CHAPTER - 3

CORDIAL LABELING OF GRAPHS

3.1 Introduction. Through this chapter, we introduce the vertex product cordial labeling for some particular family of graphs. In section 3.2, we provide definitions and theorems in support of our main results (VPC labeling). In the next two sections, we prove the existence of this labeling for some path related graphs, helm graph and gear related graphs. Section 3.5 is related to switching of a vertex and vertex product cordial labeling of graphs with this condition. In section 3.6, we show some product related graphs like $C_n\Theta K_1$, ladder graph $P_n \times P_2$ with a 1-edge path attached to its one vertex and $C_n \Theta \overline{K_m}$ admits vertex product cordial labeling. In last section 3.7, we introduce total vertex product cordial labeling and discuss the existence of this labeling for wheel graph, helm graph, fan graph etc.

3.2 Vertex Product Cordial Labeling

In 1987, Cahit [51] introduced cordial labeling and presented a paper on cordial labeling in which he examined some results on this newly defined concept.

Later in 2004, Sundaram, Somasundaram and Lee [130, 201, 202] introduced E-cordial, product cordial, prime cordial and total product cordial labeling etc. with some variations in cordial labeling. Recently, Vaidya, Shah and Barasara [208, 209, 210] have studied cordiality behavior of numerous graphs.

Remark : We write VPC labeling in place of vertex product cordial labeling.
In 2004, Sundaram, Ponraj and Somsundaram [201] defined vertex product cordial labeling as follows:

**3.2.1. Definition [201].** “A binary vertex labeling \( f \) of a graph \( G \) with induced edge labeling \( f^*: E \rightarrow \{0, 1\} \) defined by \( f^*(e=uv) = f(u) f(v) \) is called a product cordial labeling or vertex product cordial (VPC) labeling if \( |v_f(1) - v_f(0)| \leq 1 \) and \( |e_{f^*}(1) - e_{f^*}(0)| \leq 1 \). A graph \( G \) is VPC if it admits VPC labeling.”

In 2004 Sundaram, Ponraj and Somsundaram [201] defined VPC labeling for dragons, union of two paths and for unicyclic graph of odd order. They also proved that a graph with order \( p \) and size \( q \) with \( p \geq 4 \) is VPC then \( q < \frac{p^2 - 1}{4} \). In 2010, Vaidya and Kanani [209] investigated VPC labeling for the shadow graph of cycle \( C_n \). The following type of problems can be considered in the area of VPC labeling.

- how VPC labeling is affected under various graph operations.
- construct new families of VPC graph by finding suitable labeling.

Vaidya, Dani, Kanani, Barasara and Vyas [209, 210, 211] investigated VPC labeling for cycle related graphs, friendship graph, middle graph of path and tensor graph. Vaidya and Barasara [208] proved the existence of VPC labeling for friendship graph, cycle with one chord, cycle with twin chord. Recently, Vaidya, Dani, Kanani and Vyas proved the following theorems:

**3.2.2. Theorem [212].** “Graph obtained by joining apex vertices of two stars is vertex product cordial.”

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3.2.3. Remark[212]. Similar result holds for shell and wheel graphs.

For other results and theorems related to VPC labeling, see [212, 213].

3.3 Vertex Product Cordial Labeling For Some Path Related Graphs

With the definition of VPC graph, we are ready to study VPC labeling for certain graphs. First, we go through vertex product cordial labeling with path related graphs such as $P_n^2$, k copies of graph $P_n^2$ and $P_n \circ K_1$.

3.3.1. Theorem [121]. $P_n^2$ is vertex product cordial graph for odd n.

Proof. Let $P_n^2$ be the graph with vertices $v_1, v_2, ..., v_n$ and $E(P_n^2) = \{ v_i v_{i+1} : 1 \leq i \leq n-1; v_i v_{i+2} : 1 \leq i \leq n-2 \}$. We have $|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n - 3$. We define the vertex labeling $f: V(P_n^2) \rightarrow \{0, 1\}$ as follows:

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1, \\ 0 & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 < i \leq n. \end{cases}$$

In view of last defined pattern, we have

$$v_f(1) = \left\lceil \frac{n}{2} \right\rceil + 1, \quad v_f(0) = \left\lceil \frac{n}{2} \right\rceil$$

Now using definition 1.10.1, we see that for $n = 5, 7, 9, ...$,

$$e_f(1) = 3, 5, 7, ... \quad & \quad e_f(0) = 4, 6, 8, ...$$

respectively. Therefore, applying induction it can be seen that $e_f(1) = 2 \left\lceil \frac{n}{2} \right\rceil - 1 = n - 2$ and $e_f(0) = 2 \left\lceil \frac{n}{2} \right\rceil = n - 1$. So, the result holds.

3.3.2. Example[121]. Figure 3.1 shows the graph $P_5^2$ and its VPC labeling.
3.3.3. **Theorem [121]**. $P_n^2$ is not vertex product cordial graph for even $n$.

**Proof.** The graph $P_n^2$ have $n$ vertices and $2n-3$ edges. Assign 1 to first $n/2$ consecutive vertices, so that the vertex condition of the vertex product cordial graph is satisfied. The vertices with label 1 cause $n-3$ edges to have label 1 and at most $n$ edges to have label 0. Therefore $|e_f(x) - e_f(y)|=3$. So the edge condition for vertex product cordial graph is not satisfied.

Hence, $P_n^2$ is not a vertex product cordial graph for even $n$.

3.3.4. **Theorem [121]**. The path union of $k$ copies of graph $P_n^2$ admits vertex product cordial labeling.

**Proof.** Let $G$ be path union of $k$ copies $G_1, G_2, \ldots, G_k$ of $P_n^2$. Let $\{ v_{i1}, v_{i2}, \ldots, v_{in} \}$ be the vertex set and $v_{in}v_{i+1,1}$ is edge joining $G_i$ and $G_{i+1}$. Here $p=kn$, $q=2k(n-1)-1$. Define $f: V(G)\rightarrow\{0,1\}$:

**Case (1):** $k$ is even, $\forall n$.

\[
\begin{align*}
f(v_{ij}) &= 1 \text{ for } 1 \leq i \leq \frac{k}{2}, 1 \leq j \leq n , \\
f(v_{ij}) &= 0 \text{ for } \frac{k}{2} + 1 \leq i \leq k \text{ and, } 1 \leq j \leq n .
\end{align*}
\]

we have
Now using definition 1.10.1, we obtain $v_f(1) = v_f(0) = \frac{nk}{2}$, and

$$e_{f*}(1) = \left[\frac{2k(n-1)-1}{2}\right], e_{f*}(0) = \left[\frac{2k(n-1)-1}{2}\right]+1.$$  

So, the vertex condition and edge conditions of vertex product cordial labeling are satisfied.

**Case (2):** $k$ is odd and $n$ is also odd.

- $f(v_{ij}) = 1$ for $1 \leq i \leq \left[\frac{k}{2}\right] \oplus j$,  
- $f(v_{ij}) = 1$ for $i = \left[\frac{k}{2}\right]+1$ and $1 \leq j \leq \left[\frac{n}{2}\right]+1$,  
- $f(v_{ij}) = 0$ for $i = \left[\frac{k}{2}\right]+1$ and $\left[\frac{n}{2}\right]+2 \leq j \leq n$,  
- $f(v_{ij}) = 0$ for $\left[\frac{k}{2}\right]+1 \leq i \leq k$ and $1 \leq j \leq n$.

It can be easily seen that the conditions of vertex product cordial labeling are satisfied either $|v_{f}(1) - v_{f}(0)| \leq 1$ and $|e_{f*}(1) - e_{f*}(0)| \leq 1$.

3.3.5. **Remark [121].** The path union of $k$ copies of $P^2_n$ does not admit vertex product cordial labeling for odd $k$ and even $n$. Assign 1 to first consecutive $\frac{nk}{2}$ vertices, for which the vertex condition is satisfied. The vertices with label 1 cause $nk-k-2$ edges to have label 1 and at most $nk-k+1$ edge to have label 0. Therefore $|e_{f*}(1) - e_{f*}(0)|=3$. So the edge condition for vertex product cordial graph is not satisfied.

3.3.6. **Example[121].** The VPC labeling of path union of 2- copies of $P^2_5$ is depicted in Figure 3.2.
3.3.7. Theorem[121]. $P_n \circ K_1$ admits vertex product cordial labeling.

Proof. Let $G = P_n \circ K_1$, with $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ vertices and $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_1u_1, v_2u_2, \ldots, v_nu_n$ edges. Consider the vertex labeling $f: V(G) \to \{0, 1\}$ defined by

$f(v_i) = 1$ for $1 \leq i \leq n$,
$f(u_i) = 0$ for $1 \leq i \leq n$.

Thus, we have $v_f(1) = v_f(0) = n$ and using definition 1.10.1, we obtain $e_f(1) = n-1, e_f(0) = n$.

So, we have $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$. Hence, $P_n \circ K_1$ is vertex product cordial graph for all $n$.

3.3.8. Example[121]. VPC labeling for $P_4 \circ K_1$ is depicted in next Figure.
3.4. Vertex Product Cordial Labeling of Helm Graph and Gear Related Graphs

In this section, we investigate the VPC labeling for helm graph and gear graph.

3.4.1. Definition [121]. “Helm graph $H_n$ is the graph obtained from $W_n$ by attaching a pendant edge to each rim vertex. It contains three types of vertices: an apex of degree $n$, $n$ vertices of degree 4 and $n$ pendant vertices.”

3.4.2. Theorem [121]. $H_n$ is a vertex product cordial graph.

Proof. Let $v$ be the apex, $v_1,v_2,...,v_n$ be the vertices of degree 4 and $u_1,u_2,...,u_n$ be the pendant vertices of $H_n$. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. Define vertex labeling $f : V(H_n) \rightarrow \{0,1\}$ as follows:

Case 1: $n$ is an even integer.

$f(v) = 1$,

$f(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \frac{n}{2} + 1, \\ 0 & \text{for } \frac{n}{2} + 2 \leq i \leq n, \end{cases}$

and

$f(u_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \frac{n}{2} - 1, \\ 0 & \text{for } \frac{n}{2} \leq i \leq n. \end{cases}$

In view of above defined labeling pattern, we have

$v_f(1) = n+1$ and $v_f(0) = n,$
Now using definition 1.10.1, we get $e_{v*}(1) = e_{v*}(0) = \frac{3n}{2}$. So $v_{f}(1) - v_{f}(0) \leq 1$ and $|e_{v*}(0) - e_{v*}(1)| \leq 1$. Hence, $H_n$ admits vertex product cordial labeling for even $n$.

**Case 2:** $n$ is odd

$$f(v) = 1,$$

$$f(v_i) = \begin{cases} 
1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \\
0 & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n,
\end{cases}$$

and

$$f(u_i) = \begin{cases} 
1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
0 & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n.
\end{cases}$$

Thus, we have $v_f(1) = n+1$, $v_f(0) = n$ and using definition 1.10.1, we get $e_{v*}(1) = \left\lfloor \frac{3n}{2} \right\rfloor$ and $e_{v*}(0) = \left\lfloor \frac{3n}{2} \right\rfloor + 1$.

Hence, this graph is vertex product cordial graph.

**3.4.3[121]. Example.** VPC labeling for $H_8$ is depicted in next Figure.
3.4.4. Definition[121]. “Gear graph $G_n$ is obtained from $W_n$ by subdividing each of its rim edge.”

The next theorem gives vertex product cordial labeling for gear graph for odd $n$.

3.4.5. Theorem[121]. The gear graph $G_n$ is vertex product cordial graph for odd $n$.

Proof. Let $u$ be the apex vertex of gear graph $G_n$ and $v_1, v_2, ..., v_n$ and $u_1, u_2$ ...., $u_n$ be the rim vertices. Define $f : V(G_n) \rightarrow \{0,1\}$ as follows:

$$f(u) = 1,$$

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1, \\ 0 & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 < i \leq n, \end{cases}$$

and
Thus, we have

\[ v_f(1) = n+1, \quad v_f(0) = n \]

and now using definition 1.10.1, we get

\[ e_f(1) = \left\lceil \frac{3n}{2} \right\rceil, \quad e_f(0) = \left\lfloor \frac{3n}{2} \right\rfloor + 1. \]

Hence, the gear graph \( G_n \) is vertex product cordial graph for odd \( n \).

3.4.6. Example [121]. The gear graph \( G_7 \) with its VPC labelling is depicted in next Figure.

![Vertex product cordial labeling of \( G_7 \).](image-url)
3.4.7. Remarks[121]. In the above theorem we can write \( v_2, v_4, \ldots \) in place of \( u_1, u_2, \ldots \) so on.

3.4.8. Theorem[121]. The gear graph \( G_n^4 \) with a chord admits vertex product cordial labelling for even \( n \geq 4 \).

Proof. If \( u \) apex vertex of gear graph \( G_n \) and \( v_1, v_2, \ldots, v_{2n} \) the rim vertices of \( G_n \).

Let \( v_1v_5 \) be the chord in \( G_n \). Then \( |V(G_n)| = 2n+1 \) and \( |E(G)| = 3n+1 \), Define vertex labeling \( f: V(G_n) \rightarrow \{0,1\} \) as follows.

\[
\begin{align*}
    f(u) &= 1, \\
    f(v_i) &= 1 \quad \text{for } 1 \leq i \leq n-1, \\
    f(v_n) &= 0, \\
    f(v_i) &= 1 \quad \text{for } i = n+1, \\
    f(v_i) &= 0 \quad \text{for } n+2 \leq i \leq 2n.
\end{align*}
\]

Thus, \( v_l(1) = n+1, v_l(0) = n \) and \( e_{f^+}(1) = \frac{3n}{2}, e_{f^+}(0) = \frac{3n}{2} + 1 \). Hence, then graph \( G_n^4 \) admits vertex product cordial labeling.

3.4.9. Example[121]. The pattern of above defined labeling understood by VPC labeling of \( G_n^4 \) is shown in Figure 3.6. In this labeling \( n \) is even.
3.5 Switching of a Vertex and Vertex Product Cordial Labeling for Some Product Related Graphs

3.5.1 Definition[121]. "A vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$, removing all the edges incident to $v$ and adding edges joining $v$ to every other vertex which are not adjacent to $v$ in $G"."

3.5.2 Theorem[121]. Switching of a vertex in cycle $C_n$ admits vertex product cordial labeling.

Proof. Let $C_n$ be the cycle with $v_1, v_2, \ldots, v_n$ vertices and $G_{v_1}$ denotes the graph obtained by switching of a vertex $v_1$ of $G = C_n$. Then $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 5$. Define the vertex labeling $f : V(G_{v_1}) \rightarrow \{0, 1\}$ as follows:
Case 1: when \( n \) is even.

\[ f(v_1) = 1, \quad f(v_2) = 0, \]

\[ f(v_i) = \begin{cases} 
1 & \text{for } 3 \leq i \leq \frac{n}{2} + 1, \\
0 & \text{for } \frac{n}{2} + 2 \leq i \leq n.
\end{cases} \]

Thus, we have \( v_f(1) = v_f(0) = \frac{n}{2} \) and from definition 1.10.1, \( e_{f*}(1) = n-3, \)
\( e_{f*}(0) = n-2. \) So \( |v_f(1) - v_f(0)| \leq 1 \) and \( |e_{f*}(1) - e_{f*}(0)| \leq 1. \)

Case 2: when \( n \) is odd.

\[ f(v_1) = 1, \quad f(v_2) = 0, \]

\[ f(v_i) = \begin{cases} 
1 & \text{for } 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 2, \\
0 & \text{otherwise.}
\end{cases} \]

Thus, we have \( v_f(1) = \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad v_f(0) = \left\lfloor \frac{n}{2} \right\rfloor \) and from definition 1.10.1, \( e_{f*}(1) = \left\lfloor \frac{2n-5}{2} \right\rfloor + 1, \)
\( e_{f*}(0) = \left\lfloor \frac{2n-5}{2} \right\rfloor. \)

Hence, the required condition of vertex product cordial labeling is satisfied.

3.5.3. Example[121]. Switching of a vertex in cycle \( C_9 \) namely \( G_9 \) with its VPC is depicted in next figure.
In this section, we represent vertex product cordial labeling for some product related graphs like $C_n \circ K_1$, ladder graph $P_n \times P_2$ with a 1-edge path attached to its one vertex and for $C_n \circ \overline{K_m}$.

**3.6.1 [121]. Definition.** “Let $G_1$ and $G_2$ be two graphs then the corona product of $G_1$ and $G_2$ is denoted by $G_1 \circ G_2$. It is a graph obtained by one copy of $G_1$ (which has $p_1$ vertices) and $p_1$ copies of $G_2$ and joining $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$.”

**3.6.2 Theorem [121].** The corona product $C_n \circ K_1$ admits vertex product cordial labeling for all $n$.
Proof. Let \( \{ v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \} \) be the set of vertices for the graph \( C_n \Theta K_1 \), where \( u_i \) is adjacent to \( v_i \). Here, \( |V(C_n \Theta K_1)| = 2n \) and \( |E(C_n \Theta K_1)| = 2n \). Define \( f: V(C_n \Theta K_1) \rightarrow \{0, 1\} : \)

\[
f(u_i) = 1 \text{ if } 1 \leq i \leq n,
\]

\[
f(v_i) = 1 \text{ if } 1 \leq i \leq n.
\]

Thus, \( v_f(1) = v_f(0) = n \) and from definition 1.10.1, we have \( e_{f, 1}(1) = e_{f, 1}(0) = n \). So \( |v_f(1) - v_f(0)| \leq 1 \) and \( |e_{f, 1}(1) - e_{f, 1}(0)| \leq 1 \). Hence, the corona product \( C_n \Theta K_1 \) admits vertex product cordial labeling.

3.6.3. Example[121]. Figure 3.8 shows Corona product \( C_6 \Theta K_1 \) with its VPC labeling.

**Figure 3.8:** \( C_6 \Theta K_1 \) with its vertex product cordial labeling.
3.6.4. Theorem [121]. G obtained by attaching a 1-edge path to one-vertex of a
ladder graph $P_n \times P_2$ is vertex product cordial.

Proof. Let $G$ be the graph obtained by attaching 1-edge path to a vertex of ladder
graph $P_n \times P_2$. Let $V(G)=$\{v, v$_1$, v$_2$,\ldots,v$_n$, u$_1$, u$_2$,...,u$_n$\} and the edge set $E(G)=$\{v$_i$v$_{i+1}$ : 1$\leq i$ $\leq$ n-1 \} $\cup$ \{u$_i$u$_{i+1}$ : 1$\leq i$ $\leq$ n-1 \} $\cup$ \{u$_i$ v$_i$ : 1$\leq i$ $\leq$ n \} $\cup$ \{v$_n$ v \}. Clearly
\[|V(G)| = 2n+1\] and \[|E(G)| = 3n-1.\] Define $f$: $V(G)$ $\rightarrow$ \{0, 1\} as follows:

Case 1: $n$ is odd. Then
\[
f(v_i) = 1 \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,
\]
\[
f(v_i) = 0 \quad \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n,
\]
\[
f(u_i) = 1 \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,
\]
\[
f(u_i) = 0 \quad \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n,
\]
and
\[
f(v) = 0.
\]

Thus, $v_f (1) = \left\lceil \frac{2n+1}{2} \right\rceil + 1$, $v_f (0) = \left\lfloor \frac{2n+1}{2} \right\rfloor$ and using definition 1.10.1, we see
\[
e_{f*}(0) = e_{f*}(1) = \frac{3n-1}{2}.
\]

Case 2: $n$ is even. Then
\[
f(v_i) = 1 \quad \text{for } 1 \leq i \leq \frac{n}{2} + 1,
\]
\[
f(v_i) = 0 \quad \text{for } \frac{n}{2} + 2 \leq i \leq n,
\]
\[
f(u_i) = 1 \quad \text{for } 1 \leq i \leq \frac{n}{2},
\]
\[ f(u_i) = 0 \quad \text{for} \quad \frac{n}{2} + 1 \leq i \leq n, \]

and

\[ f(v) = 0. \]

Thus, \( v_{f}(1) = n + 1, \) \( v_{f}(0) = n \) and from definition 1.10.1, we have \( e_{f_0}(0) = \left\lfloor \frac{3n-1}{2} \right\rfloor + 1, \) \( e_{f_0}(1) = \left\lfloor \frac{3n-1}{2} \right\rfloor. \)

One can observe that in each case, the labeling defined above satisfies the conditions of VPC labeling and the graph is vertex product cordial.

**Illustration.** VPC labeling of ladder graph \( P_5 \times P_2 \) with one chord is depicted in next figure.

![Vertex product cordial labeling of \( P_5 \times P_2 \) with one chord.](image)

**Figure 3.9:** Vertex product cordial labeling of \( P_5 \times P_2 \) with one chord.

**3.6.5. Theorem [121].** \( C_n \Theta K_m \) admits vertex product cordial labeling for odd \( n \) and even \( m. \)
Proof. Let \( G = C_n \odot \overline{K}_m \). Then vertex set \( V = \{ u_i : 1 \leq i \leq n \} \cup \{ u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m \} \) and edge set \( E = \{ u_i, u_{i+1}, u_n u_1 : 1 \leq i \leq n-1, \} \cup \{ u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m \} \). Here \( |V| = mn + n, |E| = mn + n \). Define \( f : V(G) \to \{0,1\} \) as follows.

\[
\begin{align*}
f(u_i) &= 1 \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1; \\
f(u_i) &= 0 \quad \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n; \\
f(u_{ij}) &= 1 \quad \text{for } i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq j \leq m; \\
f(u_{ij}) &= 1 \quad \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1, 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor; \\
f(u_{ij}) &= 0 \quad \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq j \leq m; \\
f(u_{ij}) &= 0 \quad \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 2, \ldots, n; 1 \leq j \leq m ;
\end{align*}
\]

Thus, \( v_f(0) = \left\lfloor \frac{mn+n}{2} \right\rfloor \), \( v_f(1) = \left\lfloor \frac{mn+n}{2} \right\rfloor + 1 \) and from definition 1.10.1, we have \( e_{f_{v}}(1) = \left\lfloor \frac{mn+n}{2} \right\rfloor \), \( e_{f_{v}}(0) = \left\lfloor \frac{mn+n}{2} \right\rfloor + 1 \). Hence, the graph \( C_n \odot \overline{K}_m \) is vertex product cordial graph.

3.6.6. Example [121]. \( C_5 \odot \overline{K}_4 \) with its VPC labeling is depicted in Figure 3.10.
3.6.7. Theorem [121]. Let \( G = B T (n_1, n_2, n_3) \). If \( n_1 = n_2 = n_3 = n \), then \( G \) admits vertex product cordial labeling.

**Proof.** Let the vertex set \( V(G) = \{s, v, v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_n, w, w_1, w_2, \ldots, w_n\} \) and the edge set \( E(G) = \{sv_n, su_n, sw_n\} \cup \{vv_i: 1 \leq i \leq n\} \cup \{uu_i: 1 \leq i \leq n\} \cup \{ww_i: 1 \leq i \leq n\} \). Clearly \(|V(G)| = 3n+4\) & \(|E(G)| = 3n+3\). Define \( f: V(G) \rightarrow \{0, 1\} \):

\[
\begin{align*}
  f(s) &= f(v) = f(u) = 1, \quad f(w) = 0, \\
  f(v_n) &= f(w_n) = f(u_n) = 1, \\
  f(v_i) &= 1 \quad \text{for } 1 \leq i \leq n-1, \\
  f(u_i) &= 0 \quad \text{for } 1 \leq i \leq n-1,
\end{align*}
\]

Figure 3.10: \( C_5 \circ K_4 \) with its vertex product cordial labeling.
\[ f(w_i) = 0 \text{ for } 1 \leq i \leq n-1. \]

**Case 1**: \( n \) is even

\[ v_f(1) = \frac{3n+4}{2} = v_f(0) \text{ and using definition 1.10.1, we see } e_{f^*}(1) = \left[ \frac{3n+3}{2} \right], \]
\[ e_{f^*}(0) = \left[ \frac{3n+3}{2} \right] + 1. \]

Hence, the result holds for even \( n \).

**Case 2**: \( n \) is odd

\[ v_f(1) = \left[ \frac{3n+4}{2} \right] + 1, \quad v_f(0) = \left[ \frac{3n+4}{2} \right] \text{ and using definition 1.10.1, we see } e_{f^*}(1) = e_{f^*}(0) = \frac{3n+3}{2}. \]

Hence, the Banana tree \( BT(n_1, n_2, n_3), n_1 = n_2 = n_3 = n \) is a vertex product cordial graph.

**3.6.8. Example[121]**. The Banana tree \( BT(6,6,6) \) with its VPC labeling is depicted in next Figure.
3.7 Total vertex product cordial labeling

In 2004, Sundram, Somasundram [201] introduced the notion of total vertex product cordial labeling.

3.7.1. Definition[142]. “Let G be a graph, then a vertex labeling function $f : V(G) \to \{0, 1\}$ induces an edge labeling function $f^* : E(G) \to \{0, 1\}$ defined as $f^*(uv) = f(u)f(v)$ then the function $f$ is said to be total vertex product cordial labeling of $G$ if $|v_f(1)+e_{f^*}(1))-(v_f(0)+e_{f^*}(0))| \leq 1$. A graph with this labeling is called a total vertex product cordial graph.”

Total vertex product cordial labeling (TVPC labeling) of a graph is defined only for some connected graphs and for some special graphs containing isolated vertex. TVPC labeling for many graphs having some another conditions can also
be defined which we have not discussed in this chapter. Here, we discuss TVPC labeling for some connected graphs such as wheel graph, double star graph, fan graph, helm graph, closed cycle and fully binary tree.

### 3.7.2 Theorem [142]

The wheel graph $W_n$ is total vertex product cordial graph.

#### Proof

Let $W_n$ be the wheel graph with $v_1,v_2,v_3...v_n$ rim vertices and $v$ be the apex vertex.

Let $f: V(G) \to \{0, 1\}$ be defined as

**Case (i):** $n$ is an even integer.

$$f(v) = 1,$$

$$f(v_i) = \begin{cases} 
1 & \text{for } 1 \leq i \leq \frac{n}{2}, \\
0 & \text{for } \left(\frac{n}{2} + 1\right) \leq i \leq n.
\end{cases}$$

By using definition 1.10.1, we have $v_f(1) = \frac{n}{2} + 1$, $v_f(0) = \frac{n}{2}$ and $e_{f_a}(1) = n-1$, $e_{f_a}(0) = n+1$.

Thus, $|v_f(1) + e_{f_a}(1)| = \frac{3n}{2}$ and $|v_f(0) + e_{f_a}(0)| = \frac{3n}{2} + 1$. So, we have $|(v_f(1) + e_{f_a}(1)) - v_f(0) + e_{f_a}(0)| \leq 1$.

**Case (ii):** $n$ is an odd integer.

$f(v) = 1,$
\[
f(v_i) = \begin{cases} 
1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
0 & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-2, \\
1 & \text{for } i = n-1, \\
0 & \text{for } i = n.
\end{cases}
\]

Using definition 1.10.1, we see \( v_f(1) = \frac{n+3}{2} \), \( v_f(0) = \frac{n-1}{2} \) and \( e_{f*}(1) = n-1 \), \( e_{f*}(0) = n+1 \).

Thus, \( |v_f(1) + e_{f*}(1)| = \frac{3n+1}{2} \) and \( |v_f(0) + e_{f*}(0)| = \frac{3n+1}{2} \). So, we have \(|(v_f(1) + e_{f*}(1)) - (v_f(0) + e_{f*}(0))| \leq 1\).

Hence, \( W_n \) is total vertex product cordial graph.

3.7.3. **Remark.** Function \( f* \) in above theorem and in all other theorems represents an edge labeled function as defined in the definition of total vertex product cordial labeling.

3.7.4. **Example.** The wheel graphs \( W_6 \) and \( W_5 \) with its TVPC labeling are depicted in Figure 3.12.

![Figure 3.12: Total vertex product cordial labeling of \( W_6 \) and \( W_5 \).](image)
3.7.5. Theorem[142]. The double star graph $S_{m+1, n+1}$ admits total vertex product cordial labeling.

**Proof.** Let $G = S_{m+1, n+1}$ be a double star graph with $u$ and $v$ non-pendant vertices and $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$ be the other vertices. Clearly $|V(G)| = m + n + 2$ and $|E(G)| = m + n + 1$. Define $f: V(G) \rightarrow \{0, 1\}$:

**Case (i):** For $m=n$.

- $f(u) = 1$,
- $f(u_i) = 1$ for $i = 1, 2, \ldots, m$,
- $f(v) = 0$,
- $f(v_i) = 0$ for $i = 1, 2, \ldots, n$.

Using definition 1.10.1, we have $v_f(1) = m + 1$, $v_f(0) = m + 1$ and $e_{f_1}(1) = m$, $e_{f_1}(0) = m + 1$. So $|v_f(1) + e_{f_1}(1)| = 2m + 1$ and $|v_f(0) + e_{f_1}(0)| = 2m + 2$. So $|(v_f(1) + e_{f_1}(1)) - (v_f(0) + e_{f_1}(0))| \leq 1$.

**Case (ii):** For $m \neq n$, let $m \leq n$ and $n = m + m'$

- $f(u) = 1$,
- $f(u_i) = 1$ for $i = 1, 2, \ldots, m$,
- and $f(v) = 0$,
- $f(v_i) = \begin{cases} 0 & \text{for } i = 1, 2, \ldots, m, \\ 1 & \text{for } i = m + 1, \ldots, n. \end{cases}$

From definition 1.10.1, we have $v_f(1) = m + m' + 1$, $v_f(0) = m + 1$ and $e_{f_1}(1) = m$, $e_{f_1}(0) = m + m' + 1$.

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Thus, $|v_f(1)+e_{f_2}(1)| = 2m + m' + 1$ and $|v_f(0)+e_{f_2}(0)| = 2m + m' + 2$.

So, $|(v_f(1)+e_{f_2}(1)) - (v_f(0)+e_{f_2}(0))| \leq 1$. Hence, the double star graph $S_{m+1, n+1}$ is total vertex product cordial graph.

### 3.7.6. Example

Next Figure depicts TVPC labeling for double star graph $S_{4,6}$.

![Total vertex product cordial labeling of $S_{4,6}$](image)

**Figure 3.13:** Total vertex product cordial labeling of $S_{4,6}$.

### 3.7.7. Theorem [142]

The fan graph $f_n$ admits a total vertex product cordial labeling.

**Proof.** Let $v$ be an apex vertex and $v_1, v_2, \ldots, v_n$ be the other vertices of the fan graph $f_n$. Here, $p = n+1$ and $q = 2n-1$. Define $f: V(f_n) \rightarrow \{0, 1\}$ as follows:

**case (i):** For even integer $n$.

\[
 f(v)=1, \\
 f(v_i)=\begin{cases} 
 1 & \text{for } 1 \leq i \leq \frac{n}{2} \\
 0 & \text{for } \left(\frac{n}{2} + 1\right) \leq i \leq n.
\end{cases}
\]

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Applying definition 1.10.1, we obtain \( v(f)(1) = \frac{n}{2} + 1 \), \( v(f)(0) = \frac{n}{2} \) and \( e_{f*(1)} = n - 1 \), \( e_{f*(0)} = n \).

Thus, \( |v(f)(1) + e_{f*}(1)| = \frac{3n}{2} \) and \( |v(f)(0) + e_{f*}(0)| = \frac{3n}{2} \). Hence, \( |v(f)(1) + e_{f*}(1)| - (v(f)(0) + e_{f*}(0))| \leq 1 \).

**case (ii):** For odd integer \( n \).

\( f(v) = 0 \),

\[
 f(v_i) = \begin{cases} 
 0 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor, \\
 1 & \text{for } \left\lfloor \frac{n}{4} \right\rfloor + 1 \leq i \leq n.
\end{cases}
\]

It follows from definition 1.10.1, that \( v(f)(1) = \frac{3n + 1}{4} \), \( v(f)(0) = \frac{n + 3}{4} \) and \( e_{f*}(1) = \frac{3n - 3}{4} \), \( e_{f*}(0) = \frac{5n - 1}{4} \). Thus, \( |v(f)(1) + e_{f*}(1)| = \frac{3n + 1}{2} \) and \( |v(f)(0) + e_{f*}(0)| = \frac{3n - 1}{2} \). So \( |v(f)(1) + e_{f*}(1)| - (v(f)(0) + e_{f*}(0))| \leq 1 \).

Hence, the fan graph \( f_n \) admits a total vertex product cordial labeling.

**3.7.8. Example.** The fan graphs \( f_5 \) and \( f_6 \) with its TVPC labeling are depicted in next Figure 3.14.
3.7.9. Theorem [142]: The helm graph $H_n$ is total vertex product cordial graph, when $n = 3, 4$.

Proof. Let $v$ be the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 & $u_1, u_2, \ldots, u_n$ be the pendant vertices of $H_n$. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. We define $f: V(G) \to \{0, 1\}$ as follows:

**case (i):** when $n = 3, 4$

$$f(v) = 1,$$

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq n - 1, \\ 0 & \text{for } i = n. \end{cases}$$

and

$$f(v_i) = \begin{cases} 1 & \text{for } j = 1, \\ 0 & \text{for } 2 \leq j \leq n \end{cases}$$

From above defined labeling and applying definition 1.10.1, we have

$$v_{f}(1) = n+1, \ v_{f}(0) = n \ \text{and} \ e_{f^*}(1) = 2n-2, \ e_{f^*}(0) = n+2.$$
Thus, \(|v_f(1)+e_{f_{1}}(1)|=3n-1\) and \(|v_f(0)+e_{f_{0}}(0)|=2n+2\), and we get \(|(v_f(1)+e_{f_{1}}(1))-(v_f(0)+e_{f_{0}}(0))|\leq1\).

This completes the proof.

3.7.10. Example. The helm graph \(H_3\) and \(H_4\) with TVPC labeling.

3.7.11. Theorem[142]. The cycle graph \(C_n\) is total vertex product cordial graph.

Proof. Let \(v_1, v_2, ..., v_n\) be the \(n\) vertices of cycle \(C_n\). We define \(f: V(G) \rightarrow \{0, 1\}\) as follows:

\[
f(v_i) = \begin{cases} 
1 & \text{for } 1 \leq i \leq \frac{n}{2}, \\
0 & \text{for } \frac{n}{2} + 1 \leq i \leq n - 2, \\
1 & \text{for } i = n - 1, \\
0 & \text{for } i = n.
\end{cases}
\]
Applying definition 1.10.1, we have $v_f(1) = \frac{n}{2} + 1$, $v_f(0) = \frac{n}{2} - 1$ and $e_{f_a}(1) = \frac{n}{2} - 1$, $e_{f_a}(0) = \frac{n}{2} + 1$. So, $|v_f(1) + e_{f_a}(1)| = n$ and $|v_f(0) + e_{f_a}(0)| = n$.

Therefore, $|(v_f(1) + e_{f_a}(1)) - (v_f(0) + e_{f_a}(0))| \leq 1$.

**case (ii):** when $n$ is odd

$$f(v_i) = \begin{cases} 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ 0 & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n. \end{cases}$$

Applying definition 1.10.1, we have $v_f(1) = \frac{n+1}{2}$, $v_f(0) = \frac{n-1}{2}$ and $e_{f_a}(1) = \frac{n-1}{2}$, $e_{f_a}(0) = \frac{n+1}{2}$. So, $|v_f(1) + e_{f_a}(1)| = n$ and $|v_f(0) + e_{f_a}(0)| = n$. Therefore, $|(v_f(1) + e_{f_a}(1)) - (v_f(0) + e_{f_a}(0))| \leq 1$.

This completes the proof.

**3.7.12. Remark.** When $n= 2, 4$; $C_n$ is not total vertex cordial graph.

We now give examples of $C_6$ and $C_5$ with total vertex product cordial labeling.

**3.7.13. Example.** The cycle $C_5$ and $C_6$ with TVPC labeling.
Figure 3.16: Total vertex product cordial labeling of cycle $C_5$ and $C_6$.


Proof. We know that every fully binary tree has odd number of vertices and even number of edges.

Let $T$ be a fully binary tree and $v$ be the root of $T$, $v$ is called zero level vertex. If $T$ has $m$ levels then number of vertices in $T$ is $2^m + 1 - 1$ and number of edges is $2^m + 1 - 2$. Let $v_1, v_2, ..., v_r, ...$ be the vertices of $T$. Define $f: V(G) \rightarrow \{0, 1\}$ such that

$f(v) = 1$,

$f(v_i) = 1$ for $1 \leq i \leq \left\lfloor \frac{2^m - 1}{2} \right\rfloor$ and $0$ otherwise.

So, we have $|v_f(1) + e_{i_{1}}(1)| = 2^m + 1 - 1$ and $|v_f(0) + e_{i_{0}}(0)| = 2^m + 1 - 2$. Thus in this case we have $|(v_f(1) + e_{i_{1}}(1)) - (v_f(0) + e_{i_{0}}(0))| \leq 1$. Hence, every fully binary tree is total vertex product cordial graph.
3.7.15. Example. In this example we see that every fully binary tree is TVPC graph.

Figure 3.17: Total vertex product cordial labeling of fully binary tree.