Chapter 3

Analysis and Design for 3-Step SSALT under Lomax Distribution

3.1 Introduction

In Chapter 2, a 3-step SSALT model for Weibull failure time under distribution modified progressive Type-I censoring has been considered. The optimum test plan is designed by using ML procedure. The inferences on the SSALT model parameters have been drawn based on point and interval estimation by adopting both the ML and Bayesian methods. In similar fashion, the 3-step SSALT problem for Lomax distribution has been considered in this Chapter. As mentioned in the literature study, a very few researchers have used Lomax life distribution for SSALT problem.

We assume that a non-negative random variable $T$ follows Lomax distribution with two parameter $\lambda$ and $\theta$, its cumulative failure distribution given as

$$F(t; \lambda, \theta) = 1 - \left(1 + \frac{t}{\theta}\right)^{-\lambda}; \quad t > 0, \quad \lambda > 0, \quad \theta > 0, \quad (3.1)$$

where $\lambda$ and $\theta$ are the shape and scale parameters, respectively.
The main purpose of this Chapter is to analyze and design a 3-step SSALT model for Lomax distribution under modified progressive Type-I censored data. It is assumed that the relationship between mean lifetime and stress variable is log-quadratic. The inferences of the model parameters are drawn using the maximum likelihood as well as Bayesian methods. The optimum 3-step SSALT plan for Lomax distribution under modified progressive Type-I censoring is developed with an optimization criteria based on minimizing asymptotic variance of the maximum likelihood estimators of log of mean life at the design stress.

This Chapter is organized as follows. In section 3.2, test procedures and basic assumptions are presented. Section 3.3 describes maximum likelihood method and Fisher information matrix. In Section 3.4, the optimality criterion under modified progressive Type-I censoring is presented. Bayesian estimation procedure is presented in section 3.5. In Section 3.6, a numerical example is provided to illustrate the proposed SSALT Lomax model based on simulated data. Concluding remarks are highlighted for finding of proposed problem in section 3.7.

### 3.2 Test Procedure and Assumptions

The SSALT procedure under modified progressive Type-I censoring is presented in section 2.2 of Chapter 2. The basic assumption for deriving expressions of SSALT with Lomax distribution under progressive Type-I censoring are given below:

**Basic Assumptions**

(i) Under any constant stress \(x_i, i = 1, 2, 3\), the life of a test unit follows the Lomax distribution with CDF given by

\[
F_i(t) = 1 - \left(1 + \frac{t}{\theta_i}\right)^{-\lambda}; \quad t > 0, \; \theta_i > 0. \tag{3.2}
\]

(ii) At the stress level \(x_i, i = 1, 2, 3\), the scale parameter \(\theta_i\), is a log-quadratic function of stress, i.e.
\[ \log(\theta_i) = \beta_0 + \beta_1 x_i + \beta_2 x_i^2, \quad i = 1, 2, 3. \]

where \( \beta_0, \beta_1 \) and \( \beta_2 \) are unknown parameters, and are obtained from test data.

(iii) A cumulative exposure (CE) model holds.

(iv) For all stress levels, the shape \( \lambda \) is common, constant and independent of time and stress.

(v) The lifetimes of test units are independent and identically distributed.

Thus, from the assumptions (i) and (iii), the CDF of 3-step SSALT Lomax model is given by

\[
F(t) = \begin{cases} 
1 - \left(1 + \frac{t}{\theta_1}\right)^{-\lambda}, & 0 \leq t < \tau_1 \\
1 - \left(1 + \frac{t - \tau_3}{\theta_2} + \frac{\tau_2 - \tau_3}{\theta_1}\right)^{-\lambda}, & \tau_1 \leq t < \tau_2 \\
1 - \left(1 + \frac{t - \tau_5}{\theta_3} + \frac{\tau_4 - \tau_5}{\theta_2} + \frac{\tau_3 - \tau_4}{\theta_1}\right)^{-\lambda}, & \tau_2 \leq t < \infty 
\end{cases} 
\] (3.3)

Taking the first derivative of the CDF in equation (3.3), the probability density function (p.d.f.) of the random variable \( T \) is obtained as follows:

\[
f(t) = \begin{cases} 
\frac{\lambda}{\theta_1} \left(1 + \frac{t}{\theta_1}\right)^{-\lambda-1}, & 0 \leq t < \tau_1 \\
\frac{\lambda}{\theta_2} \left(1 + \frac{t - \tau_3}{\theta_2} + \frac{\tau_2 - \tau_3}{\theta_1}\right)^{-\lambda-1}, & \tau_1 \leq t < \tau_2 \\
\frac{\lambda}{\theta_3} \left(1 + \frac{t - \tau_5}{\theta_3} + \frac{\tau_4 - \tau_5}{\theta_2} + \frac{\tau_3 - \tau_4}{\theta_1}\right)^{-\lambda-1}, & \tau_2 \leq t < \infty 
\end{cases} 
\] (3.4)

3.3 Maximum Likelihood Estimation (MLE) and Fisher Information Matrix

The likelihood function of the sample observations under SSALT modified progressive Type-I censoring and Lomax failure time observations \( t_{ij}, i = 1, 2, 3; \ j = 1, 2, \ldots, n_i \)
is obtained as:

\[
L = L(t_{ij}|\lambda, \theta_1, \theta_2, \theta_3) = \prod_{j=1}^{n_1} \frac{\lambda}{\theta_1} \left(1 + \frac{t_{1j}}{\theta_1} \right)^{-(\lambda + 1)} \times \left(1 + \frac{\tau_1}{\theta_1} \right)^{-\lambda R_1^*} \\
\times \prod_{j=1}^{n_2} \frac{\lambda}{\theta_2} \left(1 + \frac{t_{2j} - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 1)} \times \left(1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-\lambda R_2^*} \\
\times \prod_{j=1}^{n_3} \frac{\lambda}{\theta_3} \left(1 + \frac{t_{3j} - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 1)} \times \left(1 + \frac{\tau_3 - \tau_2 + \tau_2 - \tau_1 + \tau_1}{\theta_3 + \theta_2 + \theta_1} \right)^{-\lambda R_3^*} \\
\left(1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-\lambda R_3^*} \\
\left(1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-\lambda R_3^*} \right) \\
(3.5)
\]

where, \( R_i^* \) is defined in equation (2.5). Now, using the assumption (ii), the logarithm of the likelihood function in equation (3.5) is obtained as

\[
\log L = \sum_{i=1}^{3} \log \left\{ \log(\lambda) - (\beta_0 + \beta_1 x_i + \beta_2 x_i^2) \right\} \\
-(\lambda + 1) \sum_{j=1}^{n_1} \log \left(1 + \frac{t_{1j}}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \right) - \lambda R_1^* \log \left(1 + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \right) \\
-(\lambda + 1) \sum_{j=1}^{n_1} \log \left(1 + \frac{t_{2j} - \tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \right) \\
-\lambda R_2^* \log \left(1 + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \right) + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \\
-(\lambda + 1) \sum_{j=1}^{n_1} \log \left(1 + \frac{t_{3j} - \tau_2}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \right) \\
-\lambda R_3^* \log \left(1 + \frac{\tau_3 - \tau_2}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \right) + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_i + \beta_2 x_i^2)}} \right) \\
(3.6)
\]

The MLE for \( \lambda, \beta_0, \beta_1, \beta_2 \) can be obtained from the first partial derivative of the log-likelihood function in equation (3.6) with respect to these parameters and setting them equal to zero, we have
\[
\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{3} n_i \lambda \\
- \sum_{j=1}^{n_1} \log \left( 1 + \frac{t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) \\
-R_1^* \log \left( 1 + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} \right) \\
- \sum_{j=1}^{n_2} \log \left( 1 + \frac{t_{2j} - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) \\
-R_2^* \log \left( 1 + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) \\
- \sum_{j=1}^{n_3} \log \left( 1 + \frac{t_{3j} - \tau_2}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) \\
-R_3^* \log \left( 1 + \frac{\tau_3 - \tau_2}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) = 0 ~ (3.7)
\]

\[
\frac{\partial \log L}{\partial \beta_0} = -(n_1 + n_2 + n_3) \\
+(\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_1 \\
+\lambda R_1^* \left( \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_1 \\
+(\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{t_{2j} - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_2 \\
+\lambda R_2^* \left( \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_2 \\
+(\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{t_{3j} - \tau_2}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_3 \\
+\lambda R_3^* \left( \frac{\tau_3 - \tau_2}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{\tau_2 - \tau_1}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_3 = 0 ~ (3.8)
\]
\[
\frac{\partial \log L}{\partial \beta_1} = -(n_1x_1 + n_2x_2 + n_3x_3)
+ (\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{x_1t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_1
+ \lambda R_1^* \left( \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_1
+ (\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{x_2(t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_2
+ \lambda R_2^* \left( \frac{x_2(\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_2
+ (\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{x_3(t_{3j} - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_3^2)}} + \frac{x_2(t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_3
+ \lambda R_3^* \left( \frac{x_3(\tau_3 - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_3^2)}} + \frac{x_2(\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_3 = 0
\]  

\[
\frac{\partial \log L}{\partial \beta_2} = -(n_1x_1^2 + n_2x_2^2 + n_3x_3^2) + (\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{x_1^2t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_1
+ \lambda R_1^* \left( \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_1
+ (\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{x_2^2(t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_2
+ \lambda R_2^* \left( \frac{x_2^2(\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_2
+ (\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{x_3^2(t_{3j} - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_3^2)}} + \frac{x_2^2(t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) A_3
+ \lambda R_3^* \left( \frac{x_3^2(\tau_3 - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_3^2)}} + \frac{x_2^2(\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_1^2)}} \right) B_3 = 0
\]  

It is observed that likelihood equations in equations (3.7) to (3.10) are not in closed form, analytical solution is not possible. Therefore, numerical iterative procedures can be used to obtain the MLEs (\(\hat{\lambda}, \hat{\beta}_0, \hat{\beta}_1, \) and \(\hat{\beta}_2\)). The Newton–Raphson algorithm implemented in R package maxLik is used to maximize the log-likelihood function. After obtaining MLEs the confidence intervals are constructed using Hessian matrix.
The Fisher information matrix can be obtained as the negative of the second and mixed partial derivatives of the log-likelihood function given in equation (3.6) with respect to the parameters \( \lambda, \beta_0, \beta_1 \) and \( \beta_2 \), as follows

\[
F = \begin{bmatrix}
-\frac{\partial^2 \log L}{\partial \lambda^2} & -\frac{\partial^2 \log L}{\partial \lambda \partial \beta_0} & -\frac{\partial^2 \log L}{\partial \lambda \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \lambda \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_0 \partial \lambda} & -\frac{\partial^2 \log L}{\partial \beta_0^2} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_1 \partial \lambda} & -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_1^2} & -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_2 \partial \lambda} & -\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_2^2}
\end{bmatrix}
\]  

(3.11)

where the elements of the Fisher information matrix are given as

\[
\frac{\partial^2 \log L}{\partial \lambda^2} = -\sum_{i=1}^{3} \frac{n_i}{\lambda^2}
\]  

(3.12)

\[
\frac{\partial^2 \log L}{\partial \lambda \partial \beta_0} = \frac{\partial^2 \log L}{\partial \beta_0 \partial \lambda} = \sum_{j=1}^{n_1} \left( \frac{t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2)} A_1}
+ R_1^* \left( \frac{\tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2)}} \right) B_1
\right)
\]

(3.13)
\[
\frac{\partial^2 \log L}{\partial \lambda \partial \beta_1} = \frac{\partial^2 \log L}{\partial \beta_1 \partial \lambda} = \sum_{j=1}^{n_1} \left( \frac{x_{1j} t_{1j}}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j}^2)} \right) A_1 + R_1^* \left( \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}^2)} \right) B_1 \\
+ \sum_{j=1}^{n_2} \left( \frac{x_{2}(t_{2j} - \tau_1)}{e(\beta_0 + \beta_1 x_{2j} + \beta_2 x_{2j}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j}^2)} \right) A_2 \\
+ R_2^* \left( \frac{x_{2}(\tau_2 - \tau_1)}{e(\beta_0 + \beta_1 x_{22} + \beta_2 x_{22}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) B_2 \\
+ \sum_{j=1}^{n_3} \left( \frac{x_{3}(t_{3j} - \tau_2)}{e(\beta_0 + \beta_1 x_{3j} + \beta_2 x_{3j}^2)} + \frac{x_{2}(\tau_2 - \tau_1)}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) A_3 \\
+ R_3^* \left( \frac{x_{3}(\tau_3 - \tau_2)}{e(\beta_0 + \beta_1 x_{32} + \beta_2 x_{32}^2)} + \frac{x_{2}(\tau_2 - \tau_1)}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) B_3 
\] (3.14)

\[
\frac{\partial^2 \log L}{\partial \lambda \partial \beta_2} = \frac{\partial^2 \log L}{\partial \beta_2 \partial \lambda} = \sum_{j=1}^{n_1} \left( \frac{x_{1j} t_{1j}}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j}^2)} \right) A_1 + R_1^* \left( \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}^2)} \right) B_1 \\
+ \sum_{j=1}^{n_2} \left( \frac{x_{2}^2(t_{2j} - \tau_1)}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j2}^2)} \right) A_2 \\
+ R_2^* \left( \frac{x_{2}^2(\tau_2 - \tau_1)}{e(\beta_0 + \beta_1 x_{22} + \beta_2 x_{22}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) B_2 \\
+ \sum_{j=1}^{n_3} \left( \frac{x_{3}^2(t_{3j} - \tau_2)}{e(\beta_0 + \beta_1 x_{3j} + \beta_2 x_{3j}^2)} + \frac{x_{2}^2(\tau_2 - \tau_1)}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) A_3 \\
+ R_3^* \left( \frac{x_{3}^2(\tau_3 - \tau_2)}{e(\beta_0 + \beta_1 x_{32} + \beta_2 x_{32}^2)} + \frac{x_{2}^2(\tau_2 - \tau_1)}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{x_{11} \tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) B_3 
\] (3.15)

\[
\frac{\partial^2 \log L}{\partial \beta_0^2} = - (\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{t_{1j}}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j}^2)} \right) A_1^2 - \lambda R_1^* \left( \frac{\tau_1}{e(\beta_0 + \beta_1 x_{11} + \beta_2 x_{21}^2)} \right) B_1^2 \\
- (\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{t_{2j} - \tau_1}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{\tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j2}^2)} \right) A_2^2 \\
- \lambda R_2^* \left( \frac{\tau_2 - \tau_1}{e(\beta_0 + \beta_1 x_{22} + \beta_2 x_{22}^2)} + \frac{\tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) B_2^2 \\
- (\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{t_{3j} - \tau_2}{e(\beta_0 + \beta_1 x_{3j} + \beta_2 x_{3j}^2)} + \frac{\tau_2 - \tau_1}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{\tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) A_3^2 \\
- \lambda R_3^* \left( \frac{\tau_3 - \tau_2}{e(\beta_0 + \beta_1 x_{32} + \beta_2 x_{32}^2)} + \frac{\tau_2 - \tau_1}{e(\beta_0 + \beta_1 x_{2j2} + \beta_2 x_{2j2}^2)} + \frac{\tau_1}{e(\beta_0 + \beta_1 x_{1j} + \beta_2 x_{22}^2)} \right) B_3^2 
\] (3.16)
\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} = \frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0} = - (\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{x_1 t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_1^2 - \lambda R_1^* \left( \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_1^2 \\
- (\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{x_2 (t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_2^2 \\
- \lambda R_2^* \left( \frac{x_2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_2^2 \\
- (\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{x_3 (t_{3j} - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_3^2 \\
- \lambda R_3^* \left( \frac{x_3 (\tau_3 - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_3^2 \tag{3.17}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} = \frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_0} = - (\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{x_1^2 t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_1^2 - \lambda R_1^* \left( \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_1^2 \\
- (\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{x_2^2 (t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_2^2 \\
- \lambda R_2^* \left( \frac{x_2^2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_2^2 \\
- (\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{x_3^2 (t_{3j} - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_2^2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_3^2 \\
- \lambda R_3^* \left( \frac{x_3^2 (\tau_3 - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_2^2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_3^2 \tag{3.18}
\]

\[
\frac{\partial^2 \log L}{\partial \beta_1^2} = - (\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{x_1^2 t_{1j}}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_1^2 - \lambda R_1^* \left( \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_1^2 \\
- (\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{x_2^2 (t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_2^2 \\
- \lambda R_2^* \left( \frac{x_2^2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_2^2 \\
- (\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{x_3^2 (t_{3j} - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_2^2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_3^2 \\
- \lambda R_3^* \left( \frac{x_3^2 (\tau_3 - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_2^2 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_1^2 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_3^2 \tag{3.19}
\]

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\[
\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_1} = -(\lambda + 1) \sum_{j=1}^{n_1} \left( \frac{x_1^3 t_{1j}}{e^{(\beta_0 + \beta_j x_1 + \beta_2 x_2^2)}} \right) A_1^2 - \lambda R_1^* \left( \frac{x_1^3 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_1^2 \\
-(\lambda + 1) \sum_{j=1}^{n_2} \left( \frac{x_2^3 (t_{2j} - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_2^3 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_2^2 \\
-\lambda R_2^* \left( \frac{x_2^3 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_2^3 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_2^2 \\
-(\lambda + 1) \sum_{j=1}^{n_3} \left( \frac{x_3^3 (t_{3j} - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_3^3 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_3^3 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) A_3^2 \\
-\lambda R_3^* \left( \frac{x_3^3 (\tau_3 - \tau_2)}{e^{(\beta_0 + \beta_1 x_3 + \beta_2 x_2^2)}} + \frac{x_3^3 (\tau_2 - \tau_1)}{e^{(\beta_0 + \beta_1 x_2 + \beta_2 x_2^2)}} + \frac{x_3^3 \tau_1}{e^{(\beta_0 + \beta_1 x_1 + \beta_2 x_2^2)}} \right) B_3^2
\] (3.20)
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\[ B_1 = \left(1 + \frac{\tau_1}{e^{(\beta_0+\beta_1x_1+\beta_2x_1^2)}}\right)^{-1} \]

\[ B_2 = \left(1 + \frac{\tau_2 - \tau_1}{e^{(\beta_0+\beta_1x_2+\beta_2x_2^2)}} e^{(\beta_0+\beta_1x_1+\beta_2x_1^2)}\right)^{-1} \]

\[ B_3 = \left(1 + \frac{\tau_3 - \tau_1}{e^{(\beta_0+\beta_1x_3+\beta_2x_3^2)}} + \frac{\tau_2 - \tau_1}{e^{(\beta_0+\beta_1x_2+\beta_2x_2^2)}} + \frac{\tau_1}{e^{(\beta_0+\beta_1x_1+\beta_2x_1^2)}}\right)^{-1} \]

Since the MLEs of the model parameters are not in closed-form, it is not possible to derive the exact confidence intervals (CI), so asymptotic CIs instead of exact CIs are derived here. Now, variance-covariance matrix can be obtained as the inverse of Fisher information matrix in equation (3.11), as follows

\[ \sum = \begin{bmatrix} AV(\lambda) & AC(\lambda, \hat{\beta}_0) & AC(\lambda, \hat{\beta}_1) & AC(\lambda, \hat{\beta}_2) \\
AC(\hat{\beta}_0, \lambda) & AV(\hat{\beta}_0) & AC(\hat{\beta}_0, \hat{\beta}_1) & AC(\hat{\beta}_0, \hat{\beta}_2) \\
AC(\hat{\beta}_1, \lambda) & AC(\hat{\beta}_1, \hat{\beta}_0) & AV(\hat{\beta}_1) & AC(\hat{\beta}_1, \hat{\beta}_2) \\
AC(\hat{\beta}_2, \lambda) & AC(\hat{\beta}_2, \hat{\beta}_0) & AC(\hat{\beta}_2, \hat{\beta}_1) & AV(\hat{\beta}_2) \end{bmatrix} = \hat{F}^{-1} \]  

(3.22)

Then the two sided 100(1 - \alpha)% asymptotic CIs of the model parameters \( \lambda, \beta_0, \beta_1 \) and \( \beta_2 \) can be obtained from

\[ \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{AV(\lambda)} \]  

(3.23)

where, \( Z_{\frac{\alpha}{2}} \) is the \( (1 - \frac{\alpha}{2}) \)th quantile of the standard normal distribution. Similarly, the two sided 100(1 - \alpha)% CIs for parameters \( \beta_0, \beta_1 \) and \( \beta_2 \) can be obtained.

### 3.4 Optimality Criterion

The objective in the optimum test criterion is to minimize the asymptotic-variance (AV) of the MLEs of the log of mean time to failure under usual operating conditions,
which is the function of $\beta_0$, $\beta_1$ and $\beta_2$, and in order to obtain the AV of $\beta_0$, $\beta_1$ and $\beta_2$, the Fisher information matrix is defined as

$$F_2 = \begin{bmatrix}
-\frac{\partial^2 \log L}{\partial \beta_0^2} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_1^2} & -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_0} & -\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_2^2}
\end{bmatrix}
$$

(3.24)

To obtain the expectations of the elements of $F_2$, we need the following properties of the count and order statistics:

**Properties:**

(1) The random variable $n_1$ has a binomial distribution with parameters $(n, F(\tau_1))$.

For $i = 2, 3$, given $n_1, \ldots, n_{i-1}$, the random variable $n_i$ has a binomial distribution with parameters $(N_i, F_i(\tau))$, where

$$F_i(\tau) = \frac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})}
$$

(3.25)

is the probability that a unit fail in the interval $(\tau_{i-1}, \tau_i]$ with $\tau_0 = 0$, and $F(\tau_i)$ is as given in (3.3).

(2) For each $i = 1, 2, 3$, the random variables $t_{ij}$, $j = 1, 2, \ldots, n_i$ constitute a random sample from a truncated Lomax distribution on $(\tau_{i-1}, \tau_i]$, where $\tau_0 = 0$, with the p.d.f

$$f_{i, \tau}(z) = \frac{f_i(z)}{F_i(z) - F(\tau_{i-1})} \text{ for } \tau_{i-1} \leq z \leq \tau_i.
$$

where, $N_i = n - \sum_{j=1}^{i-1} (n_j - R_j^*)$ is the number of non-removed unfailed units at the beginning of the $i$th stage. Using property (1) and the property of conditional expectation, we get $E(n_i) = E(N_i)F_i(\tau)$.

Let us compute the expectation of $N_i$ and $R_i^*$, $i = 1, 2, 3$. Beginning with $E(N_1) = n$ and $N_{i+1} = N_i - n_i - R_i^*$, we obtain, by induction,

$$E(R_i^*) = E(N_i) \left[ 1 - F_i(\tau) \right] \pi_i
$$

(3.26)
\[ E(N_i) = n \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*), \quad (3.27) \]

\[ E(n_i) = E(N_i) F_i(\tau) \quad (3.28) \]

Hence, the expectations of the elements of the Fisher information matrix \( F_2 \) are given as:

\[
E \left[ -\frac{\partial^2 \log L}{\partial \beta^2} \right] = \frac{n\lambda(\lambda + 1)}{(\lambda + 2)} C_1 \\
+ n\lambda \pi_1^* \frac{\tau_1}{\theta_1} \left( 1 + \frac{\tau_1}{\theta_1} \right) -(\lambda + 2) \\
+ \frac{n\lambda(\lambda + 1)(1 - \pi_1^*)}{(\lambda + 2)} [C_2 + C_3] \\
+ n\lambda \pi_2^*(1 - \pi_1^*) \left( \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right) \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right) -(\lambda + 2) \\
+ \frac{n\lambda(\lambda + 1)(1 - \pi_1^*)(1 - \pi_2^*)}{(\lambda + 2)} [C_4 + \left( \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right) C_5] \\
+ n\lambda \pi_3^*(1 - \pi_2^*) \left( \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right) \times \\
\left( 1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right) -(\lambda + 2) \quad (3.29) 
\]
\[ E \left[ -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} \right] = E \left[ -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_0} \right] \]
\[ = \frac{n\lambda(\lambda + 1)}{\lambda + 2} [x_1 C_1] + n\lambda \pi_1^* x_1 \frac{\tau_1}{\theta_1} \left( 1 + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 2)} \]
\[ + \frac{n\lambda(\lambda + 1)(1 - \pi_1^*)}{\lambda + 2} [x_2 C_2 + x_1 C_3] \]
\[ + n\lambda\pi_2^*(1 - \pi_1^*) \left( x_2 \frac{\tau_2 - \tau_1}{\theta_2} + x_1 \frac{\tau}{\theta_1} \right) \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right)^{-(\lambda + 2)} \]
\[ + \frac{n\lambda(\lambda + 1)(1 - \pi_1^*)(1 - \pi_2^*)}{\lambda + 2} \left[ x_3 C_4 + \left( x_2 \frac{\tau_2 - \tau_1}{\theta_2} + x_1 \frac{\tau}{\theta_1} \right) C_5 \right] \]
\[ + n\lambda\pi_3^*(1 - \pi_2^*) \left( x_3 \frac{\tau_3 - \tau_2}{\theta_3} + x_2 \frac{\tau_2 - \tau_1}{\theta_2} + x_1 \frac{\tau_1}{\theta_1} \right) \times \]
\[ \left( 1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 2)} \] (3.30)

\[ E \left[ -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \right] = E \left[ -\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_0} \right] \]
\[ = \frac{n\lambda(\lambda + 1)}{\lambda + 2} [x_2^2 C_1] + n\lambda \pi_1^* x_1^2 \frac{\tau_1}{\theta_1} \left( 1 + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 2)} \]
\[ + \frac{n\lambda(\lambda + 1)(1 - \pi_1^*)}{\lambda + 2} [x_2^2 C_2 + x_1^2 C_3] \]
\[ + n\lambda\pi_2^*(1 - \pi_1^*) \left( x_2^2 \frac{\tau_2 - \tau_1}{\theta_2} + x_1^2 \frac{\tau}{\theta_1} \right) \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right)^{-(\lambda + 2)} \]
\[ + \frac{n\lambda(\lambda + 1)(1 - \pi_1^*)(1 - \pi_2^*)}{\lambda + 2} \left[ x_3^2 C_4 + \left( x_2^2 \frac{\tau_2 - \tau_1}{\theta_2} + x_1^2 \frac{\tau}{\theta_1} \right) C_5 \right] \]
\[ + n\lambda\pi_3^*(1 - \pi_2^*) \left( x_3^2 \frac{\tau_3 - \tau_2}{\theta_3} + x_2^2 \frac{\tau_2 - \tau_1}{\theta_2} + x_1^2 \frac{\tau_1}{\theta_1} \right) \times \]
\[ \left( 1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 2)} \] (3.31)

\[ E \left[ -\frac{\partial^2 \log L}{\partial \beta_2^2} \right] = E \left[ -\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} \right] \] (3.32)
\[ E \left[ -\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} \right] = E \left[ -\frac{\partial^2 \log L}{\partial \beta_2 \partial \beta_1} \right] \]
\[ = \frac{n\lambda(\lambda + 1)}{(\lambda + 2)} [x_1^3 C_1] \]
\[ + n\pi_1^* x_1^4 \frac{\tau_1}{\theta_1} \left( 1 + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 2)} \]
\[ + \frac{n\pi_1^* (1 - \pi_1^*)}{(\lambda + 2)} \left[ x_2^4 C_2 + x_3^4 C_3 \right] \]
\[ + n\pi_2^* (1 - \pi_1^*) \left( x_3^4 \frac{\tau_2 - \tau_1}{\theta_2} + x_4^4 \frac{\tau_2 - \tau_1}{\theta_1} \right) \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 2)} \]
\[ + \frac{n\lambda(\lambda + 1)(1 - \pi_1^*)}{(\lambda + 2)} \left[ x_3^4 C_4 + \left( x_3^4 \frac{\tau_2 - \tau_1}{\theta_2} + x_4^4 \frac{\tau_2 - \tau_1}{\theta_1} \right) C_5 \right] \]
\[ + n\pi_3^* (1 - \pi_2^*) \left( x_3^4 \frac{\tau_3 - \tau_2}{\theta_3} + x_4^4 \frac{\tau_3 - \tau_2}{\theta_2} + x_5^4 \frac{\tau_1}{\theta_1} \right) \times \]
\[ \left( 1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda + 2)} \]
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where,

\[
C_1 = \frac{1}{\lambda + 1} - \frac{1}{\lambda + 1} \left( 1 + \frac{\tau_1}{\theta_1} \right)^{-(\lambda+1)} - \frac{\tau_1}{\theta_1} \left( 1 + \frac{\tau_1}{\theta_1} \right)^{-(\lambda+2)}
\]

\[
C_2 = \frac{1}{\lambda + 1} \left( 1 + \frac{\tau_1}{\theta_1} \right)^{-(\lambda+1)} - \frac{1}{\lambda + 1} \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right)^{-(\lambda+2)}
\]

\[
C_3 = \left( 1 + \frac{\tau}{\theta_1} \right)^{-(\lambda+2)} - \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right)^{-(\lambda+2)}
\]

\[
C_4 = \frac{1}{\lambda + 1} \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda+1)} - \frac{1}{\lambda + 1} \left( 1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda+1)}
\]

\[
C_5 = \left( 1 + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau}{\theta_1} \right)^{-(\lambda+2)} - \left( 1 + \frac{\tau_3 - \tau_2}{\theta_3} + \frac{\tau_2 - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1} \right)^{-(\lambda+2)}
\]

\[\theta_i = \exp (\beta_0 + \beta_1 x_i + \beta_2 x_i^2), \quad i = 1, 2, 3.\]

Thus, the optimal plans under accelerated conditions that minimize the AV of the MLEs of the log of mean time-to-failure at design stress condition can be obtained as follows:

\[
\text{AV} \left( \log \hat{\theta}_0 \right) = \text{AV} \left( \hat{\beta}_0 + \hat{\beta}_1 x_0 + \hat{\beta}_2 x_0^2 \right)
\]

\[
= \begin{pmatrix} 1 & x_0 & x_0^2 \end{pmatrix} \hat{F}_2^{-1} \left( \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2 \right) \begin{pmatrix} 1 & x_0 & x_0^2 \end{pmatrix}^T \quad (3.35)
\]

where, \( \hat{F}_2^{-1} \left( \hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2 \right) \) is the asymptotic variance-covariance matrix, which is obtained from inverse of Fisher information matrix given in equation (3.24) at \( (\beta_0, \beta_1, \beta_2) \).
3.5 Bayesian Inference

In this section we consider Bayes estimation of parameters. Based on Sinha (1998), the Jeffrey’s rule for choosing the non-informative prior (NIP) density functions for the independent random variables $\beta_0$, $\beta_1$, $\beta_2$ and $\lambda$ are assumed as uniform distribution, given by

$$g_1(\beta_0) = k_1; \quad -b_1 \leq \beta_0 \leq b_1,$$  \hfill (3.36)

$$g_2(\beta_1) = k_2; \quad -b_2 \leq \beta_1 \leq b_2,$$  \hfill (3.37)

$$g_3(\beta_2) = k_3; \quad -b_3 \leq \beta_2 \leq b_3,$$  \hfill (3.38)

and

$$g_4(\lambda) = k_4; \quad -b_4 \leq \lambda \leq b_4.$$  \hfill (3.39)

Therefore, the joint prior density function of the random parameters is obtained as

$$g(\lambda, \beta_0, \beta_1, \beta_2) = \prod_{l=1}^4 k_l; \quad -b_1 \leq \beta_0 \leq b_1, \quad -b_2 \leq \beta_1 \leq b_2,$$

$$-b_3 \leq \beta_2 \leq b_3, \quad -b_4 \leq \lambda \leq b_4. \hfill (3.40)$$

Using Bayes theorem, the joint posterior distribution of $\lambda$, $\beta_0$, $\beta_1$ and $\beta_2$ is obtained by combining the likelihood function of $(\lambda, \beta_0, \beta_1, \beta_2)$ given in equation (3.5) and the joint prior distribution $g(\lambda, \beta_0, \beta_1, \beta_2)$ in equation (3.40), as

$$\pi(\lambda, \beta_0, \beta_1, \beta_2 | t) \propto L(\lambda, \beta_0, \beta_1, \beta_2 | t) \times g(\lambda, \beta_0, \beta_1, \beta_2) \hfill (3.41)$$

The marginal posterior density function of the parameters $\lambda$, $\beta_0$, $\beta_1$ and $\beta_2$ can be obtained based on equation (3.41). The normalizing constants of the posterior functions
and the marginal posterior densities $\pi_1^*(\lambda|t_{ij})$, $\pi_2^*(\beta_0|t_{ij})$, $\pi_3^*(\beta_1|t_{ij})$ and $\pi_4^*(\beta_2|t_{ij})$ have complicated integrations often analytically not solvable and sometimes even a numerical integration cannot be directly obtained. However, Markov Chain Monte Carlo (MCMC) simulation, which is particularly useful in high dimensional problems, is the easiest alternative way to get reliable results.

The $100(1-\alpha)\%$ Bayesian credible intervals for the model parameters $\lambda$, $\beta_0$, $\beta_1$ and $\beta_2$ are $(L_\lambda, U_\lambda)$, $(L_{\beta_0}, U_{\beta_0})$, $(L_{\beta_1}, U_{\beta_1})$ and $(L_{\beta_2}, U_{\beta_2})$, respectively satisfying

\begin{align*}
P(L_\lambda \leq \lambda \leq U_\lambda) &= 1 - \alpha = \int_{L_\lambda}^{U_\lambda} \pi_1(\lambda|t) dt \\
P(L_{\beta_0} \leq \beta_0 \leq U_{\beta_0}) &= 1 - \alpha = \int_{L_{\beta_0}}^{U_{\beta_0}} \pi_1(\beta_0|t) dt \\
P(L_{\beta_1} \leq \beta_1 \leq U_{\beta_1}) &= 1 - \alpha = \int_{L_{\beta_1}}^{U_{\beta_1}} \pi_1(\beta_1|t) dt \\
P(L_{\beta_2} \leq \beta_2 \leq U_{\beta_2}) &= 1 - \alpha = \int_{L_{\beta_2}}^{U_{\beta_2}} \pi_1(\beta_2|t) dt
\end{align*}

where the $L$ and $U$ denote the lower and upper limits of the intervals, respectively. One can choose $L$ and $U$ as

\begin{align*}
P(\lambda \leq U_\lambda) &= \frac{\alpha}{2} = P(\lambda \geq L_\lambda) \\
P(\beta_0 \leq U_{\beta_0}) &= \frac{\alpha}{2} = P(\beta_0 \geq L_{\beta_0}) \\
P(\beta_1 \leq U_{\beta_1}) &= \frac{\alpha}{2} = P(\beta_1 \geq L_{\beta_1}) \\
P(\beta_2 \leq U_{\beta_2}) &= \frac{\alpha}{2} = P(\beta_2 \geq L_{\beta_2})
\end{align*}

Gibbs sampling is used to generate random samples from the marginal posterior distributions of the model parameters $\lambda$, $\beta_0$, $\beta_1$ and $\beta_2$ and then the lower and upper limit are the value of $\left(\frac{\alpha}{2}\right)$ th and $\left(1 - \frac{\alpha}{2}\right)$ th are percentile of the sample.
3.6 Numerical Illustration

In this section, a simulated example is considered to illustrate the procedure developed in this chapter.

3.6.1 An Example

The following initial values of the parameters are considered.
\[ n = 40, \ x_0 = 0.1, \ x_1 = 0.3, \ x_2 = 0.6, \ x_3 = 1.2, \ \lambda = 1.5, \ \beta_0 = 7, \ \beta_1 = -1, \ \beta_2 = -0.5 \]
with average progressive censoring (ACP) \( \pi_0 = 0.10, \ 0.20 \).

3.6.2 Optimum Plan

The optimum plan for stress change times is determined by assuming \( n = 40, \ x_0 = 0.1, \ x_1 = 0.3, \ x_2 = 0.6, \ x_3 = 1.2, \ \lambda = 1.5, \ \beta_0 = 7, \ \beta_1 = -1, \ \beta_2 = -0.5 \) under modified progressive Type-I censoring with ACP \( \pi_0 = 0.10 \) and 0.20. The several values of stress changing times \( \tau_1 \) and \( \tau_2 \) are chosen to plot the relation between \( AV(\log \hat{\theta}_0) \) vs. \( \tau_1 \) and \( \tau_2 \), and then the optimum value of stress changing times are obtained from the plot as given in Figure 3.1. Hence, obtained value of stress change times are \( \tau_1^* = 310 \) and \( \tau_2^* = 554 \) when \( \pi_0 = 0.10 \) and \( \tau_1^* = 240 \) and \( \tau_2^* = 414 \) when \( \pi_0 = 0.20 \).

3.6.3 Simulated Data

The data given in Table 3.1 is simulated data based on the initial values \( n = 40, \ x_0 = 0.1, \ x_1 = 0.3, \ x_2 = 0.6, \ x_3 = 1.2, \ \lambda = 1.5, \ \beta_0 = 7, \ \beta_1 = -1, \ \beta_2 = -0.5 \), with optimum values of stress changing times \( \tau_1^* = 310 \) and \( \tau_2^* = 554 \) when \( \pi_0 = 0.10 \) and \( \tau_1^* = 240 \) and \( \tau_2^* = 414 \) when \( \pi_0 = 0.20 \), respectively.
Analysis and Design for 3-Step SSALT under Lomax Distribution

Figure 3.1: AV(log ˆθ0) verses stress change times τ1 and τ2

Table 3.1: Simulated data under modified progressive Type-I censoring

<table>
<thead>
<tr>
<th>Stress levels</th>
<th>Failure times</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1 = 1.4</td>
<td>19.71831 58.28802 75.62287 94.97198 143.95849 161.22345 171.57242 210.00235 247.09146 263.84874 286.50673 298.54911</td>
</tr>
<tr>
<td>x2 = 0.7</td>
<td>336.51766 341.41473 344.31648 421.33212 453.41999 480.65165 495.84907 521.36082</td>
</tr>
<tr>
<td>x3 = 1.1</td>
<td>555.12888 560.35749 574.50328 585.69265 586.26672 618.64821 662.15453 663.38508</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Stress levels</th>
<th>Failure times</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1 = 1.4</td>
<td>17.98831 38.84301 39.01320 46.11954 60.49102 62.69209 89.21155 108.51691 110.39855 114.04113 160.07626 165.22532 217.20793</td>
</tr>
<tr>
<td>x2 = 0.7</td>
<td>281.75125 313.87395 317.70356 337.14876 371.11188 391.97284 402.19804</td>
</tr>
<tr>
<td>x3 = 1.1</td>
<td>418.43364 422.33746 434.89091 677.08685</td>
</tr>
</tbody>
</table>

3.6.4 MLEs of the Model Parameters

The MLEs and 95% confidence intervals of the model parameters λ, β0, β1 and β2 obtained using simulated data given in Table 3.2 with ACP π0 = 0.10 and 0.20, are given below:

The MLEs of the model parameters are obtained using R software by directly maximizing the log-likelihood function and the confidence intervals estimates for the...
Table 3.2: The MLEs and 95% confidence intervals of the model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MLE</th>
<th>SD</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>1.53162</td>
<td>0.44151</td>
<td>(0.66627, 2.39697)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>7.56257</td>
<td>0.91376</td>
<td>(5.77161, 9.35352)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-1.16586</td>
<td>2.98859</td>
<td>(-7.02348, 4.69177)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-1.11915</td>
<td>1.95186</td>
<td>(-4.94480, 2.70649)</td>
</tr>
<tr>
<td>$\pi_0 = 0.10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_0 = 0.20$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

model parameters are obtained using Hessian matrix. From Table 3.2, it is observed that the model parameters have smaller standard deviation for $\pi_0 = 0.10$ as compared to $\pi_0 = 0.20$.

3.6.5 Bayes Estimates of Model Parameters

We start with two MCMC chains with different initial values like $\lambda = 1$, $\beta_0 = 6$, $\beta_1 = 0$, $\beta_2 = 0$ for first chain and $\lambda = 3$, $\beta_0 = 10$, $\beta_1 = -1$, $\beta_2 = -2$ for second chain and simultaneously each MCMC chain run for 40000 iterations in single simulation. For checking the convergence, form Figure 3.3 and 3.5 it can be observed that the Gelman-Rubin convergence statistic of $\beta_0$, $\beta_1$, $\beta_2$ and $\lambda$, is converged to one under modified progressive Type-I censoring with ACP $\pi_0 = 0.10$, and 0.20, respectively. Figure 3.2 and 3.4 shows marginal posterior distributions of $\beta_0$, $\beta_1$, $\beta_2$ and $\lambda$ with ACP $\pi_0 = 0.10$, and 0.20, respectively. The summary for the posterior sampling results concerning the unknown parameters ($\lambda$, $\beta_0$, $\beta_1$, $\beta_2$) is displayed in Table 3.3.

In Table 3.3, the model parameters $\beta_1$ and $\beta_2$ have least standard deviation for $\pi_0 = 0.10$ as compared to that for $\pi_0 = 0.20$, while for other two parameters $\lambda$ and $\beta_0$ having least standard deviation for $\pi_0 = 0.20$ as compared to $\pi_0 = 0.10$. 
3.6.6 Comparative study

A numerical comparison between proposed models under modified progressive Type-I censoring and Type-I censoring in terms of the optimum plan is performed, results are given in Table 3.4.
Table 3.3: Estimates of the Posterior parameters $\lambda$, $\beta_0$, $\beta_1$ and $\beta_2$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>SD</th>
<th>MC error</th>
<th>Median (50%)</th>
<th>95% Credible Intervals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>1.4470</td>
<td>1.1840</td>
<td>0.03976</td>
<td>1.0900</td>
<td>(0.1355, 3.8410)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>8.3810</td>
<td>1.1050</td>
<td>0.03940</td>
<td>8.5470</td>
<td>(6.2950, 10.090)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-0.8766</td>
<td>0.5397</td>
<td>0.00775</td>
<td>-0.8316</td>
<td>(-1.906, -0.0419)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-0.7226</td>
<td>0.4974</td>
<td>0.01201</td>
<td>-0.6336</td>
<td>(-1.843, -0.0326)</td>
</tr>
</tbody>
</table>

Table 3.4: Comparative study with Type-I censoring

<table>
<thead>
<tr>
<th>SSALT Models</th>
<th>$\pi_0 = 0.10$</th>
<th>$\pi_0 = 0.20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under progressive Type-I censoring</td>
<td>$\tau_1^* = 310$ and $\tau_2^* = 554$</td>
<td>$\tau_1^* = 240$ and $\tau_2^* = 414$</td>
</tr>
<tr>
<td>Under Type-I censoring</td>
<td>$\tau_1^* = 355$ and $\tau_2^* = 644$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4 shows that the optimal stress change times for proposed optimum test plan under modified progressive Type-I censoring are reduced as compared to Type-I censoring. Thus, proposed plan is better than the plan under Type-I censoring for a given data set.

3.6.7 Sensitivity Analysis

The main purpose of sensitivity analysis is to observe the effect of changes in the initially estimated model parameters $\lambda$, $\beta_0$, $\beta_1$ and $\beta_2$ on the optimum value of stress change times ($\tau_1^*$ and $\tau_2^*$), sensitivity analysis is carried out for all parameters and the results are displayed in Table 3.5.

As shown in Table 3.5, the proposed optimum plan derived in this example is robust to the deviation in the values of model parameters. Especially, the test plan is robust...
### 3.5 Sensitivity Analysis of 3-step SSALT Plan

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Deviation(%)</th>
<th>$\pi_0 = 0.10$</th>
<th>$\pi_0 = 0.20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>-5%</td>
<td>$\tau_1^0 = 320$</td>
<td>$\tau_2^0 = 250$</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>$\tau_1^0 = 300$</td>
<td>$\tau_2^0 = 235$</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>-5%</td>
<td>$\tau_1^0 = 310$</td>
<td>$\tau_2^0 = 240$</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>$\tau_1^0 = 310$</td>
<td>$\tau_2^0 = 225$</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-5%</td>
<td>$\tau_1^0 = 310$</td>
<td>$\tau_2^0 = 255$</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>$\tau_1^0 = 310$</td>
<td>$\tau_2^0 = 235$</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-5%</td>
<td>$\tau_1^0 = 310$</td>
<td>$\tau_2^0 = 240$</td>
</tr>
<tr>
<td></td>
<td>+5%</td>
<td>$\tau_1^0 = 310$</td>
<td>$\tau_2^0 = 225$</td>
</tr>
</tbody>
</table>

The results show that changes in the model parameters $\beta_0$, $\beta_1$, and $\lambda$ affect the optimum stress change times, with $\beta_0$ being the most robust parameter. The pre-estimates have a small effect on the optimum values.

### 3.7 Conclusion

In this chapter, we have derived a 3-step SSALT model for Lomax failure time under modified progressive Type-I censoring. The cumulative exposure model and the mean life that is a log-quadratic function of stress levels are assumed. Maximum likelihood as well as Bayesian Method is used for parameters estimation. The main idea behind the proposed optimum test plan is to determine the optimal stress change times by using variance optimality criteria. Here, we considered modified progressive Type-I censoring with $\pi_0 = 0.10$, and $0.20$.

A simulated numerical example is provided to illustrate the proposed SSALT model. Hence, it is observed that the optimum stress change times under modified progressive Type-I censoring decreases as the amount of ACP increases. For Bayesian analysis, we used MCMC technique and WinBUGS software that enhance the flexibility of Bayesian approach in the proposed model. The simulation used for Bayesian analysis have proved to be converged as can be seen in Figures 3.3 and 3.5. A numerical comparison...
Analysis and Design for 3-Step SSALT under Lomax Distribution

for optimum test plan under modified progressive Type-I censoring with ACP $\pi_0 = 0.10$ and 0.20 and Type-I censoring is performed, and it shows that the proposed plan under modified progressive Type-I censoring perform better than plan under Type-I censoring. Sensitivity analysis shows that the proposed optimum plan is robust for small deviation in initial value of model parameter.