Chapter 4

Hybrid Function Projective Synchronization of Chaotic Systems via Adaptive Control

4.1 Introduction

Synchronization and anti-synchronization between a Lü chaotic system, which is derived from the unified chaotic system and Bhalekar-Gejji chaotic system, which cannot be derived from the unified chaotic system have been established [79] on 2014. Also, in most of the reported papers of HFPS between the two chaotic systems is being achieved where both of them are a part of the unified chaotic system. But Bhalekar-Gejji chaotic system and Pehlivan chaotic system does not fit in unified chaotic system. Very few manuscript have been reported where HFPS between two chaotic systems is performed when none of the chaotic system is a part of the unified chaotic system and the present chapter is one of them. Motivated by above discussions in this chapter, we have considered the problem of HFPS via adaptive control where hybrid projective synchronization (HPS), projective synchronization (PS), complete synchronization (CS) and anti-synchronization (AS) are the subcases of HFPS. In this chapter, firstly we discuss problem statement for synchronization of chaotic systems. Next fundamental dynamical properties such as phase portrait, Lyapunov exponents, Kaplan-Yorke dimension, equilibrium points and invariance of the Bhalekar-Gejji and Pehlivan chaotic systems are described. Then, the hybrid function projective synchronization (HFPS) of Bhalekar-Gejji and Pehlivan chaotic systems via adaptive control is performed. Theoretical results are
supported with the numerical simulations. Finally concluding remarks are given.

4.2 Problem Statement for Synchronization of Chaotic System

Consider a chaotic system as a master system having state vector $U_m \in \mathbb{R}^n$ and $P \in \mathbb{R}^{n \times n}$ is system matrix given by

$$
U_m = PU_m + f(U_m)
$$

(4.2.1)

where $f(U_m) : \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear part of the system.

Consider another chaotic system as a slave system having state vector $V_s \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ is system matrix with controller given by

$$
V_s = QV_s + g(V_s) + \sigma(U_m, V_s)
$$

(4.2.2)

where $g(V_s) : \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear part of the slave system and $\sigma$ is the adaptive controller added to slave system.

For HFPS, the error $e \in \mathbb{R}^n$ between states $U_m$ and $V_s$ is defined as:

$$
e = V_s - A(t)U_m
$$

(4.2.3)

where $A(t) = \text{diag}(\eta_1(t), \eta_2(t), ..., \eta_n(t))$ is the diagonal matrix and $\eta_i(t) : \mathbb{R}^n \to \mathbb{R}$; $(i = 1, 2, ..., n)$ are continuously differentiable and bounded functions, $\eta_i(t) \neq 0 \forall t$.

From (4.2.1) - (4.2.3), we can write error dynamics as

$$
\dot{e} = QV_s + g(V_s) + \sigma(U_m, V_s) - A(t)P(U_m) - A(t)f(U_m)
$$

Therefore, for HFPS the goal is to determine controller $\sigma(U_m, V_s)$, so that $\lim_{t \to \infty} \| e(t) \| = 0$, $\forall \ e \in \mathbb{R}^n$
4.3 Fundamental Dynamical Properties of the Systems

4.3.1 Bhalekar-Gejji Chaotic System

The dynamics of Bhalekar-Gejji system [80] is given by

\[
\begin{align*}
\dot{u}_1 &= \omega u_1 - u_2^2 \\
\dot{u}_2 &= \mu (u_3 - u_2) \\
\dot{u}_3 &= \alpha u_2 - \beta u_3 + u_1 u_2
\end{align*}
\]  \hspace{1cm} (4.3.1)

where \(u_1, u_2, u_3 \in \mathbb{R}\) are the state variables and \(\omega, \mu, \alpha, \beta \in \mathbb{R}\) are the parameters.

For \(\omega = -2.667, \mu = 10, \alpha = 27.3, \beta = 1\), the system shows chaotic behaviour. The Lyapunov exponents for the given parameters values are \(\gamma_1 = 0.95823, \gamma_2 = -0.000045403, \gamma_3 = -14.6171\) as shown in Figure (4.1). Thus the maximum Lyapunov exponent (MLE) of the system 4 is obtained as \(\gamma_1 = 0.95823\).

![Figure 4.1: Lyapunov exponents of the Bhalekar-Gejji chaotic system](image)

Further, since \(\sum_{i=1}^{3} \gamma_i = -13.6589 < 0\), so the system is dissipative.

Also, the Kaplan-Yorke dimension of the system is calculated as

\[
D_{KY} = 2 + \frac{\gamma_1 + \gamma_2}{|\gamma_3|} = 2.0655
\]
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This means that the chaotic attractor has fractal dimension.

The system has three equilibrium points \((0, 0, 0), (-26.3, 8.375, 8.375)\) and \((-26.3, -8.375, -8.375)\) and the corresponding eigenvalues are \((11.6245, -22.6245, -2.667), (1.5975 + 8.9803i, 1.5975 - 8.9803i, -16.8613)\) and \((1.5975 + 8.9803i, 1.5975 - 8.9803i, -16.8613)\) respectively.

Since corresponding to each equilibrium point there is at least one positive real part of eigenvalue so, each equilibrium point is unstable. Since the system has three equilibrium points, so the system orbits around the three unstable equilibrium points.

Since the system is invariant under the transformation \((u_1, u_2, u_3) \rightarrow (u_1, -u_2, -u_3)\), so the system is symmetric about \(u_1\)-axis.

This system generate the two-scroll butterfly shaped chaotic attractors simultaneously as shown in Figure 4.2. The two-scroll attractor in this system is formed from two one-scroll attractor [82].

![3D Phase Portrait and Projections](image)

Figure 4.2: Phase portrait of the Bhalekar-Gejji chaotic system
4.3.2 Pehlivan Chaotic System

The dynamics of Pehlivan system [81] is given by

\[
\begin{align*}
\dot{v}_1 &= v_2 - v_1 \\
\dot{v}_2 &= a v_2 - v_1 v_3 \\
\dot{v}_3 &= v_1 v_2 - b
\end{align*}
\]

(4.3.2)

where \( v_1, v_2, v_3 \in \mathbb{R} \) are the state variables and \( a, b \in \mathbb{R} \) are the parameters.

For \( a = 0.5 \), \( b = 0.5 \), the system exhibits chaotic behaviour. The Lyapunov exponents for the given parameters values are \( \gamma_1 = 0.20392 \), \( \gamma_2 = -0.010771 \), \( \gamma_3 = -0.69315 \) are shown in Figure (4.3). Thus, the maximum Lyapunov exponent (MLE) of the system (4.3.2) is obtained as \( \gamma_1 = 0.20392 \).

![Figure 4.3: Lyapunov exponents of the Pehlivan chaotic system](image)

Further, since \( \sum_{i=1}^{3} \gamma_i = -0.50 < 0 \), so the system is dissipative.

Also, the Kaplan-Yorke dimension of the system is calculated as

\[
D_{KY} = 2 + \frac{\gamma_1 + \gamma_2}{|\gamma_3|} = 2.2686
\]

This means that the chaotic attractor has fractal dimension.

The system has two equilibrium points \((-0.7071, -0.7071, 0.5)\) and \((0.7071, 0.7071, 0.5)\).
and the eigenvalue \((-1, 0.25 + 0.9682i, 0.25 - 0.9682i)\) for both equilibrium points are same. Since corresponding to each equilibrium point there is at least one positive real part of eigenvalue so, each equilibrium point is unstable. Since the system has two equilibrium points, so the system orbits around the two unstable equilibrium points. As the system is invariant under the transformation \((v_1, v_2, v_3) \rightarrow (-v_1, -v_2, v_3)\), therefore the system is symmetric about \(v_3\)-axis.

This system generate the complex 2-scroll chaotic attractors simultaneously as shown in Figure (4.4).

![Phase portrait of the Pehlivan chaotic system](image)

**Figure 4.4:** Phase portrait of the Pehlivan chaotic system

Both the systems are neither diffeomorphic nor topological equivalent with the chaotic systems within unified chaotic system [80, 81].
4.4 HFPS of Bhalekar-Gejji and Pehlivan Chaotic systems via adaptive control

Our purpose herein is to achieve HFPS between Bhalekar-Gejji and Pehlivan chaotic systems by the adaptive control method.

We consider Bhalekar-Gejji system as the master system which is described in (4.3.1). Pehlivan system is considered as slave system which is described in (4.3.2). Now adding controller to the slave system, we get

\[
\begin{align*}
\dot{v}_1 &= v_2 - v_1 + \sigma_1 \\
\dot{v}_2 &= av_2 - v_1v_3 + \sigma_2 \\
\dot{v}_3 &= v_1v_2 - b + \sigma_3
\end{align*}
\]

(4.4.1)

where \( \sigma_i \), \( i = 1,2,3 \) are controllers which are to be determined such that the master system and the slave system can be synchronized in the sense of HFPS via adaptive control. The synchronization error \( e \in \mathbb{R}^3 \) for HFPS is defined as

\[
e_i = v_i - \eta_i(t)u_i, \quad \text{where} \quad i = 1,2,3
\]

(4.4.2)

and \( \eta_i'(t) \)s\( i = 1,2,3 \) are the non-zero scaling functions which are continuously differentiable and bounded. The time derivative of error states (4.4.2) is

\[
\dot{e}_i = v_i - \dot{\eta}_i(t)u_i - \eta_i(t)\dot{u}_i
\]

(4.4.3)

Substituting (4.3.1) and (4.4.1) into (4.4.3), we obtain

\[
\begin{align*}
\dot{e}_1 &= v_2 - v_1 - \tilde{\eta}_1u_1 - \alpha \eta_1(t)u_1 + \eta_1(t)u_2^2 + \sigma_1 \\
\dot{e}_2 &= av_2 - v_1v_3 - \tilde{\eta}_2(t)u_2 - \mu \eta_2(t)u_3 + \mu \eta_2(t)u_2 + \sigma_2 \\
\dot{e}_3 &= v_1v_2 - b - \tilde{\eta}_3(t)u_3 - \alpha \eta_3(t)u_2 + \beta \eta_3(t)u_3 - \eta_3(t)u_1u_2 + \sigma_3
\end{align*}
\]

(4.4.4)

In order to achieve HFPS between Bhalekar-Gejji and Pehlivan chaotic systems with fully uncertain parameters for arbitrary initial conditions and identifying the unknown parameters simultaneously, we design the appropriate controllers \( \sigma_i(t) \) \( i = 1,2,3 \) and
corresponding parameter update rule such that the error dynamical system (4.4.4) is asymptotically stable in the origin, namely, HFPS between the master system (4.3.1) and the slave system (4.4.1) is achieved and the unknown parameters can be estimated at the same time. Hence, the synchronization problem between the master and slave systems becomes the stability problem of the error dynamical system (4.4.4). Then we have the following main theorem.

**Theorem 4.4.1.** For a given scaling function matrix \( A(t) \), HFPS between systems (4.3.2) and (4.4.1) can be achieved and the uncertain parameters \( a, b, \omega, \mu, \alpha \) and \( \beta \) can be identified if the controllers are designed as follows:

\[
\begin{align*}
\sigma_1 &= -v_2 + v_1 + \eta_1(t)u_1 + \dot{\omega}\eta_1(t)u_1 - \eta_1(t)u_2^2 - k_1 e_1 \\
\sigma_2 &= -\dot{\alpha}v_2 + v_1 v_3 + \eta_2(t)u_2 + \dot{\mu}\eta_2(t)u_2 - \dot{\mu}\eta_2(t)u_2 - k_2 e_2 \\
\sigma_3 &= -v_1 v_2 + \dot{\beta} + \eta_3(t)u_3 + \dot{\alpha}\eta_3(t)u_3 - \dot{\beta}\eta_3(t)u_3 + \eta_3(t)u_1 u_2 - k_3 e_3
\end{align*}
\]

and the parameters update rules are designed as below:

\[
\begin{align*}
\dot{\alpha} &= v_2 e_2 - k_4 e_\alpha \\
\dot{\beta} &= -e_3 - k_5 e_\beta \\
\dot{\omega} &= -\eta_1(t)u_1 e_1 - k_6 e_\omega \\
\dot{\mu} &= -\eta_2 u_3 e_2 + \eta_2 u_2 e_2 - k_7 e_\mu \\
\dot{\alpha} &= -\eta_3 u_2 e_3 - k_8 e_\alpha \\
\dot{\beta} &= \eta_3 u_3 e_3 - k_9 e_\beta
\end{align*}
\]

where the control gains \( k_i > 0 \) (\( i = 1, 2, \ldots, 9 \)), \( \dot{\alpha}, \dot{\beta}, \dot{\omega}, \dot{\mu}, \dot{\alpha} \) and \( \dot{\beta} \) are the estimated variables of the unknown parameters, \( e_\alpha = \dot{\alpha} - a, e_\beta = \dot{\beta} - b, e_\omega = \dot{\omega} - \omega, e_\mu = \dot{\mu} - \mu, e_\alpha = \dot{\alpha} - \alpha \) and \( e_\beta = \dot{\beta} - \beta \), are the corresponding parameter errors.

**Proof.** We choose Lyapunov function in such a way that it satisfies the conditions of Lyapunov stability theory for the parameter update rules designed above, which in turn shows the stability of error dynamical system and hence required synchronization is
obtained. So, we choose the following Lyapunov function candidate for the error system (4.4.4) as follows:

\[ V(t) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_\omega^2 + e_\mu^2 + e_\alpha^2 + e_\beta^2) \]

Obviously, \( V(t) > 0 \). The time derivative of \( V(t) \) along the trajectories of the error system (4.4.4) is

\[
\dot{V}(t) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_a\dot{e}_a + e_b\dot{e}_b + e_\omega\dot{e}_\omega + e_\mu\dot{e}_\mu + e_\alpha\dot{e}_\alpha + e_\beta\dot{e}_\beta \\
= e_1(v_2 - v_1 - \eta_1 u_1 - \omega\eta_1(t) u_1 + \eta_1(t) u_2^2 + \sigma_1) \\
+ e_2(\alpha v_2 - v_1 v_3 - \eta_2(t) u_2 - \mu\eta_2(t) u_3 + \mu\eta_2(t) u_2 + \sigma_2) \\
+ e_3(\nu_1 v_2 - b + \eta_3(t) u_3 - \alpha\eta_3(t) u_2 - \beta\eta_3(t) u_3 - \eta_3(t) u_1 u_2 + \sigma_3) \\
+ e_a\dot{e}_a + e_b\dot{e}_b + e_\omega\dot{e}_\omega + e_\mu\dot{e}_\mu + e_\alpha\dot{e}_\alpha + e_\beta\dot{e}_\beta 
\]  
(4.4.7)

Substituting (4.4.5) and (4.4.6) into (4.4.7), we obtain

\[
\dot{V}(t) = e_1(e_\omega\eta_1(t) u_1 - k_1 e_1) + e_2(-e_\alpha v_2 + e_\mu\eta_2(t) u_3 - e_\mu\eta_2(t) - k_2 e_2) \\
+ e_3(e_b + e_\alpha\eta_3 u_2 - e_\beta\eta_3 u_3 - k_3 e_3) + e_a(v_2 e_2 - k_4 e_a) \\
+ e_b(-e_3 - k_5 e_b) + e_\omega(-\eta_1(t) u_1 e_1 - k_6 e_\omega) + e_\mu(-\eta_2 u_3 e_2) \\
+ \eta_2 u_2 e_2 - k_7 e_\mu) + e_\alpha(-\eta_3 u_3 e_3 - k_8 e_\alpha) + e_\beta(\eta_3 u_3 e_3 - k_9 e_\beta) \\
= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_a^2 - k_5 e_b^2 - k_6 e_\omega^2 - k_7 e_\mu^2 - k_8 e_\alpha^2 - k_9 e_\beta^2 \\
= -e Ke < 0
\]

where \( e = (e_1, e_2, e_3, e_a, e_b, e_\omega, e_\mu, e_\alpha, e_\beta) \) and \( K = \text{diag}(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9) \).

Since \( \dot{V}(t) < 0 \), on the basis of the Lyapunov stability theory, the error vector \( e \) asymptotically converges to zero, i.e., \( \lim_{t \to \infty} ||e(t)|| = 0 \). It implies that the master and slave systems are globally and asymptotically hybrid function projective synchronized, and the unknown parameters are also estimated simultaneously. This completes the proof.
4.5 Numerical Simulations

Numerical simulations are performed to illustrate the validity and feasibility of the presented synchronization technique. The parameter of the Bhalekar-Gejji system are chosen as $\omega = -2.667$, $\mu = 10$, $\alpha = 27.3$, $\beta = 0.5$ and the parameter of the pehlivan system are chosen as $a = 0.5$, $b = 0.5$, so that the systems behave chaotically without the controller. The initial conditions of the Bhalekar-Gejji system are chosen to as $u_1 = -2$, $u_2 = 2$, $u_3 = 3$ and the Pehlivan system are chosen as $v_1 = 0.01$, $v_2 = 0.01$, $v_3 = -0.01$.

Case 1. We arbitrarily choose the scaling functions $\eta_1 = 2 + 3\sin(t)$, $\eta_2 = \cos(5t)$ and $\eta_3 = 3e^{-t}$. The control gains are taken to be $k_i = 2$, $\forall i = 1, 2, \ldots, 9$. and the simulation results are shown in Figure (4.5) and Figure (4.6). The error system tends to zero as $t$ tends to infinity as displayed in Figure (4.5). Also the estimated values of the unknown parameters tends to $\hat{\omega} \to \omega$, $\hat{\mu} \to \mu$, $\hat{\alpha} \to \alpha$, $\hat{\beta} \to \beta$, $\hat{a} \to a$, $\hat{b} \to b$ as displayed in Figure (4.6). Thus, the desired HFPS between master and slave system is established.

![Figure 4.5: Synchronization error between states of master and slave system](image)
Figure 4.6: The estimated values of the unknown parameters $\hat{a}$, $\hat{b}$, $\hat{\omega}$, $\hat{\mu}$, $\hat{\alpha}$ and $\hat{\beta}$ as HFPS occurs.

**Case 2.** In this case we achieve the projective synchronization between master and slave systems. For this we choose the scaling functions as $\eta_i = 3 \quad \forall \quad i = 1, 2, 3$. The control gain for this case are chosen to be $k_i = 2 \quad \forall \quad i = 1, 2, 3$. The error dynamics tends to zero as $t$ tends to infinity as displayed in Figure (4.7). Also the estimated values of the unknown parameters tend to $\hat{\omega} \rightarrow \omega$, $\hat{\mu} \rightarrow \mu$, $\hat{\alpha} \rightarrow \alpha$, $\hat{\beta} \rightarrow \beta$, $\hat{a} \rightarrow a$, $\hat{b} \rightarrow b$ as displayed in Figure (4.9). The corresponding time series showing the projective synchronization is shown in Figure (4.9). Thus, the desired HFPS between master and slave system is established.
Figure 4.7: Synchronization error between states of master and slave system

Figure 4.8: The estimated values of the unknown parameters \( \hat{\alpha}, \hat{b}, \hat{\omega}, \hat{\mu}, \hat{\alpha} \) and \( \hat{\beta} \) as HFPS occurs
Figure 4.9: Time series showing projective synchronization with scaling factor 3.

4.6 Conclusion

In this chapter, HFPS between the Bhalekar-Gejji chaotic system (taken as master system) and the Pehlivan chaotic system (taken as slave system) is realized where both the systems cannot be derived from the unified chaotic system. The case of projective synchronization is also discussed. Finally, simulations are presented to demonstrate the effectiveness of the proposed HFPS. HFPS has many applications in secure communication and neural network. Also other control techniques can also be applied to achieve HFPS between Bhalekar-Gejji and Pehlivan chaotic systems.