Chapter 2

Mixed Tracking and Projective Synchronization Of 6D Hyperchaotic System Using Active Control

2.1 Introduction

Chaotic behaviours can be undesirable in some engineering, biological and other physical applications (e.g., chaos in the brain [72], cardiac chaos [73]) and therefore it is often desired that chaos should be controlled, so as to improve the system’s performance. The control of chaos is concerned with using some designed control input(s) to modify the characteristics of a parameterized nonlinear system. There might be needed for different components of a chaotic system to follow different trajectories when controlled, therefore, there is need for mixed tracking or control. Since hyperchaotic system shows more complex dynamics, so it is a crucial issue to consider higher dimensional systems to control and synchronize which is considered in this chapter.

In this chapter, firstly we examine mixed tracking control of 6D hyperchaotic system. The designed control functions for the mixed tracking enable each of the system state variables to stabilize at different chosen positions and control each state variables of the system to track different desired smooth functions of time. Next projective synchronization is performed between two identical 6D hyperchaotic systems via active control technique. Also, we have shown that the coupling strength is inversely proportional to the synchronization time. Simulations results in Mathematica are executed to verify the theoretical results. Lastly concluding remarks are given.
2.2 System Description

The Lorenz system [74] was the first chaotic system to be modeled and one of the most widely studied. The original system was modified into a 4D hyperchaotic system by introducing a linear feedback controller to the second equation of the Lorenz system. In 2009, Hu [75] constructed a 5D hyperchaotic Lorenz system by introducing a linear feedback controller and a nonlinear feedback controller to the Lorenz system. In 2015, Yang constructed a 6D system [76], which can generate hyperchaotic attractors with four positive Lyapunov exponents which is obtained by coupling a 1D linear system and a 5D hyperchaotic system. As it is well known, a system with more than one positive Lyapunov exponents is called a hyperchaotic system [77]. Using MATLAB we have computed the six Lyapunov exponents of the 6D hyperchaotic system at time $t = 300$, which are $\lambda_1 = 1.0034$, $\lambda_2 = 0.57515$, $\lambda_3 = 0.32785$, $\lambda_4 = 0.020937$, $\lambda_5 = -0.12087$, $\lambda_6 = -12.4713$ as shown in the Figure (2.1). Since it has four positive Lyapunov exponents, so the system is hyperchaotic.

![Figure 2.1: Lyapunov exponents of the 6D hyperchaotic system](image_url)
The 6D hyperchaotic system is given as follows:

\[
\begin{align*}
\dot{x} &= a(y-x) + u \\
\dot{y} &= cx - y - xz + v \\
\dot{z} &= -bz + xy \\
\dot{u} &= du - xz \\
\dot{v} &= -ky \\
\dot{w} &= hw + ly
\end{align*}
\]  
(2.2.1)

where \( abhkl \neq 0 \) and \( a, b, c, h \) are the constant parameters, \( l \) is the coupling parameter, \( d \) and \( k \) are two control parameters, determining the hyperchaotic behaviors of the system. The coupling parameter \( l \), controllers \( u \) and \( v \) have made the classical Lorenz chaotic system become a 6D hyperchaotic system (2.2.1) with four positive Lyapunov exponents. The system (2.2.1) has a hyperchaotic attractor when \( (a, b, c, d, k, h, l) = (10, 8/3, 28, 2, 8.4, 1, 1) \), as depicted in Figure (2.2).
Figure 2.2: Phase portraits of some 6D hyperchaotic system with parameter values
(a, b, c, d, k, h, l) = (10, 8/3, 28, 2, 8.4, 1, 1)

2.3 Tracking Control of 6D Hyperchaotic System

2.3.1 Design of controllers

We aim to design controllers that enable (2.2.1) to be controlled to a predefined rule. To make it more flexible and adaptable, we employ the controls on each component of the 6D hyperchaotic system to different functions. The system (2.2.1) with the control
parameters is given as

\[
\begin{aligned}
\dot{x} &= a(y - x) + u + \mu_1 \\
\dot{y} &= cx - y - xz + v + \mu_2 \\
\dot{z} &= -bz + xy + \mu_3 \\
\dot{u} &= du - xz + \mu_4 \\
\dot{v} &= -ky + \mu_5 \\
\dot{w} &= hw + ly + \mu_6
\end{aligned}
\]

(2.3.1)

The error function is defined as

\[
\begin{aligned}
e_1 &= x - f_1 \\
e_2 &= y - f_2 \\
e_3 &= z - f_3 \\
e_4 &= u - f_4 \\
e_5 &= v - f_5 \\
e_6 &= w - f_6
\end{aligned}
\]

(2.3.2)

where \( f_i \)'s, \( i = 1 \) to \( 6 \) are the functions to be determined.

Differentiating equation (2.3.2), we have

\[
\begin{aligned}
\dot{e}_1 &= \dot{x} - \dot{f}_1 \\
\dot{e}_2 &= \dot{y} - \dot{f}_2 \\
\dot{e}_3 &= \dot{z} - \dot{f}_3 \\
\dot{e}_4 &= \dot{u} - \dot{f}_4 \\
\dot{e}_5 &= \dot{v} - \dot{f}_5 \\
\dot{e}_6 &= \dot{w} - \dot{f}_6
\end{aligned}
\]

(2.3.3)
Substituting (2.3.2) and (2.3.3) into (2.2.1), we obtain

\[
\begin{aligned}
\dot{e}_1 &= -a(e_1 + f_1) + a(e_2 + f_2) + (e_4 + f_4) + \mu_1 - \dot{f}_1 \\
\dot{e}_2 &= c(e_1 + f_1) - (e_2 + f_2) - xz + (e_5 + f_5) + \mu_2 - \dot{f}_2 \\
\dot{e}_3 &= -b(e_3 + f_3) + xy + \mu_3 - \dot{f}_3 \\
\dot{e}_4 &= d(e_4 + f_4) - xz + \mu_4 - \dot{f}_4 \\
\dot{e}_5 &= -k(e_2 + f_2) + \mu_5 - \dot{f}_5 \\
\dot{e}_6 &= l(e_2 + f_2) + h(e_6 + f_6) + \mu_6 - \dot{f}_6 \\
\end{aligned}
\]

Eliminating non-linear terms in \(e_1, e_2, e_3, e_4, e_5, e_6\) and solving for \(u(t)\), we have

\[
\begin{aligned}
\mu_1 &= af_1 - af_2 - f_4 + \dot{f}_1 + v_1 \\
\mu_2 &= -cf_1 + f_2 + xz - f_3 + \dot{f}_2 + v_2 \\
\mu_3 &= bf_3 - xy + \dot{f}_3 + v_3 \\
\mu_4 &= -df_4 + xz + \dot{f}_4 + v_4 \\
\mu_5 &= kf_2 + f_5 + v_5 \\
\mu_6 &= -hf_6 - lf_2 + \dot{f}_6 + v_6 \\
\end{aligned}
\]

the parameters \(v_i's\), \(i = 1\) to 6) will be obtained later.

Substituting (2.3.5) into (2.3.4), the error dynamics becomes

\[
\begin{aligned}
\dot{e}_1 &= -ae_1 + ae_2 + e_4 + v_1 \\
\dot{e}_2 &= ce_1 - e_2 + e_5 + v_2 \\
\dot{e}_3 &= -be_3 + v_3 \\
\dot{e}_4 &= de_4 + v_4 \\
\dot{e}_5 &= -ke_2 + v_5 \\
\dot{e}_6 &= he_6 + le_2 + v_6 \\
\end{aligned}
\]
Using the active control method, a constant matrix $A$ is chosen which will control the error dynamics (2.3.6) such that the feedback matrix is

$$
\begin{bmatrix}
    v_1 \\
    v_2 \\
    v_3 \\
    v_4 \\
    v_5 \\
    v_6
\end{bmatrix}
= A
\begin{bmatrix}
    e_1 \\
    e_2 \\
    e_3 \\
    e_4 \\
    e_5 \\
    e_6
\end{bmatrix}
$$

(2.3.7)

Thus, the matrix $A$ is chosen to be of the form

$$
A =
\begin{bmatrix}
    \lambda_1 + a & -a & 0 & -1 & 0 & 0 \\
    -c & \lambda_2 & 0 & 0 & -1 & 0 \\
    0 & 0 & \lambda_3 + b & 0 & 0 & 0 \\
    0 & 0 & 0 & \lambda_4 - d & 0 & 0 \\
    0 & k & 0 & 0 & \lambda_5 & 0 \\
    0 & -l & 0 & 0 & 0 & \lambda_6 - h
\end{bmatrix}
$$

(2.3.8)

So that the eigenvalues $\lambda_i$'s, $i = 1$ to 6) are negative.

### 2.3.2 Numerical Simulations

For simulation, the Mathematica is used to solve the differential equation with the following initial conditions $(x, y, z, u, v, w) = (1, 0, 10, 1, 1)$. The system parameters are chosen as $a = 10$, $b = 8/3$, $c = 28$, $d = 2$, $k = 8.4$, $h = 1$, $l = 1$, so the system behaves hyperchaotically as shown in Figure (2.2). We set $f_1 = \beta \sin(t)$, $f_2 = \alpha t^2$, $f_3 = \gamma$, $f_4 = \delta + \theta \sin(t)$ and $f_5 = \xi t$, $f_6 = \beta \cos(t)$ where $\beta = 8$, $\alpha = 0.01$, $\gamma = 2$, $\delta = 4$, $\theta = 60$, $\xi = 0.1$. The results are presented in Figure (2.3). The effectiveness of the control can be seen as various components of the system converges to the preset functions when the controls are applied at time $t \geq 20$. Before the activation of the controls, the system behaves chaotically while the trajectory was changed to the present function on the activation of the control.
Figure 2.3: Components of 6D hyperchaotic Lorenz system when the controls are applied at time $t = 20$ to different functions. The control functions are (a) tracking of $f_1 = \beta \sin(t)$ by $x$ (b) tracking of $f_2 = \alpha t^2$ by $y$ (c) tracking of $f_3 = \gamma$ by $z$ (d) tracking of $f_4 = \delta + \theta \sin(t)$ by $u$ (e) tracking of $f_5 = \xi t$ by $v$ and (f) tracking of $f_6 = \beta \cos(t)$ by $w$. The insert zooms the region under tracking control.
2.4 Projective synchronization of 6D system

We consider the identical hyperchaotic 6D systems with subscript 1 and 2 described as the master (2.4.1) and the slave (2.4.2) systems respectively,

\[
\begin{aligned}
\dot{x}_1 &= a(y_1 - x_1) + u_1 \\
\dot{y}_1 &= cx_1 - y_1 - x_1z_1 + v_1 \\
\dot{z}_1 &= -bz_1 + x_1y_1 \\
\dot{u}_1 &= du_1 - x_1z_1 \\
\dot{v}_1 &= -ky_1 \\
\dot{w}_1 &= hw_1 + ly_1
\end{aligned}
\]

\[
\begin{aligned}
\dot{x}_2 &= a(y_2 - x_2) + u_2 + \mu_1 \\
\dot{y}_2 &= cx_2 - y_2 - x_2z_2 + v_2 + \mu_2 \\
\dot{z}_2 &= -bz_2 + x_2y_2 + \mu_3 \\
\dot{u}_2 &= du_2 - x_2z_2 + \mu_4 \\
\dot{v}_2 &= -ky_2 + \mu_5 \\
\dot{w}_2 &= hw_2 + ly_2 + \mu_6
\end{aligned}
\]

where \(x_i, y_i, z_i, u_i, v_i\) and \(w_i\), \((i = 1, 2)\) are the state vectors and \(a, b, c, d, h, k, l\) are the parameters of the system and \(\mu = [\mu_1 \ \mu_2 \ \mu_3 \ \mu_4 \ \mu_5 \ \mu_6]^T\) is the nonlinear controller to be designed.

The projective synchronization error is defined as

\[
\begin{aligned}
\dot{e}_1 &= x_2 - \alpha x_1 \\
\dot{e}_2 &= y_2 - \alpha y_1 \\
\dot{e}_3 &= z_2 - \alpha z_1 \\
\dot{e}_4 &= u_2 - \alpha u_1 \\
\dot{e}_5 &= v_2 - \alpha v_1 \\
\dot{e}_6 &= w_2 - \alpha w_1
\end{aligned}
\]

\[
\begin{aligned}
\dot{e}_1 &= x_2 - \alpha x_1 \\
\dot{e}_2 &= y_2 - \alpha y_1 \\
\dot{e}_3 &= z_2 - \alpha z_1 \\
\dot{e}_4 &= u_2 - \alpha u_1 \\
\dot{e}_5 &= v_2 - \alpha v_1 \\
\dot{e}_6 &= w_2 - \alpha w_1
\end{aligned}
\]
where $\alpha$ is the scaling factor.

### 2.4.1 Design of Control Function

The error dynamics is obtained as

\[
\begin{align*}
    \dot{e}_1 &= -ae_1 + ae_2 + e_4 + \mu_1 \\
    \dot{e}_2 &= ce_1 - e_2 + e_5 - x_2z_2 + \alpha x_1z_1 + \mu_2 \\
    \dot{e}_3 &= -be_3 + x_2y_2 - \alpha x_1y_1 + \mu_3 \\
    \dot{e}_4 &= de_4 - x_2z_2 + \alpha x_1z_1 + \mu_4 \\
    \dot{e}_5 &= -ke_5 + \mu_5 \\
    \dot{e}_6 &= h e_6 + le_2 + \mu_6
\end{align*}
\]  

(2.4.4)

To achieve asymptotic stability of system (2.4.3), we eliminate terms which cannot be expressed as linear terms in $e_1$, $e_2$, $e_3$, $e_4$, $e_5$, $e_6$ as follows:

\[
\begin{align*}
    \mu_1 &= v_1 \\
    \mu_2 &= x_2z_2 - \alpha x_1z_1 + v_2 \\
    \mu_3 &= -x_2y_2 + \alpha x_1y_1 + v_3 \\
    \mu_4 &= x_2z_2 - \alpha x_1z_1 + v_4 \\
    \mu_5 &= v_5 \\
    \mu_6 &= v_6
\end{align*}
\]  

(2.4.5)

The parameters $v_i's$, ($i = 1$ to 6) will be obtained later.

Substituting (2.4.5) into (2.4.4), we get

\[
\begin{align*}
    \dot{e}_1 &= -ae_1 + ae_2 + e_4 + v_1 \\
    \dot{e}_2 &= ce_1 - e_2 + e_5 + v_2 \\
    \dot{e}_3 &= -be_3 + v_3 \\
    \dot{e}_4 &= de_4 + v_4 \\
    \dot{e}_5 &= -ke_5 + v_5 \\
    \dot{e}_6 &= h e_6 + le_2 + v_6
\end{align*}
\]  

(2.4.6)
Using the active control method, a constant matrix $A$ is chosen such that the error dynamics (2.4.4) is controlled. For that the feedback matrix is

$$
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
\end{bmatrix}
= A 
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6 \\
\end{bmatrix}
$$

(2.4.7)

with

$$
A = 
\begin{bmatrix}
\lambda_1 + a & -a & 0 & -1 & 0 & 0 \\
-c & \lambda_2 & 0 & 0 & -1 & 0 \\
0 & 0 & \lambda_3 + b & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 - d & 0 & 0 \\
0 & k & 0 & 0 & \lambda_5 & 0 \\
0 & -l & 0 & 0 & 0 & \lambda_6 - h \\
\end{bmatrix}
$$

(2.4.8)

In (2.4.7) the six eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, $\lambda_5$, $\lambda_6$ are chosen to be negative in order to achieve a stable projective synchronization between two identical 6D hyperchaotic system.

### 2.4.2 Numerical Simulations

Numerical solutions were carried out in Mathematica to solve systems (2.4.1) and (2.4.2) with the following initial conditions $(x_1, y_1, z_1, u_1, v_1, w_1) = (1, 0, 1, 10, 1, 1)$ and $(x_2, y_2, z_2, u_2, v_2, w_2) = (4, 4, 3, 4, 5, -2)$. The system parameters are chosen as $a = 10, b = 8/3, c = 28, d = 2, h = 8.4, k = 1, l = 1$, so the system behaves hyperchaotically as shown in Figure (2.2). The error dynamics of the system when the controls are activated at time $t \geq 10$ as shown in Figure (2.4). Then, the synchronization errors between the two systems are seen to converge to zero. Figure (2.5) shows the dynamics of the state variables of the systems when compared after activation of control.
at time $t = 0$ and value of $\alpha = 2.0$. The trajectory of the master system is seen to be twice that of the slave as expected. A quantity called synchronization time which gives a value of speed of synchronization when the error between the two synchronization approaches zero was also computed. Figure 2.6 depicts the time it takes for synchronization to occur as the coupling strength is increased. From the graph, an exponential decrease is seen. This synchronization time-coupling strength graph can be used as a measure of the speed of synchronization. In effect, for the system under consideration the synchronization time is seen to decrease with increasing coupling strength.

![Graphs showing synchronization dynamics](image)

Figure 2.4: Error dynamics of the state variables when the control functions are activated for $t \geq 10$
Figure 2.5: Dynamics of the state variables when the control functions are activated at $t = 0$ and $\alpha = 2.0$
Figure 2.6: Dependence of the synchronization time on the coupling parameter $\lambda$

### 2.5 Conclusion

This work demonstrates that chaos synchronization between identical 6D hyperchaotic systems using active control method is achieved. We have used the same control technique to enable tracking of a desired trajectories which are achieved in a systematic way. Numerical simulations are used to verify the effectiveness of the proposed active control technique. Computational and analytical results are in excellent agreement.