta} \pi_1(\delta|t)$$

where $L$ and $U$ denote the lower and upper limits of the intervals, respectively. One can choose $L$ and $U$ as

$$P(\delta \leq U_\delta) = \frac{\alpha}{2} = P(\delta \geq L_\delta) \quad (4.51)$$

Similarly, 100(1 - $\alpha$)% credible intervals for model parameters $\beta_0$, $\beta_1$ and $\beta_2$ can be obtained. Gibbs sampling is used to generate random samples from the marginal posterior distributions of the model parameters $\delta$, $\beta_0$, $\beta_1$ and $\beta_2$ and then the lower and upper limit are the value of $\left( \frac{\alpha}{2} \right)$ th and $\left( 1 - \frac{\alpha}{2} \right)$ th percentile of the sample.
4.6 Numerical Illustrations

In this section, a simulated example is considered to illustrate the derived bivariate SSALT Weibull optimum plan and also to assess the performance of model parameters using both ML and Bayesian method under modified progressive Type-I censoring.

4.6.1 An Example

The following initial values of the parameters are considered for further used in analysis and test design.

\[ n = 40, \ x_{10} = 0.1, \ x_{20} = 0.5, \ x_{11} = 0.4, \ x_{12} = 0.7, \ x_{21} = 1.2, \ x_{23} = 2.5, \ \delta = 1.5, \ \beta_0 = 6, \ \beta_1 = -1, \ \text{and} \ \beta_2 = -0.5 \] with average progressive censoring (ACP) \[ \pi_0 = 0.10, \ 0.20. \]

\[ \hat{\theta}_0 \]

\[ \text{AV(log} \hat{\theta}_0 \text{)} \text{ verses stress change times } \tau_1 \text{ and } \tau_2 \]

\[ \text{Figure 4.1: AV(log} \hat{\theta}_0 \text{)} \text{ verses stress change times } \tau_1 \text{ and } \tau_2 \]

4.6.2 Optimum Plan

For \[ n = 40, \ x_{10} = 0.1, \ x_{20} = 0.5, \ x_{11} = 0.4, \ x_{12} = 0.7, \ x_{21} = 1.2, \ x_{23} = 2.5, \ \delta = 1.5, \ \beta_0 = 6, \ \beta_1 = -1, \ \text{and} \ \beta_2 = -0.5 \] with average progressive censoring (ACP) \[ \pi_0 = 0.10, \ 0.20, \] the optimum values of the stress changing times \( \tau_1 \) and \( \tau_2 \) are obtained by using the variance-optimality criterion. The several values of stress changing
times \( \tau_1 \) and \( \tau_2 \) are chosen to plot the relation between \( AV(\log \hat{\theta}_0) \) vs. \( \tau_1 \) and \( \tau_2 \), and then the optimum value of stress changing times are obtained from the plot as given in Figure 4.1. The values obtained are \( \tau_1^* = 107.5 \) and \( \tau_2^* = 152.0 \) for \( \pi_0 = 0.10 \) and \( \tau_1^* = 85.0 \) and \( \tau_2^* = 116.0 \) for \( \pi_0 = 0.20 \).

### 4.6.3 Simulated Data

The Table 4.1 presents 40 simulated observations based on the initial values \( n = 40 \), \( x_{10} = 0.1 \), \( x_{20} = 0.5 \), \( x_{11} = 0.4 \), \( x_{12} = 0.7 \), \( x_{21} = 1.2 \), \( x_{23} = 2.5 \), \( \delta = 1.5 \), \( \beta_0 = 6 \), \( \beta_1 = -1 \) and \( \beta_2 = -0.5 \), under modified progressive Type-I censoring with optimum stress change times values \( \tau_1^* = 107.5 \) and \( \tau_2^* = 152.0 \) when \( \pi_0 = 0.10 \) and \( \tau_1^* = 85.0 \) and \( \tau_2^* = 116.0 \) when \( \pi_0 = 0.20 \), respectively.

<table>
<thead>
<tr>
<th>\pi_0 = 0.10</th>
<th>Failure times</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stress levels</td>
<td>x_{1} = 0.4</td>
</tr>
<tr>
<td></td>
<td>x_{2} = 0.7</td>
</tr>
<tr>
<td></td>
<td>x_{3} = 1.1</td>
</tr>
<tr>
<td>\pi_0 = 0.20</td>
<td>Failure times</td>
</tr>
<tr>
<td>Stress levels</td>
<td>x_{1} = 0.4</td>
</tr>
<tr>
<td></td>
<td>x_{2} = 0.7</td>
</tr>
<tr>
<td></td>
<td>x_{3} = 1.1</td>
</tr>
</tbody>
</table>

The ordered failure time data given Table 4.1 is used further for parameter estimation.
4.6.4 MLEs of the Model Parameters

The MLEs and 95% confidence intervals of the model parameters $\delta$, $\beta_0$, $\beta_1$ and $\beta_2$ obtained using simulated data given in Table 4.1 under modified progressive Type-I censoring with ACP $\pi_0 = 0.10$ and 0.20, are given below:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MLE</th>
<th>SD</th>
<th>Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>1.7486</td>
<td>0.4361</td>
<td>(0.8910, 2.6033)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>5.7759</td>
<td>0.7667</td>
<td>(4.2733, 7.2786)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-1.0716</td>
<td>1.2553</td>
<td>(-3.5319, 1.3887)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-0.2315</td>
<td>0.2746</td>
<td>(-0.7698, 0.3068)</td>
</tr>
</tbody>
</table>

Table 4.2: The MLEs and 95% confidence intervals of the model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MLE</th>
<th>SD</th>
<th>Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>1.3678</td>
<td>0.2880</td>
<td>(0.8032, 1.9323)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>5.2880</td>
<td>0.7697</td>
<td>(3.7793, 6.7966)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.5736</td>
<td>1.6825</td>
<td>(-2.7242, 3.8714)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-0.6243</td>
<td>0.4814</td>
<td>(-1.5679, 0.3193)</td>
</tr>
</tbody>
</table>

The MLEs of the model parameters are obtained using R software by directly maximizing the log-likelihood function and the confidence intervals estimates are obtained using Hessian matrix.

From Table 4.2, it is observed that the model parameters $\beta_0$, $\beta_1$ and $\beta_2$ have smaller SD for $\pi_0 = 0.10$ in compare to SD for $\pi_0 = 0.20$. On the other hand, shape parameter $\delta$ have smaller SD for $\pi_0 = 0.20$ as compared to $\pi_0 = 0.10$.

4.6.5 Bayes Estimates of the Model Parameters

In this case, we assume two MCMC chains with different initial values (for chain 1: $\delta = 1$, $\beta_0 = 6$, $\beta_1 = -2$, $\beta_2 = -1$ and for chain 2: $\delta = 3$, $\beta_0 = 10$, $\beta_1 = -3$, $\beta_2 = -2$) are run simultaneously in one simulation. Each chain continues for 40000 iterations. From WinBUGS software a simple summary on posterior mean, median and 95% posterior
credible interval can be obtained as displayed in Table 4.3. In Figure 4.3 and 4.5, Gelman-Rubin convergence statistic of $\delta$, $\beta_0$, $\beta_1$ and $\beta_2$ shows that the simulation is believed to be convergent for ACP $\pi_0 = 0.10$, and 0.20, respectively. Figure 4.2 and 4.4 shows marginal posterior distributions of $\delta$, $\beta_0$, $\beta_1$ and $\beta_2$ for ACP $\pi_0 = 0.10$ and 0.20, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>SD</th>
<th>MC error</th>
<th>Median (50%)</th>
<th>Credible Intervals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>1.315</td>
<td>0.1192</td>
<td>0.00227</td>
<td>1.316</td>
<td>(1.078, 1.545)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>9.648</td>
<td>0.3235</td>
<td>0.00493</td>
<td>9.739</td>
<td>(8.814, 9.99)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-2.689</td>
<td>0.8661</td>
<td>0.01394</td>
<td>-2.811</td>
<td>(-3.927, -0.729)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-0.5944</td>
<td>0.3692</td>
<td>0.00697</td>
<td>-0.5576</td>
<td>(-1.386, -0.0349)</td>
</tr>
</tbody>
</table>

The results presented in Table 4.3 shows that posterior parameters $\lambda$, $\beta_0$ and $\beta_2$ have smaller SD for first value of ACP as compared to SD for second value of ACP. But for parameter $\beta_1$ second value of ACP have smaller SD as compared to SD for first value of ACP.

### 4.6.6 Comparative study

We have compared the proposed SSALT model under progressive Type-I censoring with Type-I censoring in terms of the optimum plan, results are given in Table 4.4.

<table>
<thead>
<tr>
<th>SSALT Models</th>
<th>$\pi_0 = 0.10$</th>
<th>$\pi_0 = 0.20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under progressive Type-I censoring</td>
<td>$\tau_1^* = 107.5$ and $\tau_2^* = 152.0$</td>
<td>$\tau_1^* = 85.0$ and $\tau_2^* = 116.0$</td>
</tr>
<tr>
<td>Under Type-I censoring</td>
<td>$\tau_1^* = 135$ and $\tau_2^* = 196$</td>
<td>$\tau_1^* = 135$ and $\tau_2^* = 196$</td>
</tr>
</tbody>
</table>
Table 4.4 shows that the optimal stress change times for proposed optimum test plan under modified progressive Type-I censoring are smaller as compared to Type-I censoring. Thus, proposed plan is better than the plan under Type-I censoring for a given data set.
4.6.7 Sensitivity Analysis

The sensitivity analysis is presented to observe the effect of changes in the value of initially estimated model parameters $\delta$, $\beta_0$, $\beta_1$ and $\beta_2$ on the optimum value of stress change times ($\tau_1^*$ and $\tau_2^*$), and the results are displayed in Table 4.5.

Table 4.5: Sensitivity analysis of bivariate SSALT plan

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Deviation(%)</th>
<th>$\pi_0 = 0.10$</th>
<th>$\pi_0 = 0.20$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\tau_1^*$</td>
<td>$\tau_2^*$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>-5%</td>
<td>107.5</td>
<td>152.0</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>107.5</td>
<td>152.0</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>-5%</td>
<td>105.0</td>
<td>148.0</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>110.0</td>
<td>156.0</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-5%</td>
<td>105.0</td>
<td>148.0</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>107.5</td>
<td>152.0</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-5%</td>
<td>107.5</td>
<td>152.0</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>107.5</td>
<td>152.0</td>
</tr>
</tbody>
</table>

The results in Table 4.5 shows that the proposed optimum test plan derived in this chapter is robust to the deviations in true values of the model parameters. Especially, the test plan is robust to changes in the model parameters $\beta_0$ and $\beta_1$ and strongly robust to the changes in the parameter $\beta_2$ and $\delta$. Therefore, the proposed optimum plan is robust.

4.7 Conclusion

This chapter presents a bivariate SSALT model with two stress variables for Weibull distribution under modified progressive Type-I censoring. For parameter estimation maximum likelihood method and Bayesian approach is considered. In Bayesian analysis, the estimates of the posterior parameters is obtained by using WinBUGS software. The optimum test plan is developed by minimizing the asymptotic variance of the MLEs of the log of mean life at the design stress.
We provide a simulated numerical example to illustrate the proposed methodology under modified progressive Type-I censoring with average censoring proportions (ACP), \( \pi_0 = 0.10 \) and 0.20. In parameter estimation, it is observed that estimates of the model parameters through ML method for \( \pi_0 = 0.10 \) performs better in compare to \( \pi_0 = 0.20 \) and the corresponding 95% confidence intervals are also presented in Table 4.2. The point and interval estimates of model parameters in the Bayesian method are presented in Table 4.3. The simulation used for posterior analysis being converged and it can also be seen in Figures 4.3 and 4.5 under modified progressive Type-I censoring for ACP, \( \pi_0 = 0.10 \) and 0.20, respectively. Sensitivity analysis has been performed and it is observed that the optimum test plan are robust for small deviations in the true value of the model parameters. A comparative study for proposed plan is also performed with Type-I censoring and it is notice that the proposed plan under modified progressive Type-I censoring performs better than the plan under Type-I censoring, for simulated data set. Finally, it is observed that the optimum test design have smaller value of stress change times for ACP, \( \pi_0 = 0.20 \) as compare to first value of ACP, \( \pi_0 = 0.10 \).