CHAPTER-2

CONVERGENCE OF ITERATIVE ALGORITHMS FOR NONEXPANSIVE MAPPINGS

In this Chapter, we study the convergence of iterative procedures for nonexpansive mappings in Hadamard manifolds and Hyperbolic spaces. This chapter is divided into four sections. Section 2.1 is introductory in nature. In section 2.2, we study the strong convergence of Ishikawa iterative procedure (a two-step iterative procedure) in Hadamard manifolds. The main result has been supported by a numerical example. The results presented in this section are published in the journal “Communications in Optimization Theory, vol. 2014 (4), 1-14”.

In section 2.3, we establish the convergence of Noor iterative procedure (a three-step procedure) for nonexpansive mappings on Hadamard manifolds. Our results improve and extend the recent research in the framework of Hadamard manifolds. The results included in this section are accepted for publication in “Journal of Engineering and Applied Sciences, vol. 12, (2017)”.

In section 2.4, we analyze the convergence of Noor implicit algorithm for nonexpansive mappings in Hyperbolic spaces. The results of this section are published in the journal “Mathematical Sciences International Research Journal, vol. 3(1), (2014), 287-290”.

2.1 INTRODUCTION

In 1953, Mann [181] introduced the most general iterative formula for approximation of fixed points of non-expansive mapping which is called Krasnoselskii–Mann iterative procedure. This procedure has been extensively studied by many authors [72,150,157,162]. Then, Halpern [19] gave an iterative procedure for a fixed point in 1967. Further in 1974, Ishikawa iteration procedure for approximating fixed points in Hilbert space has been introduced by Ishikawa [151]. In 1993, Tan and Xu [80] showed weak and strong convergence of Ishikawa iterative procedure for non-expansive mappings. Further, Ishikawa iterative scheme has been studied extensively by many authors to solve the nonlinear equations in Hilbert space and Banach space.
In recent years, some algorithms for solving variational inequalities and minimization problems have been extended from the Hilbert space framework to the more general setting of Riemannian manifolds.

First of all, we present some basic results which are useful for proving the main results of this chapter. We assume that $M$ is an $m$-dimensional Hadamard manifold.

**Proposition 2.1.1** [167]. “Let $p \in M$. Then $\exp_p : T_p M \to M$ is a diffeomorphism and for any two points $p, q \in M$ there exists a unique normalized geodesic joining $p$ to $q$, which is in fact a minimal geodesic. This result shows that $M$ has the topology and differential structure similar to $\mathbb{R}^m$. Thus Hadamard manifolds and Euclidean spaces have some similar geometrical properties.”

**Proposition 2.1.2** [167]. “A subset $K \subseteq M$ is said to be convex if for any two points $p$ and $q$ in $K$, the geodesic joining $p$ to $q$ is contained in $K$, i.e., if $\gamma : [a, b] \to M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1 - t) a + t b) \in K$ for all $t \in [0, 1]$. Now $K$ will denote a nonempty, closed and convex set in $M$.”

“A real valued function $f$ defined on $M$ is said to be convex if for any geodesic $\gamma$ of $M$, the composition function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is convex, that is,

$$f(\gamma(t) a + (1-t)b) \leq t f(\gamma(a)) + (1-t)f(\gamma(b))$$

for any $a, b \in \mathbb{R}$, and $0 \leq t \leq 1$.”

**Proposition 2.1.3** [167]. “Let $d : M \times M \to \mathbb{R}$ be a distance function. Then $d$ is a convex function with respect to the product Riemannian metric, i.e., given any pair of geodesics $\gamma_1 : [0,1] \to M$ and $\gamma_2 : [0,1] \to M$ the following inequality holds for all $t \in [0,1]$: 

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1))$$

In particular, for each $p \in M$, the function $d(\cdot, p) : M \to \mathbb{R}$ is a convex function.”

**Definition 2.1.4** [131]. “Let $X$ be a complete metric space and $F \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subseteq X$ is called Fejer convergent to $F$ if $d(x_{n+1}, y) \leq d(x_n, y)$ for all $y \in F$ and $n \geq 0$.”

**Lemma 2.1.5** [131]. “Let $X$ be a complete metric space. If $\{x_n\} \subseteq X$ is Fejer convergent to a nonempty set $F \subseteq X$, then $\{x_n\}$ is bounded. Moreover, if a cluster point $x$ of $\{x_n\}$ belongs to $F$, then $\{x_n\}$ converges to $x$.”
Definition 2.1.6 [150, 151]. “Let $x_0 \in X$ be arbitrary. If the sequence $\{x_n\}_{n=0}^\infty$ satisfies the condition

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Ty_n$$
$$y_n = \beta_n x_n + (1 - \beta_n) Tx_n$$

(2.1.1)

for $n = 0, 1, 2, \ldots$ then this is called the Ishikawa iteration, where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences of positive numbers that satisfy the following conditions:

(i) $0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1$, for all positive integers $n$,

(ii) $\lim \alpha_n = 0$,

(iii) $\sum \alpha_n \beta_n = \infty$.”

In 1993, Tan and Xu [80] proved some result- “for non-expansive mapping $T : K \rightarrow K$, where $K$ is a bounded closed subset of a uniformly convex Banach space $X$, that if

$$\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty, \quad \sum_{n=0}^{\infty} \beta_n (1 - \alpha_n) < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \beta_n < 1$$

(2.1.2)

then, $\|Tx_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$ which implies the convergence of $\{x_n\}$ to a fixed point of $T$ if the range of $T$ lies in a compact subset of $X$. In 1984, Goebel and Reich [78] studied the behaviour of the sequence of iterates $x_{n+1} = T(x_n)$ in Hyperbolic metric spaces.”

In 2010, Li et al. [24] studied the Mann and Halpern iterative algorithms for non-expansive mappings on Hadamard manifolds. Motivated by the results of Li et al., we study the convergence of Ishikawa iterative procedure for approximating a fixed point of nonexpansive mappings in Hadamard manifolds (i.e. complete simply connected Riemannian manifolds of non-positive sectional curvature).

### 2.2 TWO-STEP ITERATIVE PROCEDURE FOR NONEXPANSIVE MAPPINGS IN HADAMARD MANIFOLDS

The two step iteration (2.1.1) in Hadamard manifolds $M$ is as follows:

$$x_{n+1} = \exp_{x_n} (1 - \alpha_n) \exp_{x_n}^{-1} T(y_n),$$
$$y_n = \exp_{x_n} (1 - \beta_n) \exp_{x_n}^{-1} T(x_n)$$

(2.2.1)
for all \( n \geq 0 \), where \( 0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1 \).

In our main result, we will assume that \( M \) is an \( m \)-dimensional Hadamard Manifold.

**Theorem 2.2.1.** Let \( K \) be a closed convex subset of \( M \) and \( T : K \to K \) a non-expansive mapping with \( F = \text{Fix}(T) \neq \emptyset \). Suppose that \( \{\alpha_n\} \subset (0,1) \) and \( \{\beta_n\} \subset (0,1) \) satisfy the condition (2.1.2). Let \( x_0 \in M \) and let \( \{x_n\} \) be the sequence generated by the algorithm (2.2.1). Then \( \{x_n\} \) converges to a fixed point of \( T \).

**Proof.** We know that \( K \) is a closed convex subset of \( M \), thus \( K \) is a complete metric space. Using Lemma 2.1.5, it is sufficient to prove that \( \{x_n\} \) is Fejer convergent to \( F \) and that all cluster points of \( \{x_n\} \) belong to \( F \).

Now we suppose that \( n \geq 0 \) and \( p \in F \) be fixed and \( \gamma_1 \) and \( \gamma_2 \) denote the geodesic joining \( x_n \) to \( T(y_n) \) and \( y_n \) to \( x_n \). Then \( x_{n+1} = \gamma_1(1-\alpha_n) \) and \( y_n = \gamma_2(1-\beta_n) \).

By using the convexity of distance function and the nonexpansivity of \( T \), we have

\[
d(x_{n+1}, p) = d(\gamma_1(1-\alpha_n), p) \leq \alpha_n d(x_n, p) + (1-\alpha_n) d(Ty_n, p)
\]

\[
\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(y_n, p)
\]

(2.2.2)

And

\[
d(y_n, p) = d(\gamma_2(1-\beta_n), p) \leq \beta_n d(x_n, p) + (1-\beta_n) d(Tx_n, p)
\]

\[
\leq \beta_n d(x_n, p) + (1-\beta_n) d(x_n, p)
\]

\[
\Rightarrow d(y_n, p) \leq d(x_n, p)
\]

(2.2.3)

By (2.2.2) and (2.2.3), we obtain

\[
d(x_{n+1}, p) \leq \alpha_n d(x_n, p) + (1-\alpha_n) d(x_n, p) \leq d(x_n, p)
\]

Hence \( \{x_n\} \) is Fejer convergent to \( F \). Suppose \( x \) is a cluster point of \( \{x_n\} \). Then there exists a subsequence \( \{n_k\} \) of \( n \) such that \( x_{n_k} \to x \).

Now we prove that

\[
\lim_{n \to \infty} d(x_n, Tx_n) = 0.
\]

(2.2.4)

For this, let \( p \in F \) and \( n \geq 0 \). Let \( \Delta(x_n, q, p) \) be the geodesic triangle with vertices \( x_n, q = Ty_n \) and \( p \). From Lemma 3.3 [24, p. 546] there exists a comparison triangle \( \Delta(x_n', q', p') \) which conserves the length of edge. Also we have \( x_{n+1} = \gamma_1(1-\alpha_n) \). Set
\( x'_{n+1} = \alpha_n x'_n + (1 - \alpha_n) T y'_n = \alpha_n x'_n + (1 - \alpha_n) q' \) as its comparison point. By Lemma 3.5(2) [24, p. 547].

\[
d^2(x_{n+1}, p) \leq \|x'_{n+1} - p\|^2 = \|\alpha_n (x'_n - p') + (1 - \alpha_n)(q' - p')\|^2
\]

\[
= \alpha_n \|x'_n - p\|^2 + (1 - \alpha_n) \|q' - p\|^2 - \alpha_n(1 - \alpha_n) \|x'_n - q\|^2
\]

\[
= \alpha_n d^2(x_n, p) + (1 - \alpha_n) d^2(T y_n, p) - \alpha_n(1 - \alpha_n) d^2(x_n, T y_n)
\]

\[
\leq \alpha_n d^2(x_n, p) + (1 - \alpha_n) d^2(y_n, p) - \alpha_n(1 - \alpha_n) d^2(x_n, T y_n) \quad (2.2.5)
\]

Now, let \( \Delta(x_n, \ell, p) \) be the geodesic triangle with vertices \( x_n, \ell = T x_n \) and \( p \). From Lemma 3.3 [24, p. 546] there exists a comparison triangle \( \Delta(x'_n, \ell', p') \) which conserves the length of edge. Also we have \( y_n = \gamma_2(1 - \beta_n) \) and set \( y' = \beta_n x'_n + (1 - \beta_n) T x'_n = \beta_n x'_n + (1 - \beta_n) \ell' \)

Similarly, we can obtain

\[
d^2(y_n, p) \leq \|y'_n - p\|^2 = \|\beta_n (x'_n - p') + (1 - \beta_n)(\ell' - p')\|^2
\]

\[
= \beta_n \|x'_n - p\|^2 + (1 - \beta_n) \|\ell' - p\|^2 - \beta_n(1 - \beta_n) \|x'_n - \ell\|^2
\]

\[
= \beta_n d^2(x_n, p) + (1 - \beta_n) d^2(T x_n, p) - \beta_n(1 - \beta_n) d^2(x_n, T x_n)
\]

\[
\leq \beta_n d^2(x_n, p) + (1 - \beta_n) d^2(y_n, p) - \beta_n(1 - \beta_n) d^2(x_n, T x_n)
\]

\[
\leq d^2(x_n, p) - \beta_n(1 - \beta_n) d^2(x_n, T x_n) \quad (2.2.6)
\]

Combining (2.2.5) and (2.2.6), we obtain

\[
d^2(x_{n+1}, p) \leq \alpha_n d^2(x_n, p) + (1 - \alpha_n)[d^2(x_n, p) - \beta_n(1 - \beta_n) d^2(x_n, T x_n)]
\]

\[
- \alpha_n(1 - \alpha_n) d^2(x_n, T y_n)
\]

\[
\leq \alpha_n d^2(x_n, p) + (1 - \alpha_n) d^2(x_n, p) - \beta_n(1 - \alpha_n)(1 - \beta_n) d^2(x_n, T x_n)
\]

\[
- \alpha_n(1 - \alpha_n) d^2(x_n, T y_n)
\]

\[
\leq d^2(x_n, p) - \beta_n(1 - \alpha_n)(1 - \beta_n) d^2(x_n, T x_n) - \alpha_n(1 - \alpha_n) d^2(x_n, T y_n)
\]

It follows that
\[ \alpha_n(1 - \alpha_n)d^2(x_n, Ty_n) + \beta_n(1 - \alpha_n)(1 - \beta_n)d^2(x_n, Tx_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p) \]

and \[ \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)d^2(x_n, Ty_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \beta_n(1 - \alpha_n)(1 - \beta_n)d^2(x_n, Tx_n) < \infty \]

(2.2.7)

which implies that \[ \liminf_{n \to \infty} d(x_n, Ty_n) = 0 \quad \text{and} \quad \liminf_{n \to \infty} d(x_n, Tx_n) = 0 \] (2.2.8)

because otherwise \[ d(x_n, T(x_n)) \geq a \] and \[ d(x_n, T(y_n)) \geq b \] for all \( n \geq 0 \) and for some \( a, b > 0 \) and then

\[ \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)d^2(x_n, Ty_n) \geq b \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty \]

and \[ \sum_{n=1}^{\infty} \beta_n(1 - \alpha_n)(1 - \beta_n)d^2(x_n, Tx_n) \geq a \sum_{n=1}^{\infty} \beta_n(1 - \alpha_n) < \infty \]

which is a contradiction with (2.2.7).

On the other hand, the nonexpansivity of \( T \) and convexity of the distance function, implies that

\[ d(x_{n+1}, T(x_{n+1})) \leq d(x_{n+1}, T(x_n)) + d(T(x_n), T(x_{n+1})) \]

\[ \leq d(x_{n+1}, T(x_n)) + d(x_n, x_{n+1}) \]

\[ \leq \alpha_n d(x_n, T(x_n)) + (1 - \alpha_n) d(T(y_n), T(x_n)) \]

\[ \leq \alpha_n d(x_n, T(x_n)) + (1 - \alpha_n) d(T(y_n), x_n) + (1 - \alpha_n) d(y_n, T(x_n)) \]

Now,\[ d(Ty_n, x_n) \leq \beta_n d(x_n, x_n) + (1 - \beta_n) d(Tx_n, x_n) \]

\[ \leq \beta_n d(x_n, x_n) + (1 - \beta_n) d(x_n, x_n) \]

\[ \leq d(x_n, x_n) \]

Therefore, \[ d(x_{n+1}, T(x_{n+1})) \leq \alpha_n d(x_n, T(x_n)) + (1 - \alpha_n) d(x_n, T(x_{n+1})) \leq d(x_n, T(x_n)) \]

This means that \( \{d(x_n, T(x_n))\} \) is a monotone sequence. Combining this and (2.2.8) we get that (2.2.4) holds. Then, since

\[ d(x, T(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(x)) \]

25
\[ \leq 2d(x_{nk}, x) + d(x_{nk}, T(x_{nk})) \]

by taking limit, we deduce that \( d(x, T(x)) = 0 \), which means that \( x \in \text{Fix}(T) \).

**Example 2.2.2** Let \( X = H^3 \) and let \( N : X \rightarrow X \) be a non-expansive mapping defined by
\[ N(x) = (-x_1, -x_2, -x_3, x_4) \]
for any \( x = (x_1, x_2, x_3, x_4) \in H^3 \). Also \( \text{Fix}(N) = (0,0,0,1) \).

To study the convergence to a fixed point for the non-expansive mapping \( N : X \rightarrow X \) in the above example the following algorithms for Mann, Ishikawa and Halpern Iterative procedures have been taken:

**Mann’s algorithm** [24]
\[
x_{n+1} = (\cosh(1-\alpha_n)r(x_n, x_n)) x_n + (\sinh(1-\alpha_n)r(x_n, x_n)) V(x_n, x_n), \quad \text{for all } n \geq 0,
\]

**Ishikawa’s algorithm** (2.2.6)
\[
y_n = (\cosh(1-\beta_n)r(x_n, x_n)) x_n + (\sinh(1-\beta_n)r(x_n, x_n)) V(x_n, x_n),
\]
\[
x_{n+1} = (\cosh(1-\alpha_n)r(x_n, y_n)) x_n + (\sinh(1-\alpha_n) r(x_n, y_n)) V(x_n, y_n), \quad \text{for all } n \geq 0,
\]

**Halpern’s algorithm** [24]
\[
x_{n+1} = (\cosh(1-\alpha_n)r(u, x_n)) u + (\sinh(1-\alpha_n) r(u, x_n)) V(u, x_n) \quad \text{for all } n \geq 0,
\]
where \( r(x, y) = \arccosh(-\langle x, T(y) \rangle) \), \( V(x, y) = \frac{T(y)+(x.T(y)x)}{\sqrt{(x.T(y))^2-1}} \) for all \( x, y \in H^m \)
and \( \alpha_n = \beta_n = \frac{1}{n+4} \) for each \( n = 0, 1, 2, 3, \ldots \ldots \)

Following tables denotes the error rates, where \( e_n = d\ (x_n, z) \) is the error of nth step where \( d(x, y) = \arccosh(-\langle x, y \rangle) \) for all \( x, y \in H^m \) and \( z = (0, 0, 0, 1) \) is the unique fixed point, \( u = (0.60379247919382, 0.27218792496996, 0.19881426776106, 1.21580374135624) \) have been taken as fixed point for Halpern algorithm. Further, we take two random initial points \( x_0^1 \) and \( x_0^2 \) as follows:

\[
x_0^1 = (0.69445440978475, 1.01382609280137, 0.99360871330745, 1.87012527625153)
\]
\[
x_0^2 = (0.82054041398189, 1.78729906182707, 0.11578260956854, 2.0932797928782)
\]
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**Table 2**

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<td>$e_{20}$</td>
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<td>$e_{21}$</td>
<td>0.017802516548</td>
<td>0.034112000043</td>
<td>0.169534343224</td>
</tr>
</tbody>
</table>
Fig. 1

Fig. 2
Remark 2.2.3. In the computational study, a program has been generated in Mathematica to study the convergence to a fixed point for the non-expansive mappings using Mann, Ishikawa and Halpern iterative procedures and the following have been analyzed:

- Table 1 and Table 2 show the outcomes of error rates produced by Mann, Ishikawa and Halpern iterative procedures for the random initial points $x_0^1$ and $x_0^2$.
- In Fig.1, we analyzed that the Ishikawa iterative procedure gives lesser error rates than the Halpern iterative procedure, i.e., the convergence rate to a fixed point for non-expansive mapping is faster than the convergence rate of Mann and Halpern iterative procedures. (see also Fig. 2)

### 2.3 NOOR ITERATIVE ALGORITHM FOR NONEXPANSIVE MAPPINGS IN HADAMARD MANIFOLDS

In 2000, Noor [92] introduced a three-step iterative process and studied the approximate solution of variational inclusion in Hilbert spaces. Many researchers studied this iteration process to approximate fixed points for various classes of nonlinear operators [54,148,159,192]. In many cases, it is observed that a three-step iterative process is better than a two-step and a one-step iterative process for finding numerical results under different conditions [94,139,149]. Thus we found that it is important to study three-step iterative processes in solving various numerical problems in the field of pure and applied sciences. In this section, we prove the convergence of the Noor iteration procedure (a three-step iterative procedure) to a fixed point for non-expansive mappings on Hadamard manifolds. The Noor iteration in Hadamard manifolds $M$ is as follows:

$$
x_{n+1} = \exp_{x_n}((1-\alpha_n)\exp_{x_n}^{-1}T(y_n))
$$

$$
y_n = \exp_{x_n}((1-\beta_n)\exp_{x_n}^{-1}T(z_n))
$$

$$
z_n = \exp_{x_n}((1-\lambda_n)\exp_{x_n}^{-1}T(x_n))
$$

for all $n \geq 0$, where $0 < \{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} < 1$ and satisfy the following conditions:
\[\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty, \quad \sum_{n=1}^{\infty} \beta_n (1-\beta_n) = \infty, \quad \sum_{n=1}^{\infty} \lambda_n (1-\lambda_n) = \infty \quad (2.3.2)\]

**Theorem 2.3.1.** Let \( K \) be a closed convex subset of \( M \) and \( T : K \rightarrow K \) a non-expansive mapping with \( F = \text{Fix}(T) \neq \emptyset \). Let \( x_0 \in M \) and let \( \{x_n\} \) be the sequence generated by the algorithm (2.3.1). Then \( \{x_n\} \) converges to a fixed point of \( T \).

**Proof.** We know that \( K \) is a closed convex subset of \( M \), thus \( K \) is a complete metric space. Using Lemma 2.1.5, it is sufficient to prove that \( \{x_n\} \) is Fejer convergent to \( F \) and that all cluster points of \( \{x_n\} \) belong to \( F \). Now we suppose that \( n \geq 0 \) and \( p \in F \) be fixed and \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) denote the geodesic joining \( x_n \) to \( T(y_n) \), \( y_n \) to \( T(z_n) \) and \( z_n \) to \( T(x_n) \).

Then \( x_{n+1} = \gamma_1 (1-\alpha_n) \), \( y_n = \gamma_2 (1-\beta_n) \) and \( z_n = \gamma_3 (1-\lambda_n) \).

By using the convexity of distance function and the nonexpansivity of \( T \), we have

\[d(x_{n+1}, p) = d(\gamma_1 (1-\alpha_n), p) \leq \alpha_n d(x_n, p) + (1-\alpha_n) d(Ty_n, p) \leq \alpha_n d(x_n, p) + (1-\alpha_n) d(y_n, p) \quad (2.3.3)\]

And

\[d(y_n, p) = d(\gamma_2 (1-\beta_n), p) \leq \beta_n d(x_n, p) + (1-\beta_n) d(Tz_n, p) \leq \beta_n d(x_n, p) + (1-\beta_n) d(z_n, p) \quad (2.3.4)\]

\[d(z_n, p) = d(\gamma_3 (1-\lambda_n), p) \leq \lambda_n d(x_n, p) + (1-\lambda_n) d(Tx_n, p) \leq \lambda_n d(x_n, p) + (1-\lambda_n) d(x_n, p) \]

\[\Rightarrow d(z_n, p) \leq d(x_n, p) \quad (2.3.5)\]

By (2.3.3), (2.3.4) and (2.3.5), we obtain

\[d(x_{n+1}, p) \leq d(x_n, p)\]

Hence \( \{x_n\} \) is Fejer convergent to \( F \). Suppose \( x \) is a cluster point of \( \{x_n\} \). Then there exists a subsequence \( \{n_k\} \) of \( n \) such that \( x_{n_k} \rightarrow x \). Now we prove that

\[\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \quad (2.3.6)\]
For this, let \( p \in F \) and \( n \geq 0 \). Let \( \Delta(x_n, q, p) \) be the geodesic triangle with vertices \( x_n, q = Ty_n \) and \( p \). From Lemma 3.3 [24, p. 546] there exists a comparison triangle \( \Delta(x'_n, q', p') \) which conserves the length of edge. Also we have \( x_{n+1} = \gamma_1(1-\alpha_n) \). Set \( x'_{n+1} = \alpha_n x'_{n} + (1-\alpha_n)Ty'_n = \alpha_n x'_{n} + (1-\alpha_n)q' \) as its comparison point. By Lemma 3.5(2) [24, p. 547], we have

\[
\begin{align*}
\|x'_{n+1} - p\|^2 &\leq \|\alpha_n(x'_n - p) + (1-\alpha_n)(q' - p')\|^2 \\
&= \alpha_n \|x'_n - p\|^2 + (1-\alpha_n)\|q' - p\|^2 - \alpha_n(1-\alpha_n)\|x'_n - q\|^2 \\
&= \alpha_n d^2(x_n, p) + (1-\alpha_n)d^2(Ty_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, Ty_n) \\
&\leq \alpha_n d^2(x_n, p) + (1-\alpha_n)d^2(y_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, Ty_n)
\end{align*}
\]

(2.3.7)

Now, let \( \Delta(x_n, \ell, p) \) be the geodesic triangle with vertices \( x_n, \ell = Tz_n \) and \( p \). From Lemma 3.3 [24, p. 546] there exists a comparison triangle \( \Delta(x'_n, \ell', p') \) which conserves the length of edge.

Also we have \( y_n = \gamma_2(1-\beta_n) \) and set \( y' = \beta_n x'_n + (1-\beta_n)Tz'_n = \beta_n x'_n + (1-\beta_n)\ell' \)

Similarly, we can obtain

\[
\begin{align*}
\|y' - p\|^2 &\leq \|\beta_n(x'_n - p) + (1-\beta_n)(\ell' - p')\|^2 \\
&= \beta_n \|x'_n - p\|^2 + (1-\beta_n)\|\ell' - p\|^2 - \beta_n(1-\beta_n)\|x'_n - \ell\|^2 \\
&= \beta_n d^2(x_n, p) + (1-\beta_n)d^2(Tz_n, p) - \beta_n(1-\beta_n)d^2(x_n, Tz_n) \\
&\leq \beta_n d^2(x_n, p) + (1-\beta_n)d^2(z_n, p) - \beta_n(1-\beta_n)d^2(x_n, Tz_n)
\end{align*}
\]

(2.3.8)

Now, let \( \Delta(x_n, m, p) \) be the geodesic triangle with vertices \( x_n, m = Tx_n \) and \( p \). From Lemma 3.3 [24, p. 546] there exists a comparison triangle \( \Delta(x'_n, m', p') \) which conserves the length of edge.

Also we have \( z_n = \gamma_3(1-\lambda_n) \) and set \( z' = \lambda_n x'_n + (1-\lambda_n)Tx'_n = \lambda_n x'_n + (1-\lambda_n)\ell' \)

Similarly, we can obtain
\[ d^2(z_n, p) \leq \|z_n' - p\|^2 = \|\lambda_n (x_n' - p') + (1 - \lambda_n)(m' - p')\|^2 \]
\[ = \lambda_n \|x_n' - p\|^2 + (1 - \lambda_n)\|m' - p'\|^2 - \lambda_n (1 - \lambda_n)\|z_n' - m'\|^2 \]
\[ = \lambda_n d^2(x_n, p) + (1 - \lambda_n)d^2(Tx_n, p) - \lambda_n (1 - \lambda_n)d^2(x_n, Tx_n) \]
\[ \leq \lambda_n d^2(x_n, p) + (1 - \lambda_n)d^2(x_n, p) - \lambda_n (1 - \lambda_n)d^2(x_n, Tx_n) \]
\[ \leq d^2(x_n, p) - \lambda_n (1 - \lambda_n)d^2(x_n, Tx_n) \] (2.3.9)

Combining (2.3.8) and (2.3.9), we obtain
\[ d^2(y_n, p) \leq \beta_n d^2(x_n, p) + (1 - \beta_n)\left[ d^2(x_n, p) - \lambda_n (1 - \lambda_n)d^2(x_n, Tx_n) \right] \]
\[ - \beta_n (1 - \beta_n)d^2(x_n, Tz_n) \]
\[ \leq \beta_n d^2(x_n, p) + (1 - \beta_n)d^2(x_n, p) - \lambda_n (1- \lambda_n)(1 - \beta_n)d^2(x_n, Tx_n) \]
\[ - \beta_n (1 - \beta_n)d^2(x_n, Tz_n) \]
\[ \leq d^2(x_n, p) - \lambda_n (1 - \lambda_n)(1 - \beta_n)d^2(x_n, Tx_n) - \beta_n (1 - \beta_n)d^2(x_n, Tz_n) \] (2.3.10)

Combining (2.3.7) and (2.3.10), we get
\[ d^2(x_{n+1}, p) \leq \alpha_n d^2(x_n, p) + (1 - \alpha_n)(d^2(x_n, p) - \lambda_n (1 - \lambda_n)(1 - \alpha_n)d^2(x_n, Tx_n) \]
\[ - \beta_n (1 - \beta_n)d^2(x_n, Tz_n) - \alpha_n (1 - \alpha_n)d^2(x_n, Ty_n) \]
\[ \leq \alpha_n d^2(x_n, p) + (1 - \alpha_n)d^2(x_n, p) - \lambda_n (1 - \lambda_n)(1 - \alpha_n)\|d^2(x_n, Tx_n) \]
\[ - \beta_n (1 - \beta_n)(1 - \alpha_n)d^2(x_n, Tz_n) - \alpha_n (1 - \alpha_n)d^2(x_n, Ty_n) \]
\[ \leq d^2(x_n, p) - \lambda_n (1 - \lambda_n)(1 - \alpha_n)(1 - \beta_n)d^2(x_n, Tx_n) \]
\[ - \beta_n (1 - \beta_n)(1 - \alpha_n)d^2(x_n, Tz_n) - \alpha_n (1 - \alpha_n)d^2(x_n, Ty_n) \]

It follows that
\[ \lambda_n (1 - \lambda_n)(1 - \alpha_n)(1 - \beta_n)d^2(x_n, Tx_n) - \beta_n (1 - \beta_n)(1 - \alpha_n)d^2(x_n, Tz_n) \]
\[ - \alpha_n (1 - \alpha_n)d^2(x_n, Ty_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p) \]

and
\[ \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n)d^2(x_n, Ty_n) < \infty \quad \sum_{n=1}^{\infty} \beta_n (1 - \alpha_n)(1 - \beta_n)d^2(x_n, Tz_n) < \infty \quad \text{and} \]
\[ \sum_{n=1}^{\infty} \lambda_n (1 - \lambda_n)(1 - \alpha_n)(1 - \beta_n) d^2(x_n, Tx_n) < \infty \] (2.3.11)

which implies that

\[ \lim \inf_{n \to \infty} d(x_n, Ty_n) = 0, \quad \lim \inf_{n \to \infty} d(x_n, Tz_n) = 0 \quad \text{and} \quad \lim \inf_{n \to \infty} d(x_n, Tx_n) = 0 \] (2.3.12)

because otherwise \( d(x_n, T(x_n)) \geq a, \quad d(x_n, T(y_n)) \geq b \quad \text{and} \quad d(x_n, T(z_n)) \geq c \) for all \( n \geq 0 \) and for some \( a, b, c > 0 \) and then

\[ \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) d^2(x_n, Ty_n) \geq b \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \]

and

\[ \sum_{n=1}^{\infty} \beta_n (1 - \alpha_n)(1 - \beta_n) d^2(x_n, Tz_n) \geq c \sum_{n=1}^{\infty} \beta_n (1 - \alpha_n) < \infty \]

and

\[ \sum_{n=1}^{\infty} \lambda_n (1 - \lambda_n)(1 - \alpha_n)(1 - \beta_n) d^2(x_n, Tx_n) \geq a \sum_{n=1}^{\infty} \lambda_n (1 - \alpha_n)(1 - \beta_n) < \infty \]

which is a contradiction with (2.3.11).

On the other hand, the nonexpansivity of \( T \) and convexity of the distance function, implies that

\[
d(x_{n+1}, T(x_{n+1})) \leq d(x_{n+1}, T(x_n)) + d(T(x_n), T(x_{n+1})) \leq d(x_{n+1}, T(x_n)) + d(x_n, x_{n+1}) \leq \alpha_n d(x_n, T(x_n)) + (1 - \alpha_n) d(T(y_n), T(x_n)) \leq \alpha_n d(x_n, T(x_n)) + (1 - \alpha_n) d(T(y_n), x_n) + (1 - \alpha_n) d(y_n, T(x_n))
\]

Now, \( d(Ty_n, x_n) \leq \beta_n d(Tz_n, x_n) + (1 - \beta_n) d(Tz_n, x_n) \leq \beta_n d(Tz_n, x_n) + (1 - \beta_n) d(Tz_n, x_n) \leq d(Tz_n, x_n) \)

and \( d(Tz_n, x) \leq \lambda d(x, x) + (1 - \lambda) d(Tx, x) \)
\[ \leq \lambda_n d(x_n, x_n) + (1 - \lambda_n) d(x_n, x_n) \]
\[ \leq d(x_n, x_n) \]

Therefore,
\[ d(x_{n+1}, T(x_{n+1})) \leq \alpha_n d(x_n, T(x_n)) + (1 - \alpha_n) d(x_n, T(x_n)) \leq d(x_n, T(x_n)) \]  
(2.3.13)

This means that \{d(x_n, T(x_n))\} is a monotone sequence. Combining (2.3.11) and (2.3.13),
we found that (2.3.6) holds. Then, since
\[ d(x, T(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(x)) \]
\[ \leq 2d(x_{n_k}, x) + d(x_{n_k}, T(x_{n_k})) \]
by taking limit, we deduce that \(d(x, T(x)) = 0\), which means that \(x \in \text{Fix}(T)\).

**Corollary 2.3.2.** Let \(K\) be a closed convex subset of \(M\) and \(T: K \to K\) a non-expansive mapping with \(F = \text{Fix}(T) \neq \emptyset\). Let \(x_0 \in M\) and let \(\{x_n\}\) be the sequence generated by the algorithm

\[ x_{n+1} = \exp_{x_n}(1 - \alpha_n) \exp_{x_n}^{-1} T(y_n), \]
\[ y_n = \exp_{x_n}(1 - \beta_n) \exp_{x_n}^{-1} T(x_n) \]

for all \(n \geq 0\), where \(0 < \{\alpha_n\}, \{\beta_n\} < 1\). Then \(\{x_n\}\) converges to a fixed point of \(T\).

**Proof.** If we put \(\lambda_n = 0\) in (2.3.1) in Theorem 2.3.1, then we get the desired result.

**Corollary 2.3.3.** Let \(K\) be a closed convex subset of \(M\) and \(T: K \to K\) a non-expansive mapping with \(F = \text{Fix}(T) \neq \emptyset\). Suppose that \(\{\alpha_n\} \subseteq (0, 1)\) satisfy the condition
\[ \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \pm \). Let \(x_0 \in M\) and let \(\{x_n\}\) be the sequence generated by the algorithm
\[ x_{n+1} = \exp_{x_n}(1 - \alpha_n) \exp_{x_n}^{-1} T(x_n), \]
for all \(n \geq 0\).

Then \(\{x_n\}\) converges to a fixed point of \(T\).

**Proof.** If we put \(\beta_n = \lambda_n = 0\) in (2.3.1) in Theorem 2.3.1, we get the desired result.
2.4 AN ALGORITHM FOR FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACE

In general, Hyperbolic spaces provide rich geometrical structures for different results with applications in topology, graph theory, multivalued analysis and metric fixed point theory. The class of Hyperbolic spaces includes normed spaces, CAT(0) spaces, Hadamard manifolds, R-trees and Hilbert balls equipped with Hyperbolic metric\[78,79\]. In 1970, Takahashi [182] introduced the convex metric space. A subset \( K \) of a hyperbolic space \( X \) is convex if \( W(x, y, \alpha) \in K \) for all \( x, y \in K \) and \( \alpha \in [0, 1] \).

In 2005, Kohlenbach [171] introduced a convex structure in hyperbolic space as follows:

**Definition 2.4.1.** [171] “A hyperbolic space \((X, d, W)\) is a metric space \((X, d)\) together with a map \(W: X^2 \times [0, 1] \rightarrow X\) satisfying:

1. \[ d\left(u, W(x, y, \alpha)\right) \leq (1-\alpha)d(u, x) + \alpha d(u, y) \]
2. \[ d\left(W(x, y, \alpha), W(x, y, \beta)\right) = |\alpha - \beta|d(x, y) \]
3. \[ W(x, y, \alpha) = W(y, x, (1-\alpha)) \]
4. \[ d\left(W(x, z, \alpha), W(y, w, \alpha)\right) \leq (1-\alpha)d(x, y) + \alpha d(z, w) \]

for all \( x, y, z, w \in X \) and \( a, b \in [0, 1] \).”

**Definition 2.4.2.** “A hyperbolic space \((X, d, W)\) is said to be:

(i) **Strictly Convex** [182] if for any \( x, y \in X \) and \( \lambda \in [0, 1] \), there exists a unique element \( z \in X \) such that \( d(z, x) = \lambda d(x, y) \) and \( d(z, y) = (1-\lambda)d(x, y) \).

(ii) **Uniformly Convex** [168] if for all \( u, x, y \in X \), \( r > 0 \) and \( \varepsilon \in (0, 2] \), there exists a \( \delta \in (0, 1] \) such that \( d(x, u) \leq r, d(y, u) \leq r, d(x, y) \leq \varepsilon r \)

\[ \Rightarrow d\left(W(x, y, 1/2), u\right) \leq (1-\delta)r. \]
A map $\eta:(0, \infty) \times (0, 2] \to (0, 1]$ such that $\delta = \eta(r, \varepsilon)$ for $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity of $X$. We call $\eta$ to be monotone if it decreases with $r$ (for a fixed $\varepsilon$). A uniformly convex hyperbolic space is strictly convex [86].”

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $X$. “For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}): X \to [0, \infty)$ by $r(x, \{x_n\}) = \lim_{n \to \infty} \sup d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\}$.

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset $K$ of $X$ is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$  

This is the set of minimizers of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to $X$, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that ‘bounded sequences have unique asymptotic centers with respect to closed convex subsets.’ The following lemma is due to Leustean [87] and ensures that this property also holds in a complete uniformly convex hyperbolic space.”

**Lemma 2.4.3** [87]. “Let $(X, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Then every bounded sequence $\{x_n\}$ in $X$ has a unique asymptotic center with respect to any nonempty closed convex subset $K$ of $X.””

A sequence $\{x_n\}$ in $X$ is said to $\Delta$–converge to $x \in X$ if $x$ is the unique asymptotic centre of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [175]. In this case, we write $x$ as $\Delta$–limit of $\{x_n\}$, i.e., $\Delta$–lim$_n x_n = x$.

**Lemma 2.4.4** [11]. “Let $(X, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\lim sup_{n \to \infty} d(x_n, x) \leq r, \quad \lim sup_{n \to \infty} d(y_n, x) \leq r \quad \text{and} \quad \lim_{n \to \infty} d(W(x_n, y_n, a_n), x) = r$$

for some $r \geq 0$, then $\lim_{n \to \infty} d(x_n, y_n) = 0.$”
Lemma 2.4.5 [11]. “Let \( K \) be a nonempty closed convex subset of a uniformly convex hyperbolic space and \( \{x_n\} \) a bounded sequence in \( K \) such that \( A(\{x_n\}) = \{y\} \) and \( r(\{x_n\}) = \rho \). If \( \{y_m\} \) is another sequence in \( K \) such that \( \lim_{m \to \infty} r(y_m, \{x_n\}) = \rho \), then \( \lim_{m \to \infty} y_m = y \).”

“In 1976, Lim [161] introduced the notion of asymptotic center and introduced the concept of \( \Delta \)-convergence in a general setting of a metric space. In 2008, Kirk and Panyanak [175] investigated \( \Delta \)-convergence in CAT(0) spaces and showed that \( \Delta \)-convergence coincides with the usual weak convergence in Banach spaces. Moreover, both concepts share many useful properties in uniformly convex spaces.” Many authors have studied the \( \Delta \)-convergence of various iterative schemes with different mappings in hyperbolic spaces and CAT(0) spaces [11, 12, 48, 90, 156].

The Noor iteration [21] is defined as

\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n \\
y_n &= (1 - \beta_n)x_n + \beta_nTz_n \\
z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n
\end{align*}
\]

(2.4.1)

for all \( n \geq 1 \), where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \([0, 1]\). If we take \( \beta_n = \gamma_n = 0 \) for all \( n \), (2.4.1) reduces to the Mann iteration [181] and we take \( \gamma_n = 0 \) for all \( n \), (2.4.1) reduces to the Ishikawa iteration [151].

Inspired and motivated by the work of Kirk [177], Xu and Ori [55], Khan et al. [11], Fukhar-ud-din et al. [49], we investigate \( \Delta \)-convergence through a three-step implicit algorithm for a finite family of nonexpansive maps in hyperbolic spaces. In a hyperbolic space, the three-step algorithm (2.4.1) can be defined as:

\[
\begin{align*}
x_n &= W(x_{n-1}, T_ny_n, \alpha_n) \\
y_n &= W(x_n, T_nz_n, \beta_n) \\
z_n &= W(x_n, T_nx_n, \gamma_n)
\end{align*}
\]

(2.4.2)

where \( n \geq 1, T_n = T_{n(mod N)} \)

Definition 2.4.6. “Let \( K \) be a nonempty subset of a metric space \((X, d)\), and let \( T \) be a self-mapping on \( K \). Denote by \( F(T) = \{x \in K : T(x) = x\} \) the set of fixed points of \( T \). Then \( T \) is said to be
• Nonexpansive if \( d(Tx, Ty) \leq d(x, y) \) for \( x, y \in K \).

• Quasi-nonexpansive if \( d(Tx, p) \leq d(x, p) \) for \( x \in K \) and for \( p \in F(T) \).

• Asymptotically nonexpansive if there exists a sequence \( k_n \in [0, \infty) \) and \( \lim_{n \to \infty} k_n = 0 \) and \( d(T^n x, T^n y) \leq (1 + k_n) d(x, y) \) for \( x, y \in K \) and \( n \geq 1 \).

Now, we prove a lemma useful for the main result.

Lemma 2.4.7. Let \( K \) be a nonempty closed convex subset of a hyperbolic space \( X \) and let \( \{T_i : i \in I\} \) be a finite family of non-expansive self maps on \( K \) such that \( F \) (the set of fixed points) is nonempty. Then for the sequence \( \{x_n\} \) defined implicitly in (2.4.2), we have

(i) \( \lim_{n \to \infty} d(x_n, p) \) exists for each \( p \in F \).

(ii) \( \lim_{n \to \infty} d(x_n, T_l x_n) = 0 \), for each \( l = 1, 2, 3, \ldots, N \).

Proof. (i) For any \( p \in F \), it follows from (2.4.2) that

\[
\begin{align*}
d(z_n, p) &= d(W(x_n, T_n x_n, \gamma_n), p) \\
&\leq (1 - \gamma_n) d(x_n, p) + \gamma_n d(T_n x_n, p) \\
&\leq (1 - \gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \\
&= d(x_n, p)
\end{align*}
\]

(2.4.3)

Now, using (2.4.3) we get

\[
\begin{align*}
d(y_n, p) &= d(W(x_n, T_n z_n, \beta_n), p) \\
&\leq (1 - \beta_n) d(x_n, p) + \beta_n d(T_n z_n, p) \\
&\leq (1 - \beta_n) d(x_n, p) + \beta_n d(z_n, p) \\
&\leq (1 - \beta_n) d(x_n, p) + \beta_n d(x_n, p) \\
&= d(x_n, p)
\end{align*}
\]

\[\Rightarrow d(y_n, p) \leq d(x_n, p)\] (2.4.4)

Using (2.4.4), we have

\[
d(x_n, p) = d(W(x_{n-1}, T_n y_n, \alpha_n), p)
\]
\[
\leq (1 - \alpha_n)d(x_{n-1}, p) + \alpha_n d(T_n y_n, p) \\
\leq (1 - \alpha_n)d(x_{n-1}, p) + \alpha_n d(y_n, p) \\
\leq (1 - \alpha_n)d(x_{n-1}, p) + \alpha_n d(x_n, p)
\]
\[\Rightarrow d(x_n, p) \leq d(x_{n-1}, p) \quad (2.4.5)\]

This shows that the sequence \( \{d(x_n, p)\} \) is nonincreasing and bounded below, and so \( \lim_{n \to \infty} d(x_n, p) \) exists for each for each \( p \in F \). Hence, \( \lim_{n \to \infty} d(x_n, F) \) exists. This completes the proof of part (i).

(ii). Assume that \( \lim_{n \to \infty} d(x_n, p) = c \).

For \( c = 0 \), the case is trivial.

Now we discuss the case \( c > 0 \).

By (2.4.3) and (2.4.4), we have
\[
d(z_n, p) \leq d(x_n, p) \quad \text{and} \quad d(y_n, p) \leq d(x_n, p)
\]

Now, taking \( \lim \sup \) on both sides of both inequalities, we get
\[
\lim_{n \to \infty} \sup d(z_n, p) \leq c \quad \text{and} \quad \lim_{n \to \infty} \sup d(y_n, p) \leq c
\]

As \( T_n \) is nonexpansive, \( \lim_{n \to \infty} \sup d(T_n z_n, p) \leq c \), \( \lim_{n \to \infty} \sup d(T_n y_n, p) \leq c \)

Also, \( \lim_{n \to \infty} \sup d(x_{n-1}, p) \leq c \)

Now by Lemma 2.4.4, we have
\[
\lim_{n \to \infty} d(x_{n-1}, T_n y_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(x_{n-1}, T_n z_n) = 0 \quad (2.4.6)
\]

Next, \( d(x_n, x_{n-1}) = d(W(x_{n-1}, T_n y_n, \alpha_n), x_{n-1}) \)
\[
\leq \alpha_n d(x_{n-1}, T_n y_n)
\]
taking \( \lim \sup \) on both sides in the above inequality, we get \( \lim_{n \to \infty} \sup d(x_n, x_{n-1}) \leq 0 \)

Hence, \( \lim_{n \to \infty} \sup d(x_n, x_{n-1}) = 0. \) \quad (2.4.7)

Also \( d(x_n, x_{n+l}) \leq d(x_n, x_{n+l}) + d(x_n, x_{n+l+1}) + d(x_{n+l+1}, x_{n+l+2}) + \ldots \) \( + d(x_{n+l}, x_{n+l}). \)

Taking \( \lim \sup \) on both sides in the above inequality and using (2.4.7) we get
\[
\lim_{n \to \infty} d(x_n, x_{n+l}) = 0 \quad \text{for} \quad l < N.
\]
On the other hand,
\[ d(x_n, p) \leq (1 - \alpha_n)d(x_{n-1}, p) + \alpha_n d(T_n y_n, p) \]
\[ \leq (1 - \alpha_n)d(x_{n-1}, T_n y_n) + (1 - \alpha_n)d(T_n y_n, p) + \alpha_n d(y_n, p) \]
\[ \leq (1 - \alpha_n)d(x_{n-1}, T_n y_n) + (1 - \alpha_n)\alpha_n + \alpha_n d(y_n, p) \]
\[ \leq (1 - \alpha_n)d(x_{n-1}, T_n y_n) + d(y_n, p) \]
Similarly,
\[ d(y_n, p) \leq (1 - \beta_n)d(x_{n-1}, T_n z_n) + d(z_n, p) \]

Now, using (2.4.6) and applying \( \text{lim inf} \) and \( \text{lim sup} \) on both sides in the above inequalities, we get
\[ c \leq \lim_{n \to \infty} \text{inf} d(y_n, p) \leq \lim_{n \to \infty} \text{sup} d(y_n, p) \leq c \]
and
\[ c \leq \lim_{n \to \infty} \text{inf} d(z_n, p) \leq \lim_{n \to \infty} \text{sup} d(z_n, p) \leq c \]
\[ \Rightarrow \lim_{n \to \infty} d(y_n, p) = c \text{ and } \lim_{n \to \infty} d(z_n, p) = c. \]
\[ d(x_n, T_n x_n) \leq d(x_n, T_n y_n) + d(T_n y_n, T_n x_{n-1}) + d(T_n x_{n-1}, T_n x_n) \]
\[ \leq (1 - \alpha_n)d(x_{n-1}, T_n y_n) + d(y_n, x_{n-1}) + d(x_{n-1}, x_n) \]
\[ \leq (1 - \alpha_n)d(x_{n-1}, T_n y_n) + (1 - \beta_n)d(x_{n-1}, T_n z_n) + 2d(x_{n-1}, x_n) \]
\[ \Rightarrow d(x_n, T_n x_n) = 0. \]

For each \( l \in I \), we have
\[ d(x_n, T_{n+l} x_{n+l}) \leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(T_{n+l} x_{n+l}, T_{n+l} x_n) \]
\[ \leq 2 d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) \]
Thus the sequence \( \{ d(x_n, T_l x_n) \} \) is a subsequence of \( \bigcup_{l=1}^N \{ d(x_n, T_{n+l} x_n) \} \)
and \( \lim_{n \to \infty} d(x_n, T_{n+l} x_n) = 0 \) for each \( l \in I \).

Therefore, \( \lim_{n \to \infty} d(x_n, T_l x_n) = 0 \) for each \( l \in I \).

This completes the proof of part (ii). Now, we will prove the main result.

**Theorem 2.4.8.** Let \( M \) be a nonempty closed convex subset of a complete uniformly convex hyperbolic space \( X \) with monotone modulus of uniform convexity \( \eta \) and let \( \{ T_i : i \in I \} \) be a finite family of nonexpansive selfmaps on \( M \) such that \( F \) is non-empty.
Then the sequence \( \{ x_n \} \) defined implicitly in (2.4.2), \( \Delta \)-converges to a common fixed point of \( \{ T_i : i \in I \} \).

**Proof.** Since the sequence \( \{ x_n \} \) is bounded by Lemma 2.4.7 (i). Also by Lemma 2.4.3, \( \{ x_n \} \) has a unique asymptotic center, that is , \( A(\{ x_n \}) = \{ x \} \). Let \( \{ s_n \} \) be any sequence of \( \{ x_n \} \), such that \( A(\{ s_n \}) = \{ s \} \). Then by Lemma 2.4.7 (ii), we have \( \lim_{n \to \infty} d(s_n, T_l s_n)=0 \) for each \( l \in I \). We claim that \( s \) is the fixed point of \( \{ T_i : i \in I \} \).

Now, we define a sequence \( \{ t_m \} \) in \( M \) by \( t_m = T_m s \) where \( T_m = T_m(mod N) \).

Now, we can write

\[
d(t_m, s_n) \leq d(T_m s, T_m s_n) + d(T_m s_n, T_{m-1} s_n) + \ldots + d(T s_n, s_n)
\]

\[
\leq d(s, s_n) + \sum_{i=1}^{m-1} d(s_n, T_i s_n).
\]

Therefore by above estimation, we have

\[
r(t_m, \{ s_n \}) = \lim_{n \to \infty} \sup d(t_m, s_n)
\]

\[
\leq \lim_{n \to \infty} \sup d(s, s_n) = r(s, \{ s_n \}).
\]

\[
\Rightarrow |r(t_m, \{ s_n \}) - r(s, \{ s_n \})| \to 0 \text{ as } m \to \infty.
\]

Now by Lemma 2.4.5, \( T_m(mod N) s = s \). Thus \( s \) is the common fixed point of \( \{ T_i : i \in I \} \).

Next, we claim the uniqueness of the asymptotic center ‘s’ for each subsequence \( \{ s_n \} \) of \( \{ x_n \} \). Let us assume that \( x \neq s \).

By Lemma 2.4.7 (i), \( \lim_{n \to \infty} d(x_n, s) \) exists and by the uniqueness of asymptotic center \( s \),

\[
\lim_{n \to \infty} \sup d(s_n, s) < \lim_{n \to \infty} \sup d(s_n, x)
\]

\[
\leq \lim_{n \to \infty} \sup d(x_n, x)
\]

\[
< \lim_{n \to \infty} \sup d(x_n, s)
\]

\[
= \lim_{n \to \infty} \sup d(s_n, s),
\]

which is a contradiction.

Hence \( x = s \). Since \( \{ s_n \} \) is an arbitrary sequence of \( \{ x_n \} \), therefore \( A(\{ s_n \}) = \{ s \} \) for all subsequences \( \{ s_n \} \) of \( \{ x_n \} \). This proves that \( \Delta \)-converges of sequence \( \{ x_n \} \) to a common fixed point of \( \{ T_i : i \in I \} \).
We now establish strong convergence of the iteration (2.4.2) based on lemma 2.4.7.

**Theorem 2.4.9.** Let $M, X, \{T_i : i \in I\}$ and $\{x_n\}$ be as in Theorem 2.4.8. Suppose that $T_m \in \{T_m \}_{m=1}^{r}$ is semi-compact. Then the sequence $\{x_n\}$ defined in (2.4.2) converges strongly to $s \in F$.

**Proof.** Suppose that $T_m$ is semi-compact for some positive integers $1 \leq m \leq r$. Then by lemma 2.4.7 (ii), we have $\lim_{n \to \infty} d(T_n x_n, x_n) = 0$.

Since $\{x_n\}$ is bounded and $T_m$ is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to s$ as $j \to \infty$. By continuity of $T_m$ and lemma 2.4.7 (ii), we obtain

$$d(T_m s, s) = \lim_{j \to \infty} d(T_m x_{n_j}, x_{n_j}) = 0$$

for each $m = 1, 2, 3, \ldots, r$.

This implies that $s$ is the common fixed point of $\{T_m\}_{m=1}^{r}$. The remaining proof is similar to Theorem 2.4.8.

**Remarks 2.4.10.** From the above, we can state that the result is an extended result in the general setup of uniformly convex Hyperbolic spaces. This result can be extended to strong convergence and multi-step iteration with different classes of nonlinear mappings in different fields.