CHAPTER-5
DYNAMICS OF SUPERIOR ANTIFRACTALS

The aim of Chapter 5 is to study the pattern among the antifractals in the complex dynamics of antipolynomial $z \rightarrow z^m + c$, for $m \geq 2$. This chapter is divided into three sections. In section 5.1, we give some basic introduction which is useful in understanding the next sections.

In section 5.2, we generate a new class of tricorns and multicorns using SP orbit which is an example of four-step feedback process and analyze their patterns. The results of this section are published in “International Journal of Computer Trends and Technology, vol. 43(2), January 2017, 105-112”.

In section 5.3, we generate a new class of tricorns and multicorns using a new four-step feedback process [94]. The results presented in this section are communicated in “International Journal of Advances in Mathematics, vol. 2017(6), 60-67.”.

5.1 INTRODUCTION

Anti-fractals have interesting features in the complex graphics of dynamical system. The dynamics of antiholomorphic complex polynomials $z \rightarrow z^m + c$, for $m \geq 2$, was studied and explored to visualize interesting Tricorns and Multicorns with respect to one-step feedback process [140], two-step feedback process [126,127], three-step feedback process [195] and four-step feedback process[13,27].

The connected locus of antipolynomial $z \rightarrow z^m + c$ is known as Tricorn. The term Tricorn was firstly used by Milnor. In 2003, Nakane et al.[155] described various properties of Tricorn and Multicorn by computing beautiful figures and quoted that Multicorns are the generalized Tricorns or the Tricorns of higher order. The dynamics of antipolynomial $z \rightarrow z^m + c$ where $m \geq 2$ with respect to iterative function generates amazing Tricorns and Multicorns [140,154,155]. Crowe et al. [178] considered it as a formal analogy with Mandelbrot sets and named it as Mandelbar set. They also brought their bifurcation features along arcs rather than at points. Multicorns have been found in a real slice of the cubic connectedness locus [155]. Winter [143] showed that the
boundary of the Tricorn contains arc. The symmetries of Tricorn and Multicorns have been analyzed by Lau and Schleicher [31].

**Definition 5.1.1** [140]. (Multicorn). “The multicorns $A_c$ for the quadratic function

$$A_c(z) = z^m + c$$

is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$A_c = \{ c \in C : A^n_c(0) \text{ do not tend to } \infty \}$$

where $C$ is a complex space. $A^n_c$ is the $n^{th}$ iterate of the function $A_c(z)$. An equivalent formulation is that the connectedness of loci for higher degree antihoomorphic polynomials $A_c(z) = z^m + c$ are called multicorns.”

Note that at $m = 2$, multicorns reduce to tricorn.

They have $(m+1)$-fold rotational symmetries. Also, by dividing these symmetries, the resulting multicorns are called unicorns [154].

### 5.2 GENERATION OF ANTIFRACTALS IN SP ORBIT

In 2011, Phuengrattana and Suantai [180] proposed the SP-iteration for approximating a fixed point of continuous functions on an arbitrary interval and compared the convergence speed of Mann, Ishikawa, Noor and SP-iterations using some numerical examples and proved that the SP-iteration converges faster than the other iterations. In this section, we generate a new class of Tricorns and Multicorns under SP orbit which is an example of four-step feedback process and analyze their patterns.

**Definition 5.2.1.** [180] “Let $T:X \to X$ be a mapping. Let us consider a sequence $\{z_n\}$ of iterates for initial point $z_0 \in X$ such that

$$\{z_{n+1} : z_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n;$$

$$u_n = (1 - \beta_n)v_n + \beta_nTv_n;$$

$$v_n = (1 - \gamma_n)z_n + \gamma_n Tz_n;$$

$$n = 0, 1, 2, \ldots \},$$
where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences of positive numbers. The above sequence of iterates is called as SP orbit, which is a function of five tuples $(T, z_0, \alpha_n, \beta_n, \gamma_n)$.”

Now, we will obtain a general escape criterion for polynomials of the form $G_c(z) = z^m + c$

**Theorem 5.2.2.** For a general function $G_c(z) = z^m + c$, $m = 1, 2, 3…$, where $0 < \alpha < 1$, $0 < \beta < 1, 0 < \gamma < 1$, and $c$ is a complex number. Define

$$z_1 = (1 - \alpha)u + \alpha G_c(u)$$

$$z_2 = (1 - \alpha)u_1 + \alpha G_c(u_1)$$

$$\ldots$$

$$z_m = (1 - \alpha)u_{m-1} + \alpha G_c(u_{m-1}), \quad m = 1, 2, 3, \ldots$$

Then, the general Superior escape criterion is

$$\max\{|c|, (2/\alpha)^{1/m-1}, (2/\beta)^{1/m-1}, (2/\gamma)^{1/m-1}\}.$$

**Proof:** For proving the theorem, we use the method of induction.

For $m = 1$, we have $G_c(z) = z + c$, and this implies

$$|z| > \max\{|c|, 0, 0, 0\}.$$  

For $m = 2$, we have $G_c(z) = z^2 + c$, then the escape criterion is

$$|z| > \max\{|c|, 2/\alpha, 2/\beta, 2/\gamma\}.$$  

Similarly, for $m = 3$, we get $G_c(z) = z^3 + c$. The escape criterion is

$$|z| > \max\{|c|, (2/\alpha)^{1/2}, (2/\beta)^{1/2}, (2/\gamma)^{1/2}\}.$$  

Hence the theorem is true for $m = 1, 2, 3 \ldots$
Now, suppose that theorem is true for any \( m \). We prove that the result is true for \( m+1 \).

Let \( G_c(z) = z^{m+1} + c \) and \( |z| \geq |c| > (2/\alpha)^{1/m} \), \( |z| \geq |c| > (2/\beta)^{1/m} \) and \( |z| \geq |c| > (2/\gamma)^{1/m} \).

Then, consider

\[
|v| = |(1-\gamma)z + \gamma G_c(z)|, \quad \text{where } G_c(z) = z^{m+1} + c
\]

\[
= |(1-\gamma)z + \gamma(z^{m+1} + c)|
\]

\[
= |z||\gamma z^m - \gamma + \gamma| - \gamma |c|
\]

\[
= |z|\left(\gamma z^m - 1\right) + \gamma |z| - \gamma |z| \quad (\because |z| \geq |c|)
\]

i.e. \( |v| = |z|\left(\gamma z^m - 1\right) \quad (5.2.1) \)

Also, \( |u| = |(1-\beta)v + \beta G_c(v)| \)

\[
= |(1-\beta)v + \beta(v^{m+1} + c)|
\]

\[
\geq |(1-\beta)|z|(\gamma |z|^{m} - 1| + \beta\left[|z|\left(\gamma |z|^{m} - 1\right)\right]^{m} + c)| \quad (5.2.2)
\]

Since \( |z| \geq (2/\gamma)^{1/m} \) implies \( \gamma |z|^{m} - 1 > 1 \), so \( |z|\left(\gamma |z|^{m} - 1\right) > |z| \).

(5.2.3)

Using (5.2.3) in (5.2.2), we have

\[
|u| \geq |(1-\beta)|z| + \beta\left(|z|^{m+1} + c\right)|
\]

\[
\geq \beta |z|^{m+1} + (1-\beta)|z| - \beta |c|
\]

\[
\geq \beta |z|^{m+1} + (1-\beta)|z| - \beta |z| \quad (\because |z| \geq |c|) \]
\[ z \geq |z| (\beta |z|^m - 1) \]

i.e.
\[ |u| \geq |z| (\beta |z|^m - 1) \] \hspace{1cm} (5.2.4)

Now for \( z_m = (1-\alpha)u_{m-1} + \alpha G_c(u_{m-1}) \), we have

\[
|z_1| = \left| (1-\alpha)u + \alpha G_c(u) \right| \\
= \left| (1-\alpha)u + \alpha \left( u^{m+1} + c \right) \right| \\
\geq \left| (1-\alpha)|z| (\beta |z|^m - 1) + \alpha \left( |z| (|\beta| |z|^m - 1) \right)^{m+1} + c \right| \] \hspace{1cm} (5.2.5)

Since \( |z| \geq (2/\beta)^{1/m} \) implies \( \beta |z|^m - 1 > 1 \), so \( |z| (\beta |z|^m - 1) > |z| \). \hspace{1cm} (5.2.6)

Using (5.2.6) in (5.2.5), we get
\[
|z_1| \geq \left| (1-\alpha)|z| + \alpha \left( |z|^{m+1} - 1 \right) \right|, \\
\geq \left| \alpha |z|^{m+1} + (1-\alpha)|z| - \alpha |z| \right| \\
\geq |z| \left( \alpha |z|^m - 1 \right) \\
\text{i.e.} \quad |z_1| \geq |z| \left( \alpha |z|^m - 1 \right).
\]

Since \( |z| > (2/\alpha)^{1/m} \), \( |z| > (2/\beta)^{1/m} \) and \( |z| > (2/\gamma)^{1/m} \) exist, we have \( \alpha |z|^m - 1 > 1 + \lambda > 1 \).

In particular, \( |z_1| > (1+\lambda)|z| \)
\[ \vdots \]
\[ |z_m| > (1+\lambda)^m |z| \]

Hence,
\[ |z_m| \rightarrow \infty \text{ as } m \rightarrow \infty. \]

This completes the proof.
Corollary 5.2.3. “Suppose $|c| > (2/\alpha)^{1/m-1}$, $|c| > (2/\beta)^{1/m-1}$ and $|c| > (2/\gamma)^{1/m-1}$ exists. Then the orbit $SP(G_c, 0, \alpha, \beta, \gamma)$ escapes to infinity.”

Corollary 5.2.4. (Escape Criterion). “Let us Assume that for some $k \geq 0$, $|z_k| > \max\{|c|, (2/\alpha)^{1/k-1}, (2/\beta)^{1/k-1}, (2/\gamma)^{1/k-1}\}$, then $|z_k| > \lambda |z_{k-1}|$ and $|z_m| \to \infty$ as $m \to \infty$.

This corollary gives an algorithm to generate AntiJulia sets for the functions of the type $G_c(z) = z^m + c$, $m = 2, 3, \ldots$ Thus, for visualizing Superior antifractals, the required escape criterion with respect to the SP orbit for $z \to z^m + c$ is

$$\max\left\{|c|, \left(\frac{2}{\alpha}\right)^{1/m-1}, \left(\frac{2}{\beta}\right)^{1/m-1}, \left(\frac{2}{\gamma}\right)^{1/m-1}\right\}.$$

5.2.5. Tricorns and Multicorns in SP Orbit

In this section, we generate Tricorns and Multicorns by programming the polynomial $z \to z^m + c$ in the software Mathematica 10.0 under SP orbit (see Figs. 5.2.1 – 5.2.18).

![Fig.5.2.1: $\alpha = 0.3, \beta = 0.5, \gamma = 0.6$](image1)

![Fig.5.2.2: $\alpha = 0.6, \beta = 0.3, \gamma = 0.5$](image2)

![Fig.5.2.3: $\alpha = 0.68, \beta = 0.27, \gamma = 0.95$](image3)

![Fig.5.2.4: $\alpha = 0.95, \beta = 0.27, \gamma = 0.68$](image4)
Fig. 5.2.5: $\alpha = 0.9, \beta = 0.1, \gamma = 0.1$

Fig. 5.2.6: $\alpha = 0.1, \beta = 0.1, \gamma = 0.9$

Fig. 5.2.7: $\alpha = 0.5, \beta = 0.5, \gamma = 0.5$

Fig. 5.2.8: $\alpha = 0.09, \beta = 0.7, \gamma = 0.8$

Fig. 5.2.9: $m = 3, \alpha = 0.08, \beta = \gamma = 0.6$

Fig. 5.2.10: $m = 3, \alpha = \beta = 0.6, \gamma = 0.08$

Fig. 5.2.11: $m = 3, \alpha = 0.6, \beta = 0.08, \gamma = 0.6$

Fig. 5.2.12: $m = 3, \alpha = 0.5, \beta = 0.5, \gamma = 0.5$
We have the following observations:

- The number of branches in the tricorns and multicorns is $m+1$, where $m$ is the power of $z$. Also, few branches have $m$ sub-branches.

- By taking the value of parameter $\beta$, we generate anti fractals with sharp corners (Figs. 5.2.2, 5.2.3, 5.2.5, 5.2.6, 5.2.11, 5.2.13).

- As we increase the value of $\beta$, generated Tricorns become fattier.

- We have the beautiful Rangoli Patterns (Figs. 5.2.16, 5.2.17).
We also find that higher degree multicorns become circular saw (Fig. 5.2.18).

Some authors [13,27,127] had also found the similar conclusion while generating Multicorns using two-step, three-step and four-step feedback processes. The name circular saw was, first, given by Rani and Kumar to Mandelbrot sets [124].

5.2.6. Anti Julia Sets in SP-Orbit

We compute anti Julia sets for $z \rightarrow z^m + c$ in the software Mathematica 10.0 via SP orbit. We have the following observations while generating them.

- In Figs. 5.2.19-5.2.20, we notice that as we increase the value of parameters $\alpha, \beta and \gamma$ keeping constant $c$ same anti Julia sets become fattier.
- The number of branches in anti Julia sets is $m+1$, but few branches have $m$ sub-branches (see Figs. 5.2.27 – 5.2.29).
- Also, we observe that the higher degree anti Julia sets take different shapes (like circular saw and Rangoli pattern) for different values of $m, \alpha, \beta, \gamma and c.$ (see Figs. 5.2.30 – 5.2.32)
Fig. 5.2.23: AntiJulia set for $m = 2$
$\alpha = 0.9, \beta = 0.6, \gamma = 0.3, c = 0.3 + 0.5i$

Fig. 5.2.24: AntiJulia set for $m = 2$
$\alpha = 0.5, \beta = 0.9, \gamma = 0.1, c = 0.3 - 0.5i$

Fig. 5.2.25: AntiJulia set for $m = 2$
$\alpha = 0.9, \beta = 0.1, \gamma = 0.5, c = 0.3 - 0.5i$

Fig. 5.2.26: AntiJulia set for $m = 3$
$\alpha = 0.9, \beta = 0.1, \gamma = 0.5, c = 0.1 - 0.1i$

Fig. 5.2.27: AntiJulia set for $m = 3$
$\alpha = \beta = \gamma = 0.03, c = 0.1 - 0.1i$

Fig. 5.2.28: AntiJulia set for $m = 3$
$\alpha = 0.1, \beta = 0.5, \gamma = 0.9, c = 0.1 - 0.1i$
5.2.7. REMARKS

For the antipolynomials $z \rightarrow z^m + c$, where $m > 2$, there exist many antifractals for the same value of $m$ but different values of parameters in SP orbit. In our results, we find that for higher degree polynomials, all the antifractals become circular saw. We observe that Multicorns are symmetrical about both x and y axis for odd values of $m$, but for even values of $m$, the symmetry is maintained only along x-axis.

5.3 SUPERIOR ANTIFRACTALS IN A NEW ORBIT

In 2014, Abbas and Nazir [94] introduced a new iterative process which is faster than all of Picard, Mann and Agarwal et al. processes and is independent of these iterative processes. First, we give the new orbit, which is used in this section to implement four-step feedback process in the dynamics of antipolynomial $z \rightarrow z^m + c$. 

\[ \alpha = 0.1, \beta = 0.5, \gamma = 0.9, \]
\[ c = 0.1 - 0.1I \]

\[ \alpha = 0.9, \beta = 0.1, \gamma = 0.5, \]
\[ c = 0.1 - 0.1I \]

\[ \alpha = 0.5, \beta = 0.9, \gamma = 0.1, \]
\[ c = 0.1 - 0.1I \]

\[ m = 50, \alpha = 0.1, \beta = 0.5, \gamma = 0.9, \]
\[ c = 0.05 + 0.05I \]
**Definition 5.3.1 [94]:** “Consider a sequence \{x_n\} of iterates for initial point \(x_0 \in X\) such that

\[
\begin{align*}
  x_{n+1} : x_{n+1} &= (1-\alpha_n)T_y n + \alpha_n T_z n; \\
  y_n &= (1-\beta_n)T_x n + \beta_n T_z n; \\
  z_n &= (1-\gamma_n)x_n + \gamma_n T_x n; \quad n = 0, 1, 2, ...
\end{align*}
\]

(5.3.1)

where \(\alpha_n, \beta_n, \gamma_n \in [0, 1]\) and \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are sequences of positive numbers. Then the sequence (5.3.1) is a function (New Orbit) of five tuples \((T, x_0, \alpha_n, \beta_n, \gamma_n)\).”

For visualizing new antifractals, the required escape criterion with respect to the new orbit for \(z \rightarrow z^m + c\) is \(\max\left\{ |c|, (2/\alpha)^{m-1}, (2/\beta)^{m-1}, (2/\gamma)^{m-1} \right\}\) [112].

**5.3.2. Tricorns and Multicorns in New Orbit**

In this section, we programmed the polynomial \(z \rightarrow z^m + c\) in the software Mathematica 10.0 and generate tricorns and multicorns in a new orbit (see Figs. 5.3.1-5.3.14).

**A. Tricorns for m=2:**

*Fig. 5.3.1: \(\alpha = \beta = 0.3, \ \gamma = 0.1\)*

*Fig. 5.3.2: \(\alpha = 0.3, \ \beta = 0.1, \ \gamma = 0.3\)*

*Fig. 5.3.3: \(\alpha = \beta = \gamma = 0.3\)*

*Fig. 5.3.4: \(\alpha = 0.1, \ \beta = 0.9, \ \gamma = 0.1\)*
B. Multicorns for $m=3$:

- Fig. 5.3.5: $\alpha = \beta = 0.9$, $\gamma = 0.1$
- Fig. 5.3.6: $\alpha = \beta = 0.6$, $\gamma = 0.1$
- Fig. 5.3.7: $\alpha = 0.1$, $\beta = \gamma = 0.6$
- Fig. 5.3.8: $\alpha = 0.6$, $\beta = 0.1$, $\gamma = 0.6$
- Fig. 5.3.9: $\alpha = 0.1$, $\beta = \gamma = 0.9$

C. Multicorns for higher degrees:

- Fig. 5.3.10: $m = 4$, $\alpha = 0.1$, $\beta = 0.9$, $\gamma = 0.1$
- Fig. 5.3.11: $m = 6$, $\alpha = 0.6$, $\beta = 0.1$, $\gamma = 0.6$
We find the following observations from generated multicorns:

- The number of branches in the tricorns and multicorns is \( m+1 \), but few branches have \( m \) subbranches (see Figs. 5.3.6, 5.3.7).
- Multicorns exhibit \((m+1)\)-fold rotational symmetries.
- There exist many multicorns for any \( m \).

Higher degree multicorns have taken the shape as circular saw (Figs. 5.3.13-5.3.14).

### 5.3.3. New Superior Anti-Julia Sets

Anti Julia sets have been generated for \( z \rightarrow z^m + c \) in a new orbit. In Figs. 5.3.15-5.3.17, we can see that the anti Julia sets look like tricorns or multicorns for \( m=2 \). Also, we observe that the higher degree anti Julia sets took different shapes for different values of \( m, \alpha, \beta, \gamma \) and \( c \).
Fig. 5.3.15: AntiJulia set for m = 2
α = 0.4, β = 1.0, γ = 1.0, c = 0.3 + 0.5i

Fig. 5.3.16: AntiJulia set for m = 2
α = β = γ = 0.5, c = 0.3 + 0.5i

Fig. 5.3.17: AntiJulia set for m = 2
α = β = γ = 0.5, c = 0.1 + 0.1i

Fig. 5.3.18: AntiJulia set for m = 3
α = β = γ = 0.4, c = 0.7 + 0.7i

Fig. 5.3.19: AntiJulia set for m = 3
α = β = 0.1, γ = 0.05, c = 0.6 + 0.5i

Fig. 5.3.20: AntiJulia set for m = 3
α = β = γ = 0.05, c = 0.5 + 0.4i
Fig. 5.3.21: AntiJulia set for $m = 3$
$\alpha = \beta = \gamma = 0.4, c = 0.1+0.1i$

Fig. 5.3.22: AntiJulia set for $m = 3$
$\alpha = \beta = \gamma = 0.4, c = 0.5+0.5i$

Fig. 5.3.23: AntiJulia set for $m = 4$
$\alpha = 0.1, \beta = 1.0, \gamma = 0.1, c = 0.1+0.1i$

Fig. 5.3.24: Circular saw AntiJulia set for $m = 8$
$\alpha = \beta = 0.05, \gamma = 0.7, c = 0.05+0.05i$