Chapter 1

INTRODUCTION

1.1 FIXED POINTS

Let $X$ be a normed linear space and $f$ be a self-mapping of $X$. A solution of the equation $f(x) = x$, if it exists, is called a fixed point of $f$. Existence problems of this type arise frequently in analysis. For example, the problem of solving the equation $p(x) = 0$ is equivalent to finding a fixed point of the mapping $x \mapsto x - p(x)$, provided it is a self-mapping of $X$. More generally, if $f : K \to X$ is any mapping on a subset $K$ of $X$, to show that the equation $p(x) = 0$ has a solution is equivalent to showing that the mapping $y \mapsto y - p(y)$, has a fixed point in $K$. Thus conditions on a mapping or on its domain of definition which guarantee the existence of fixed point can be frequently reinterpreted as existence theorem in analysis and therefore have considerable interest. Istratescu(1979), Dugundji and Granas(1982), Bose and Joshi(1985), Zeidler(1986) and Takahashi(2000) are very good source books for fixed point theory and its applications. On the other hand if the fixed point equation $f(x) = x$ dose not possess a solution, it is necessary to find a point $x$ in a suitable space, if one such exists, such that $d(x, f(x)) = d(f(x), K)$, where $K$ is the domain.
of the mapping \( f \) and \( d(f(x), K) = \inf_{y \in K} d(f(x), y) \). Thus there exists the theory of best approximations.

Application of fixed point to best approximation theory, mini-max problems, mathematical economics, variational inequalities, eigenvalue problems and boundary value problems are well known [Zeidler (1986)].

1.2 A SURVEY OF FIXED POINT THEOREMS

We often come across the terminology, what is called “Fixed Point Theorem”. By a fixed point theorem, we shall understand a statement which asserts that, under certain conditions, a mapping \( T \) of \( X \) into itself admits one or more fixed points. Historically, the first theorem of this type is the famous theorem of Brouwer that involves a space \( X \), which is a topologically simple subset of \( R^n \), and a continuous mapping of \( X \) into itself. In this theorem, \( X \) can be replaced by any homeomorphic thereof.

Theorem 1.1 (Brouwer’s Fixed Point Theorem) Every continuous mapping of the closed unit ball \( S = \{ x : \| x \| \leq 1 \} \) in \( R^n \) into itself has a fixed point.

An equivalent form of this theorem can be stated as “every continuous mapping of a closed bounded convex set in \( R^n \) into itself has a fixed point”. Brouwer, a Dutch Mathematician, propounded his theorem in 1912. There are several proofs of this fundamental theorem, but most of them depend on the notion of Algebraic Topology. However, for the proof, one may refer to Sasty and Bram, Bers, Kantorovich and
Akilvo. It may be remarked that such theorems, where the spaces are subsets of $\mathbb{R}^n$, are not of much use in functional analysis where one is generally concerned with infinite dimensional subsets of some function spaces. This was first investigated by Birkhoff and Kellog in 1922 during their work on Existence Theorem in analysis. They established fixed points for continuous self-mappings on compact convex subset of $C[0,1]$ and $L^2[0,1]$. Subsequently, Schauder, a Polish Mathematician, propounded his theorem in 1930. In fact, Schauder generalized these results to the case where $X$ is a compact convex subset of a normed linear space.

**Theorem 1.2** (Schauder’s Fixed Point Theorem) Let $C$ be a non-empty compact convex subset of a normed space $X$. Then every continuous mapping of $C$ into itself has a fixed point.

The proof of Schauder’s theorem consists of approximating the infinite dimensional set $C$ by a finite dimensional set, applying Brouwer’s theorem to deduce the existence of a fixed point of the finite dimensional approximation, and then taking the limit as the dimension of the approximating space tends to infinity.

The following version of Schauder’s fixed point theorem is often useful.

**Theorem 1.3** Let $C$ be a non-empty convex closed subset of a normed linear space $X$ and let $D$ be a relatively compact subset of $C$. Then every continuous mapping of $C$ into $D$ has a fixed point.

Schauder’s fixed point theorem was generalized to locally convex topological vector space by Tychonoff and this generalization is known as Schauder-Tychonoff Theorem.
Theorem 1.4 (Schauder-Tychonoff) Let $T$ be a compact and continuous mapping of a normed linear space $X$ into itself and let $T(X)$ be bounded. Then $T$ has a fixed point.

We need the following definitions and known results to make the thesis a self content.

Definition 1.1 Let $T$ be a mapping from a metric space $(X, d)$ into itself. Then mapping $T$ is said to satisfy the Lipschitz condition with Lipschitz constant $\alpha$ if

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$. $T$ is called a contraction if $\alpha < 1$ and non-expansive if $\alpha = 1$.

$T$ is said to be contractive if, for all $x, y \in X$ and $x \neq y$, we have

$$d(Tx, Ty) < \alpha d(x, y).$$

It may be remarked that any Lipschitz mapping is uniformly continuous.

In 1922, Banach formulated and proved the general contraction principle in complete metric spaces, which become soon a powerful tool in both classical and modern analysis. Due to its simplicity and generality, the contraction principle has drawn attention of a very large number of mathematicians.

The Banach’s fixed point theorem asserts that any contraction mapping on a complete metric space $X$ has a unique fixed point. It is the simplest and one of the most versatile result in the fixed point theory. Based on an iteration process, it can be implemented on a computer to find the fixed point of contraction mapping. It
produces approximations of any required accuracy and moreover even the number of
iterations needed to get a specified accuracy can be determined.

Banach’s fixed point theorem has been generalized by Boyd and Wong (1969),
Hardy and Rogers (1973), Husain and Sehgal (1975), Caristi (1976) and Downing and
Kirk (1977) in different directions. For a detailed comparison of various definitions
and fixed point theorems for contraction and contractive mappings one can refer the
source article by Rhoades (1977).

Kannan (1968) obtained existence of a common fixed point for self -mappings $f$
and $g$ of a complete metric space which satisfy the condition

$$d(f(x), g(y)) \leq \beta[d(x, f(x)) + d(y, g(y))], \quad \text{where} \quad 0 < \beta < 1/2.$$  

If $f = g$ in the above equation, then $f$ is called a Kannan mapping. This mapping
need not be continuous.

Edelstein (1962) has derived fixed point theorem in the setting of compact metric
spaces for contractive mapping. Sehgal (1969) has extended this result to the family
of mappings that satisfy the inequality

$$d(f(x), f(y)) < \max\{d(x, y), d(x, f(x)), d(y, f(y))\}, \quad \text{whenever} \quad x \neq y.$$  

On the other hand, the theory of non-expansive mappings is basically different
from that of contraction and contractive mappings. The study of existence of fixed
points for non-expansive mappings is an extension of the classical theory of successive
approximations for contraction mappings. In the case of a contraction the sequence
of iteration defined by $x_{n+1} = f^n(x)$ converges strongly to a unique fixed point of the mapping $f$. In the case of non-expansive mapping this sequence of iteration need not converge, there need not be any fixed point, nor need the fixed point be unique if it exists.

**Example 1.1** Let $X = R$, the set of real numbers. Let $f, g$ be self-mappings of $X$ defined by $f(x) = x + a$ and $g(x) = x$, where $a$ is a non-zero real number. Then both $f$ and $g$ are non-expansive. But no point of $R$ is a fixed point of $f$ and every point of $R$ is a fixed point of $g$.

Because of these reasons, it is necessary to seek additional assumptions regarding the structure of the space to ensure existence of fixed point for non-expansive mappings.

The notion of normal structure, introduced by Brodskii and Milman(1948) was employed by Kirk(1965) to prove the fundamental existential fixed point theorem for non-expansive mappings which states that any non-expansive self-mapping of a nonempty weakly compact convex subset of a Banach space possessing normal structure has a fixed point. Interestingly the same result has been proved independently and simultaneously by Browder(1965) and Gohde(1965). In fact they proved that any non-expansive self-mapping of a nonempty bounded closed convex subset of a uniformly convex Banach space has normal structure, the result of Browder(1965) and Gohde(1965) is a special case of the one of Kirk(1965). Alspach(1981) proved with the following example that the normal structure assumption in Kirk’s result is
not dispensable.

Example 1.2 Let \( K = \{ f \in L^1[0, 1] : \int_0^1 f = 1, 0 \leq f \leq 2 \ a.e \} \). Then \( K \) is a weakly compact convex subset of \( L^1[0, 1] \). Define \( T : K \to K \) by

\[
T(f(t)) = \begin{cases} 
\min\{2f(2t), 2\} & \text{if } 0 \leq t \leq 1/2 \\
\max\{2f(2t - 1) - 2, 0\} & \text{if } 1/2 \leq t \leq 1
\end{cases}
\]

Though \( T \) is non-expansive and has no fixed point in \( K \).

The fixed point theorems of Browder (1965), Gohde(1965) and Kirk (1965) has been extended to a wider class of mappings in several directions. Goebel, Kirk and Shimi(1973) have extended the result of Browder(1965) and Gohde(1965) to the mapping \( f \) that satisfy the inequality

\[
\|f(x) - f(y)\| \leq a\|x - y\| + b(\|x - f(x)\| + \|y - f(y)\|) + c(\|x - f(y)\| + \|y - f(x)\|)
\]

for all \( x, y \) in the domain of \( f \), where \( a, b, c \geq 0 \) and \( a + 2b + 2c \leq 1 \).

Further, Goebel and Kirk(1972) introduced the notion of asymptotically non-expansive mapping which is more general than non-expansive mapping and proved that any asymptotically non-expansive self-mapping of a nonempty bounded closed convex subset of a uniformly convex Banach space has a fixed point which generalized the fixed point theorems of Browder, Gohde(1965) and Kirk (1965). Vijayaraju (1988),(1995) and (1998) proved the existence of fixed points and obtained the convergence of the sequence of iterates to a fixed point for asymptotically non-expansive mappings in locally convex spaces. Schu(1991) derived some fixed point theorems in this direction.
The generalization of Pai and Veeramani(1982) is for a class of mappings which are not necessarily continuous. In fact it states that if \( K \) is a nonempty weakly compact convex subset of a Banach space having normal structure and \( f \) is a self-mapping on \( K \) satisfying the condition that for every closed convex invariant subset \( F \) of \( K \) with \( \delta(F) > 0 \), there exists \( \alpha(F) \) with \( 0 \leq \alpha(F) < 1 \) such that \( \|f(x) - f(y)\| \leq \max\{\delta(x, F), \alpha(F), \delta(F)\} \) for all \( x, y \in K \), then \( f \) has a fixed point. This result has been further generalized by Veeramani(1993) to non-expansive mappings which are not necessarily self-mappings.

A sufficient condition for the existence of fixed points for non-expansive mappings in the setting of metric spaces has been derived by Kijima and Takahashi(1969). They proved that if \( X \) is a compact metric space such that each admissible subset with positive diameter has at least one non-diameter point, then any non-expansive mapping on \( X \) has a fixed point. Later this result has been extended by Vetrivel, Veeramani and Bhattacharyya(1992) to a class of mappings introduced by Pai and Veeramani(1982), which contains the class of all non-expansive mappings. Kirk(1981), Penot(1979), Takahasi(1970) have proved fixed point theorems for non-expansive mappings in the setting of metric space with convexity structure.

1.3 FIXED POINTS OF MULTI-VALUED MAPPINGS

Let \( X \) and \( Y \) be any two nonempty sets. A multi-valued mapping or a set-valued mapping \( T \) from \( X \) to \( Y \) is a mapping that associates with each value \( x \in X \) a
nonempty subset $T(x)$ of $Y$, called image or the value of $T$ at $x$. It is denoted by $T : X \to 2^Y$ where $2^Y$ denotes nonempty subsets of $Y$. For any subset $K$ of $X$, the image of $K$ under $T$ is defined by $T(K) = \bigcup_{x \in K} T(x)$.

For any subset $K$ of $Y$, the pre image of $K$ under $T$ is defined by

$$T^{-1}(K) = \{ x \in X : T(x) \cap K \neq \phi \}.$$ 

**Definition 1.2** Let $X$ and $Y$ be any two normed linear spaces and let $2^Y$ denote the family of all nonempty subset of $Y$. A mapping $g : X \to 2^Y$ is said to be

(i) Upper semicontinuous iff $g^{-1}(K) = \{ x \in X : g(x) \cap K \neq \phi \}$ is closed for each closed subset $K$ of $Y$.

(ii) Lower semicontinuous iff $g^{-1}(K) = \{ x \in X : g(x) \cap K \neq \phi \}$ is open for each open subset $K$ of $Y$.

(iii) Continuous iff it is both upper semicontinuous and lower semicontinuous.

**Example 1.3** The multi-valued mapping $T : R \to R$ defined by

$$T(x) = \begin{cases} 
{0} & \text{if } x = 0 \\
[0, 1] & \text{if } x \neq 0 
\end{cases}$$

is lower semicontinuous, but not upper semicontinuous at $x = 0$.

**Example 1.4** The multi-valued mapping $T : R \to R$ defined by

$$T(x) = \begin{cases} 
[0, 1] & \text{if } x = 0 \\
\{0\} & \text{if } x \neq 0 
\end{cases}$$

is upper semicontinuous, but not lower semicontinuous at $x = 0$. 
Example 1.5 The multi-valued mapping $T : R^+ \rightarrow R^+$ defined by

$$T(x) = [0, x] \text{ for all } x \in R^+.$$ Then $T$ is both upper semicontinuous and lower semicontinuous and hence continuous.

Definition 1.3 Let $X$ be a nonempty set and let $T : X \rightarrow 2^X$. Then a point $x \in X$ is called a fixed point of $T$ if $x \in T(x)$.

Example 1.6 Let $X = R$, the set of real numbers, and define $T : X \rightarrow 2^X$ by $T(x) = (-\infty, x]$ for all $x \in X$. Then each point of $X$ is a fixed point of $T$.

Definition 1.4 A nonempty subset $K$ of a normed linear space $X$ is said to be approximatively compact if for every $x \in X$ and every sequence $\{x_n\}$ of point of $K$ with $\lim_{n \rightarrow \infty} \|x - x_n\| = d(x, K)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to an element of $K$. It is known that every compact set is approximatively compact but the converse is not true.

Example 1.7 Let $K = \{x = \{x_n\} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \text{ and } \|x\| \leq 1\}$ be a closed unit ball in $l_2$. Then $K$ is approximatively compact but not compact.

Definition 1.5 Let $X$ be a normed linear space and let $K$ be a nonempty subset of $X$, then metric projection of $X$ onto $K$ is a mapping $Q : X \rightarrow P(K)$ defined as $Q(x) = \{y \in K : \|x - y\| = d(x, K)\}$ where $P(K)$ denotes the set of all subsets of $K$.

Definition 1.6 A mapping $T$ is called a Kakutani multivalued mapping if there exist two topological spaces $X$ and $Y$ such that

(i) $T : X \rightarrow 2^Y$ 

(ii) $T$ is upper semicontinuous and
(iii) Either $T(x)$ is a singleton for each $x \in X$ (in which case $Y$ is required to be a Hausdorff topological vector space) or for each $x \in X$, $T(x)$ is a nonempty compact convex subset of $Y$ (in which case $Y$ is required to be a convex subset of a Hausdorff topological vector space).

**Definition 1.7** A mapping $T$ is called a *Kakutani factorizable multi-valued mapping* if there exist two topological spaces $X$ and $Y$ such that

(i) $T : X \rightarrow Y$ and

(ii) $T = T_nT_{n-1}...T_0$,

where each $T_j$ is a Kakutani multi-valued mapping for $j = 0, 1, ..., n$.

Fixed point theorems for multi-valued mappings proved natural setting for many problems in control theory involving differential equations. Fixed point theorems are also effectively used in tackling problems in economics and game theory.

The first extension of the topological theory of fixed points of continuous mappings to the case of multi-valued mappings was made by Von Neumann(1937) in connection with the proof of the fundamental theorem of game theory. Kakutani(1941) extended Browder’s fixed point theorem to an upper semicontinuous multi-valued mapping $T$ on a compact convex subset of a finite dimensional space and the corresponding extensions of Schauder’s fixed point theorem in Banach spaces were given independently by Bohnenblust and Karlin(1950) and Glicksberg(1952). Tychonoff’s theorem for locally convex topological vector spaces was proved by Ky Fan(1952). Ky Fan proved that any closed convex-valued upper semi continuous multi-valued mapping of a nonempty
compact convex subset in a Hausdorff locally topological vector space into itself has a fixed point. Further results in this direction were obtained by Browder (1968) and others.

Nadler (1969) defined multi-valued contraction mappings and extended Banach contraction principle for multi-valued mappings. Smithson (1971) defined contractive multi-valued mappings and proved fixed point theorems for such mappings. Using the Liapunov function, Smithson (1971) was able to extend many results from single-valued mappings to multi-valued mappings.

Later Himmelberg (1972) weakened the compactness condition imposed on the domain of the mapping and proved the existence of a fixed point for any closed convex-valued compact upper semicontinuous multi-valued mapping of a nonempty convex subset in a Hausdorff locally convex topological vector space into itself. It is Lassonde (1990) who proved the existence of fixed points for Kakutani factorizable multi-valued mappings which are not necessarily convex.

The study of fixed points for multivalued contractions and non-expansive maps using the Hausdorff metric was initiated by Markin (1973). Later, an interesting and rich fixed point theory for such maps was developed.

1.4 CONE METRIC SPACES

Huang and Zhang (2007) introduced the concept of a cone metric space, replacing the co-domain of the metric namely set of positive real numbers, by an ordered Banach
space. They obtained some fixed point theorems in cone metric spaces. Rezapour and Hamilbarani Haghi (2008) showed the existence of a non normal cone metric space and also obtained some fixed point theorems in cone metric spaces. Wardowski (2009) introduced the concept of multivalued contractions in cone metric spaces and using the notion of normal cones, obtained fixed point theorems for such mappings. Cone metric spaces are a special case of generalized metric spaces.

But a basic question raised as follows: “Are those spaces a real generalization of metric spaces?” This question has been investigated by various author. The authors showed that the cone metric spaces are metrizable and defined the equivalent metric in different approaches. However there was another question “Will the equivalent metric satisfy the same contractive conditions which the cone does?” Some authors answered affirmatively for a few contractive conditions but it is impossible to answer the question in general. By renorming the Banach spaces which has been partially ordered by a cone, we can obtain a new norm which converts it to normal cone, so every cone metric space is metrizable.

**Definition 1.8** Let $E$ be a Banach space and a subset $P$ of $E$ is said to be a cone if it satisfies the following conditions,

(i) $P \neq \emptyset$ and $P$ is closed;

(ii) $ax + by \in P$ for all $x, y \in P$ and $a, b$ are non-negative real numbers;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}.$
A cone $P \subset E$, we define a partial ordering $\leq$ with respect to the $P$ by $x \leq y$ if and only if $y - x \in P$. If $y - x \in \text{interior of } P$, then it is denoted by $x \ll y$.

**Definition 1.9** The cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, where $K$ is called the normal constant of $P$.

**Definition 1.10** The cone $P$ is called regular if every increasing sequence which is bounded above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Equivalently, the cone $P$ is called regular if every decreasing sequence which is bounded below is convergent.

The following examples show that there are non-normal cones.

**Example 1.8** Let $E = C^2([0,1])$ with the norm, $\|f\| = \|f\|_\infty + \|f^1\|_\infty$ and consider the cone $P = \{f \in E : f \geq 0\}$ for each $k \geq 1$, Put $f(x) = x$ and $g(x) = x^{2k}$, then, $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k + 1$. Since $K\|f\| \leq \|g\|$, $k$ is not normal constant of $P$. Therefore $P$ is a non-normal cone.

**Lemma 1.5** Every regular cone is normal.

**Proof.** Let $P$ a regular cone which is not normal. For each $n \geq 1$, choose $t_n, s_n \in P$ such that $t_n - s_n \in P$ and $n^2\|t_n\| < \|s_n\|$. For each $n \geq 1$, put $y_n = \frac{t_n}{\|t_n\|}$ and $x_n = \frac{s_n}{\|t_n\|}$ Then $x_n, y_n, y_n - x_n \in P$. $\|y_n\| = 1$ and $n^2 < \|x_n\|$, for all $n \geq 1$. Since
the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \|y_n\|$ is convergent and $P$ is closed, there is an element $y \in P$ such that $\sum_{n=1}^{\infty} \frac{1}{n^2} y_n = y$. Now, note that

$$0 \leq x_1 \leq x_1 + \frac{1}{2^2} x_2 \leq x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_3 \leq \ldots \leq y.$$  

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$ is convergent because $P$ is regular. Hence, $\lim_{n \to \infty} \frac{\|x_n\|}{n^2} = 0$, which is contraction.

**Lemma 1.6** There is not normal cone with normal constant $M < 1$.

**Proof.** Let $(X, d)$ be a cone metric space and $P$ a normal cone with normal constant $M < 1$. Choose a non-zero element $x \in P$ and $0 < \varepsilon < 1$ such that $M < 1 - \varepsilon$. Then $(1 - \varepsilon)x \leq x$, but $(1 - \varepsilon)\|x\| > M\|x\|$. This is a contraction. \qed

**Example 1.9** Let $E = C_R([0, 1])$ with the supremum norm $P = \{f \in E : f \geq 0\}$. Then, $P$ is a cone with normal constant of $M = 1$. Now, consider the following sequence of elements of $E$ which is decreasing and bounded from below but it is not convergent in $E$.

$$x \geq x^2 \geq x^3 \geq \ldots \geq 0.$$  

Therefore, the converse of the above lemma is not true.

**Example 1.10** Let $k > 1$ be given. Consider the real vector space

$$E = \left\{ ax + b : a, b \in \mathbb{R}; x \in \left(1 - \frac{1}{k}, 1\right) \right\},$$  

with supremum norm and the cone $P = \{ax + b \in E : a \geq 0, b \leq 1\}$ in $E$. The cone $P$ is regular and so normal.
Note:

(1) For $a \geq 0$ and $x \in P$, taking $b = 0$ in (ii) we have $ax \in P$.

(2) taking $a = b = 0$ in (ii) we have $0 \in P$.

**Proposition 1.7** Let $P$ be a cone in a real Banach space $E$. If $a \in P$ and $a \in ka$ for some $k \in [0, 1)$, then $a = 0$.

**Proof.** For $a \in P$, $k \in [0, 1)$ and $a \leq ka$ gives $(k - 1)a \in P$. As $0 \leq k < 1$ we have $1 - k > 0$ which gives $\frac{1}{1-k} > 0$. So by Note(1), $\frac{1}{1-k} (k - 1)a \in P$ implies $-a \in P$. Now $a \in P$ and $-a \in P$, which implies $a = 0$. 

**Proposition 1.8** Let $P$ be a cone in a real Banach space $E$. If $a \in P$ and $a \ll c$ for all $c \in intP$, then $a = 0$.

**Definition 1.11** $P$ is called minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$ and strongly minihedral if every subset of $E$ which is bounded from above has a supremum.

It is easy to see that every strongly minihedral normal cone is regular.

**Example 1.11** Let $E = C[0, 1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then $P$ is a cone with normal constant $K = 1$ which is not regular. This is clear, since the sequence $\{x_n\}$ is monotonically decreasing, but not uniformly convergent to 0. Thus, $P$ is not strongly minihedral.

**Definition 1.12** The cone $P$ is called solid if $intP \neq \emptyset$.

**Example 1.12** Let $E = \mathbb{R}^2$ with $P = \{(x_1, x_2, \ldots x_n) : x_i \geq 0 \text{ for all } i = 1, 2, \ldots n\}$. The cone $P$ is normal, minihedral, strongly minihedral and solid.
1.5 BEST APPROXIMATION AND BEST PROXIMITY POINT

Fixed point theory is an important tool for solving equations $Tx = x$ for mappings $T$ defined on subsets of metric spaces or normed linear spaces. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element $x$ which is in some sense closed to $Tx$. Best approximation theorems and best proximity point theorems are relevant in this perspective. A classical best approximation theorem, due Ky Fan [1969], states that if $A$ is a non-empty compact convex subset of a Hausdorff locally convex topological vector space $X$ and $T : A \to X$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. There have been many subsequent extensions and variant of Fan’s Theorem, including those by Reich[1978], Sehgal and Singh[1988]. Further, Vetrivel et al.[1992] has furnished a unified approach to such results.

On the other hand, though best approximation theorems ensure the existence of approximate solutions, such results need not yield optimal solutions. But best proximity point theorems furnish sufficient conditions that assure the existence of approximate solutions which are optimal as well. Indeed, if there is no exact solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then it is desirable to find an approximate solution $x$ such that $d(x, Tx)$ is minimum. In view of the fact that $d(x, Tx) \geq d(A, B)$, an absolute optimal approximate solution is an element $x$ for which $d(x, Tx)$ attains the least possible value $d(A, B)$. As result, a best proximity pair theorem offers sufficient conditions for the existence of an optimal
approximate solution $x$, called a best proximity point of the mapping $T$, satisfying the condition that $d(x, Tx) = d(A, B)$. Interestingly, best proximity theorems also serve as a natural generalization of the fixed point theorems, for a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping.

1.6 AN OUTLINE OF THE RESULTS OF THE THESIS

The thesis is divided into five chapters and a brief summary of each chapter is given below: Chapter 1 of the thesis is intended to provide references to concepts and known results which would help easy reading of the succeeding chapters. It includes an elementary knowledge of multi-valued mappings, a brief survey of results on fixed points, Lipschitzian mappings and continuous mappings. It also provides the concept of cone metric space and well known theorems.

In chapter 2, a generalization of Zhaohui Gu and et al(2008) common fixed Point of mean non-expansive mapping in the setting of uniformly convex Banach spaces. It is also proved common fixed point of pair of mappings by using Ishikawa iteration sequence and also extent the theorems for family of mappings and sequence of mappings. Some results on common fixed point theorems for contraction type and non-expansive non-self mappings are obtained which are extensions of the results of Bose and Mukherjee(1977) and that of Bose and Sahani(1987).

**Theorem 1.9** Let $X$ be a Banach space and $E$ be a nonempty closed convex subset of $X$, $S : E \to E$ and $T : E \to E$ are pair of mean non-expansive mappings with a
nonempty common fixed point set; if $b > 0$, $0 < \alpha \leq \alpha_n \leq 1/2$, $0 < \beta_n \leq \beta \leq 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S$ and $T$.

The following Zhaohui Gu et al’s theorem is a special case of the above theorem.

**Corollary 1.10** Let $X$ be a uniformly convex Banach space, $S : X \to X$ and $T : X \to X$ are a pair of mean non-expansive with a nonempty common fixed points set; if $b > 0$, $0 < \alpha \leq \alpha_n \leq 1/2$, $0 < \beta_n \leq \beta < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S$ and $T$.

The following theorem is the extension of the above theorem for the product of mean non-expansive mappings in a Banach Space

**Theorem 1.11** Let $X$ be a Banach space and $E$ be a nonempty closed convex subset of $X$. $S^m, T^n : E \to E$ are the mean non-expansive mappings with a nonempty common fixed point set; if $b > 0$, $0 < \alpha \leq \alpha_n \leq 1/2$, $0 < \beta_n \leq \beta \leq 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S^m$ and $T^n$.

**Corollary 1.12** Let $X$ be a uniformly convex Banach space, $S^m, T^n : X \to X$ are the mean non-expansive with a nonempty common fixed points set; if $b > 0$, $0 < \alpha \leq \alpha_n \leq 1/2$, $0 \leq \beta_n \leq \beta < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S^m$ and $T^n$.

The following theorem is proved for a family of mappings in the setting of Banach spaces, which is also a generalization of the Zhaohui Gu et al.,[93].

**Theorem 1.13** Let $X$ be a Banach space and $E$ be a nonempty closed convex subset of $T_i : E \to E$ for all $n$, are family of mean non-expansive mapping with a nonempty
common fixed point set; if \( b > 0, 0 < \alpha \leq \alpha_n \leq 1/2, 0 < \beta_n \leq \beta \leq 1 \), then the Ishikawa sequence \( \{x_n\} \) converges to the common fixed point of \( T_i \).

A pair of mean non-expansive non-self mappings of a closed convex subset of a Banach space \( E \) have a common fixed point with the setting of modified Ishikawa iteration sequence \( \{x_n\} \) associated with the pair, which is a generalization of Abdul Rahim Khan.

**Lemma 1.14** Let a real sequence \( \{x_n\}_{n=1}^\infty \) satisfy the following condition

\[
x_{n+1} \leq \alpha x_n + \beta_n,
\]

where \( x_n \geq 0, \beta_n \geq 0 \) and \( \lim_{n \to \infty} \beta_n = 0, 0 \leq \alpha < 1 \). Then, \( \lim_{n \to \infty} x_n = 0 \).

We construct the Ishikawa iteration scheme as follows:

\[
y_n = P(\alpha' x_n + \beta'_n Sx_n + \gamma'_n v_n),
\]

\[
x_{n+1} = P(\alpha_n x_n + \beta_n Ty_n + \gamma_n u_n)
\]

\( \forall \ n \geq 1 \) where \( x_1 \in C \); and \( T, S : C \to E \) are non-self maps, \( P : E \to C \) is a non-expansive retraction, \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\} \) are real sequences in \([0,1]\), and satisfying \( \alpha_n + \beta_n + \gamma_n = 1 \) and \( \alpha'_n + \beta'_n + \gamma'_n = 1 \); \( \{u_n\} \) and \( \{v_n\} \) are bounded sequences in \( C \), and satisfy the following conditions:

\[
\sum_{n=0}^{\infty} \gamma_n < \infty, \quad \sum_{n=0}^{\infty} \gamma'_n < \infty
\]

\( \lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \gamma'_n = 0 \)

The following lemmas are the main tool in the proof of many principal results of chapter 2.

**Lemma 1.15** Let \( E \) be a Banach space and \( C \) be a nonempty closed convex subset of \( E \), \( S : C \to E \) and \( T : C \to E \) are pair of mean non-expansive mappings with...
\(F(T) \cap F(S) \neq \phi\). If the sequence \(\{x_n\}\) is given by \((1.1)\) where \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}\) are real sequences in \([0,1]\), satisfying \(\alpha_n + \beta_n + \gamma_n = 1, \alpha'_n + \beta'_n + \gamma'_n = 1\), and \(\{u_n\}, \{v_n\}\) are bounded sequences in \(E\), then \(\sum_{n=0}^{\infty} \gamma_n < \infty, \sum_{n=0}^{\infty} \gamma'_n < \infty\) and \(\lim_{n \to \infty} \gamma_n = 0 = \lim_{n \to \infty} \gamma'_n\). Then \(\lim_{n \to \infty} ||x_n - q|| = 0\) for \(q \in F(T) \cap F(S)\).

**Lemma 1.16** Let \(E\) be a Banach space and \(C\) be a nonempty closed convex subset of \(E\), \(S : C \to E\) and \(T : C \to E\) are pair of mean non-expansive mapping with \(F(T) \cap F(S) \neq \phi\). If the sequence \(\{x_n\}\) is given by \((1.1)\) where \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}\) are real sequences in \([0,1]\), satisfying \(\alpha_n + \beta_n + \gamma_n = 1, \alpha'_n + \beta'_n + \gamma'_n = 1\), and \(\{u_n\}, \{v_n\}\) are bounded sequences in \(E\), then \(\sum_{n=0}^{\infty} \gamma_n < \infty, \sum_{n=0}^{\infty} \gamma'_n < \infty\). Then \(\lim_{n \to \infty} \gamma_n = 0 = \lim_{n \to \infty} \gamma'_n\). Then \(\lim_{n \to \infty} ||x_n - Ty_n|| = 0\) and \(\lim_{n \to \infty} ||x_n - Sx_n|| = 0\).

The following theorem ascertains the existence of the common fixed point for a pair of mean non-expansive non-self mappings.

**Theorem 1.17** Let \(E\) be a Banach space and \(C\) be a nonempty closed convex subset of \(E\), and let \(T, S\) and \(\{x_n\}\) be as in above Lemma, \(F(T) \cap F(S) \neq \phi\). Then the sequence \(\{x_n\}\) converges to the common fixed point of \(S\) and \(T\).

**Corollary 1.18** Let \(E\) be a uniformly convex Banach space and \(C\) be a nonempty closed convex subset of \(C\), \(S, T : C \to C\) are pair of mean non-expansive mapping with \(F(T) \cap F(S) \neq \phi\). Let \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}\) are real sequences in \([0,1]\), satisfying \(\alpha_n + \beta_n + \gamma_n = 1, \alpha'_n + \beta'_n + \gamma'_n = 1\), and \(\{u_n\}, \{v_n\}\) are bounded sequences in \(C\); \(\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \alpha'_n, \sum_{n=0}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \alpha'_n = \infty\). Then the sequence \(\{x_n\}\) defined by the iteration \((1.1)\) converges to the common fixed point of \(S\) and \(T\).
The following theorem generalization of the above theorem for a family of mean-
non-expansive non-self mappings.

**Theorem 1.19** Let $E$ be a Banach space and $C$ be a nonempty closed convex subset
of $E$, $T_i : C \to C$ for all $i$; be the family of mean non-expansive mapping with
$\cap F(T_i) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ be real sequences in $[0,1]$
 satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $\alpha'_n + \beta'_n + \gamma'_n = 1$; $\{u_n\}, \{v_n\}$ are bounded sequences in
$C$,
\[
\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \alpha'_n \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \alpha'_n = \infty.
\]
Then the sequence $\{x_n\}$ defined by the iteration (1.1) converges to the common fixed
point of $T_i$.

In Chapter 3, Banach contraction principal and Kannan’s fixed point theorems
in the setting of cone metric spaces are derived. Fixed point theorem for sequence
of mappings and product of mappings in the setting cone metric spaces are proved.
These results offer a generalization of Kannan’s, Banach and Chattergies’s fixed point
theorems in metric spaces. A generalization of the result of Sh.Rezapour(2008) for
contractive mappings in complete cone metric spaces is obtained.

**Theorem 1.20** Let $(X,d)$ be a complete cone metric space and the mapping
$T : X \to X$ satisfy the contractive condition
\[
d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]
\]
for all $x, y \in X$, and $a + b < \frac{1}{2}$, $a, b \in [0, \frac{1}{2})$. Then $T$ has a unique fixed point in $X$. 
Corollary 1.21  Let \((X,d)\) be a complete cone metric space and the mapping \(T : X \to X\) satisfies the contractive condition
\[
d(Tx,Ty) \leq a(d(Tx,y) + d(x,Ty))
\]
for all \(x, y \in X\), where \(a \in [0, \frac{1}{2})\) is a constant. Then \(T\) has a unique fixed point in \(X\). For each \(x \in X\), the iterative sequence \(\{T^n x\}_{n \geq 1}\) converges to the fixed point.

Theorem 1.22  Let \((X,d)\) be a complete cone metric space and the mapping \(T : X \to X\) satisfy the contractive condition
\[
d(Tx,Ty) \leq ru \quad \text{where} \quad u \in \{d(x,y), d(x,Tx), d(y,Ty)\}
\]
for all \(x, y \in X\) and \(r \in [0, 1)\). Then \(T\) has a unique fixed point in \(X\).

Corollary 1.23  Let \((X,d)\) be a complete cone metric space and the mapping \(T : X \to X\) satisfies the contractive condition
\[
d(Tx,Ty) \leq kd(x,y)
\]
for all \(x, y \in X\), where \(k \in [0, 1)\) is a constant. Then \(T\) has a unique fixed point in \(X\). For each \(x \in X\), the iterative sequence \(\{T^n x\}_{n \geq 1}\) converges to the fixed point.

The following theorem ascertains existence of fixed point of the contractive map in the setting of a normal cone with normal constant \(K\).

Theorem 1.24  Let \((X,d)\) be a complete cone metric space, and \(P\) a normal cone with normal constant \(K\). Suppose the mapping \(T : X \to X\) satisfies the contractive condition
\[
d(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)] + b[d(x,Ty) + d(Tx,y)]
\]
for all \( x, y \in X \), and \( a + b < \frac{1}{2} \), \( a, b \in [0, \frac{1}{2}) \). Then \( T \) has a unique fixed point in \( X \).

And for any \( x \in X \), iterative sequence \( \{T^n x\} \) converges to the fixed point.

**Theorem 1.25** Let \((X, d)\) be a complete cone metric space, and \( P \) a normal cone with normal constant \( K \). Suppose the mapping \( T : X \to X \) satisfies the contractive condition \( d(Tx, Ty) \leq ru \), where \( u \in \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\} \) for all \( x, y \in X \), and \( r \in [0, 1) \). Then \( T \) has a unique fixed point in \( X \). And for any \( x \in X \), iterative sequence \( \{T^n x\} \) converges to the fixed point.

The preceding theorem subsumes the result of Sh.Rezapour R.Hamlbarani

**Corollary 1.26** Let \((X, d)\) be a complete cone metric space, and \( P \) a normal cone with normal constant \( K \). Suppose the mapping \( T : X \to X \) satisfies the contractive condition \( d(Tx, Ty) \leq ru \), where \( u \in \{d(x, y), d(x, Tx), d(y, Ty)\} \) for all \( x, y \in X \), and \( r \in [0, 1) \). Then \( T \) has a unique fixed point in \( X \). And for any \( x \in X \), iterative sequence \( \{T^n x\} \) converges to the fixed point.

The following theorem ascertains existence of a unique fixed point of a contractive map in a regular cone.

**Theorem 1.27** Let \((X, d)\) be a cone metric space and \( P \subset X \) and \( P \) is a regular cone, and let \( S \) be the class of functions \( \alpha : \mathbb{R}^+ \to [0, 1) \) satisfies \( \alpha(t_n) \to 1 \) implies \( t_n \to 0 \) and \( T : X \to X \) satisfies the contractive condition

\[
d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \text{for all } x, y \in X.
\]

Then \( T \) has a unique fixed point in \( X \).
Theorem 1.28 Let \((X, d)\) be a complete cone metric space. Let \(T : X \to X\) be a mapping and \(\phi : [0, \infty) \to [0, \infty)\) be a continuous nondecreasing function such that

(i) \(\phi(\lambda) \leq \lambda\) for some \(\lambda \in [0, \infty)\) and \(\phi(r) \leq \lambda\) implies \(r = 0\),

(ii) \(\phi(l + m) \leq \phi(l) + \phi(m)\) for all \(l, m \in [0, \infty)\),

(iii) \(\phi(d(Tx, Ty)) \leq a\phi(d(x, x)) + b\phi(d(y, y))\), for all \(x, y \in X, x \neq y\), where \(a \in [0, 1)\).

Then \(T\) has a unique fixed point in \(X\).

Theorem 1.29 Let \((X, d)\) be a complete cone metric space. Let \(T : X \to X\) be a mapping and \(\phi : [0, \infty) \to [0, \infty)\) be a continuous nondecreasing function such that

(i) \(\phi(\lambda) \leq \lambda\) for some \(\lambda \in [0, \infty)\) and \(\phi(r) \leq \lambda\) implies \(r = 0\),

(ii) \(\phi(l + m) \leq \phi(l) + \phi(m)\) for all \(l, m \in [0, \infty)\),

(iii) \(\phi(d(Tx, Ty)) \leq a\phi(d(x, x)) + b\phi(d(y, y))\), for all \(x, y \in X\), \(a + b < 1\), where \(a, b \in [0, 1)\). Then \(T\) has a unique fixed point in \(X\).

Theorem 1.30 Let \((X, d)\) be a complete cone metric space. The mapping \(\phi\) is defined as in the above theorem and the mapping \(T : X \to X\) and the (iii) condition is taken as \(\phi(d(Tx, Ty)) \leq ru\), where \(u \in \{\phi(d(x, x)), \phi(d(x, x)), \phi(d(y, y))\}\) for all \(x, y \in X\) and \(r \in [0, 1)\). Then \(T\) has a unique fixed point in \(X\).

Corollary 1.31 Let \((X, d)\) be a complete cone metric space, the mapping \(\phi\) is defined as in the previous theorem and the mapping \(T : X \to X\) satisfies the contractive condition \(\phi(d(Tx, Ty)) \leq r\phi(d(x, y))\) for all \(x, y \in X\), where \(k \in [0, 1)\) is a constant. Then \(T\) has a unique fixed point in \(X\). For each \(x \in X\), the iterative sequence \(\{T^n x\}_{n \geq 1}\) converges to the fixed point.
The following theorem provide existence of fixed point of contractive map in the setting of $P$ is a normal cone with normal constant $K$ in the above theorems.

**Theorem 1.32** Let $(X, d)$ be a complete cone metric space, and $P$ a normal cone with normal constant $K$. Suppose the mapping $T : X \to X$ satisfies the contractive condition
\[
\phi(d(Tx, Ty)) \leq a\phi(d(x, Tx) + d(y, Ty)) + b\phi(d(x, Ty) + d(Tx, y))
\]
for all $x, y \in X$, and $a + b < \frac{1}{2}$, $a, b \in [0, \frac{1}{2})$. Then $T$ has a unique fixed point in $X$. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

**Corollary 1.33** Let $(X, d)$ be a complete cone metric space, and $P$ a normal cone with normal constant $K$. Suppose the mapping $T : X \to X$ satisfies the contractive condition
\[
\phi(d(Tx, Ty)) \leq b\phi(d(x, Ty) + d(Tx, y))
\]
for all $x, y \in X$, and $b \in [0, \frac{1}{2})$. Then $T$ has a unique fixed point in $X$. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

**Theorem 1.34** Let $(X, d)$ be a cone metric space and $P \subset X$ and $P$ is a regular cone, and let $S$ be the class of functions $\alpha : \mathbb{R}^+ \to [0, 1)$ satisfies $\alpha(t_n) \to 1$ implies $t_n \to 0$ and $T : X \to X$ satisfies the contractive condition
\[
\phi(d(Tx, Ty)) \leq \alpha(\phi(d(x, y)))\phi(d(x, y))
\]
for all $x, y \in X$. Then $T$ has a unique fixed point in $X$.

The following theorem provide existence of unique fixed point of a map with a comparison function in a normal cone.
1.6. AN OUTLINE OF THE RESULTS OF THE THESIS

**Theorem 1.35** Let \((X,d)\) be a complete cone metric space, where \(P\) is a normal cone with normal constant \(K\). Let \(T : X \rightarrow X\) be a function such that there exists a comparison function \(\psi : P \rightarrow P\) such that

\[
d(Tx, Ty) \leq \psi(d(x, y)),
\]

for every \(x, y \in X\). Then \(T\) has a unique fixed point.

The following theorem ascertains existence of unique common fixed point for a pair of Kannan type mappings in cone metric space.

**Theorem 1.36** Let \((X,d)\) be a complete cone metric space. If \(T, S : X \rightarrow X\) satisfy the contractive condition

\[
d(Tx, Sy) \leq a[d(x, Tx) + d(y, Sy)]
\]

for all \(x, y \in X\), and \(0 < a < \frac{1}{2}\). Then \(T\) and \(S\) have a unique common fixed point in \(X\).

**Corollary 1.37** Let \((X,d)\) be a complete cone metric space and the mapping \(T : X \rightarrow X\) satisfies the contractive condition

\[
d(Tx, Ty) \leq a(d(Tx, y) + d(x, Ty))
\]

for all \(x, y \in X\), where \(a \in \left[0, \frac{1}{2}\right)\) is a constant. Then \(T\) has a unique fixed point in \(X\). For each \(x \in X\), the iterative sequence \(\{T^n x\}_{n \geq 1}\) converges to the fixed point.

The following theorem ascertains existence of unique common fixed point for a pair of contraction type mappings in cone metric space.
Theorem 1.38 Let \((X, d)\) be a complete cone metric space. Let \(T, S\) be two self-mappings on \(X\) such that

(i) \(d(Tx, Sy) \leq ad(x, y)\) for all \(x, y \in X\), and \(0 < a < 1\),

(ii) \(S\) is a contraction on \(X\), that is, there exists \(k\) with \(0 < k < 1\) such that

\[
d(Sx, Sy) \leq kd(x, y) \quad \text{for all } x, y \in X,
\]

(iii) there exists \(x_0 \in X\) such that

\[
x_{n+1} = \begin{cases} 
  Tx_n & \text{if } n \text{ is even} \\
  Sx_n & \text{if otherwise.}
\end{cases}
\]

Then \(T\) and \(S\) have a unique common fixed point in \(X\).

Corollary 1.39 Let \((X, d)\) be a complete cone metric space. Let \(T\) be a self-mapping on \(X\) such that \(d(Tx, Ty) \leq ad(x, y)\) for all \(x, y \in X\), and \(0 < a < 1\). Then \(T\) has a unique common fixed point in \(X\).

The following theorem ascertains sufficient condition for a point to be a fixed point of a contractive type mapping in a non-complete cone metric space.

Theorem 1.40 Let \((X, d)\) be a cone metric space, and \(P\) be a normal cone with normal constant \(K\). Let \(T\) be a self-mapping of \(X\) such that

(i) \(d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]\) for all \(x, y \in X\), and \(0 < a < \frac{1}{2}\),

(ii) \(T\) is continuous at \(x \in X\),

(iii) There exists a point \(x_0 \in X\) such that the sequence of iterates \(\{T^n x_0\}\) has a subsequence \(\{T^{m_i} x_0\}\) converging to \(x\). Then \(x\) is a unique fixed point of \(T\) in \(X\).
The following theorem ascertains existence of unique fixed point of a mappings in non-complete cone metric space and $P$ is a normal cone with normal constant $K$.

**Theorem 1.41** Let $(X, d)$ be a cone metric space, and $P$ be a normal cone with normal constant $K$. Let $T$ be a self-mappings of $X$. Suppose $T$ is continuous at a point $x \in X$. If the exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n x_0\}$ converges to $x$, then $Tx = x$. if in addition, $d(Tx, Ty) \leq ad(x, y)$ for all $x, y \in X$, and $0 < a < 1$, Then $x$ is a unique fixed point of $T$ in $X$.

The following theorem ascertains existence of unique common fixed point for a sequence of mappings in complete cone metric space.

**Theorem 1.42** Let $\{T_n\}$ be a sequence of self-mappings of a complete cone metric space $X$, and $P$ be a normal cone with normal constant $K$, and for any two mappings $T_i, T_j$, we have $d(T_i^m x, T_j^m y) \leq a_{i,j}[d(x, T_i^m x) + d(y, T_j^m y)]$ for some $m > 0$ and $0 < a_{i,j} \leq k < \frac{1}{2}$, $i, j \in N$ for all $x, y \in X$. Then $\{T_n\}$ has a unique common fixed point in $X$.

The following corollary ascertains existence of unique fixed point of Kannan’s type mappings in complete cone metric space.

**Corollary 1.43** Let $X$ be a complete cone metric space, and $P$ be a normal cone with normal constant $K$, and let $T$ be a self mapping of $X$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ where $0 < a < \frac{1}{2}$. Then $T$ has a unique fixed point in $X$. 
The following theorem ascertains existence of unique point of coincidence for pair of mappings in the setting of a cone metric spaces.

**Theorem 1.44**  Let \((X, d)\) be a cone metric space, suppose that the mappings \(T, S : X \to X\) satisfy

\[
d(Tx, Ty) \leq \alpha d(Sx, Sy) + \beta [d(Sx, Tx) + d(Sy, Ty)] + \gamma [d(Sx, Ty) + d(Sy, Tx)] \quad \ldots (2)
\]

for all \(x, y \in X\), \(\alpha, \beta, \gamma > 0\), \(\alpha + 2\beta + 2\gamma < 1\). If \(T(X) \subseteq S(X)\) and \(S(X)\) or \(T(X)\) is complete subspace of \(X\), then \(T\) and \(S\) have a unique point of coincidence.

The following result ascertains existence of unique common fixed point of weakly compatible mappings in cone metric spaces.

**Theorem 1.45**  Let \((X, d)\) be a cone metric space, suppose that the mappings \(T, S : X \to X\) satisfies

\[
d(Tx, Ty) \leq \alpha d(Sx, Sy) + \beta [d(Sx, Tx) + d(Sy, Ty)] + \gamma [d(Sx, Ty) + d(Sy, Tx)]
\]

for all \(x, y \in X\), \(\alpha, \beta, \gamma > 0\), \(\alpha + 2\beta + 2\gamma < 1\). If \(T(X) \subseteq S(X)\) and \(S(X)\) or \(T(X)\) is complete subspace of \(X\), and \(T\) and \(S\) are weakly compatible then \(T\) and \(S\) have a unique common fixed point.

**Corollary 1.46**  Let \((X, d)\) be a cone metric space, suppose that the mappings \(T, S : X \to X\) satisfies

\[
d(Tx, Ty) \leq \alpha d(Sx, Sy) + \beta [d(Sx, Ty) + d(Sy, Tx)]
\]

for all \(x, y \in X\), where \(\alpha, \beta \in [0, 1)\) with \(\alpha + 2\beta < 1\). If \(T(X) \subseteq S(X)\) and \(S(X)\) is complete subspace of \(X\). Then, \(T\) and \(S\) have a unique point of coincidence. Moreover, if \(T\) and \(S\) are weakly compatible, then \(T\) and \(S\) have a unique common fixed point.
Corollary 1.47 Let $(X, d)$ be a cone metric space, suppose that the mappings $T, S : X \to X$ satisfies
\[d(Tx, Ty) \leq \alpha d(Sx, Sy) + \beta d(Sx, Ty) + \gamma d(Sy, Tx) \quad \ldots \quad (3)\]
for all $x, y \in X$, where $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$. If $T(X) \subseteq S(X)$ and $S(X)$ is complete subspace of $X$. Then, $T$ and $S$ have a unique point of coincidence. Moreover, if $T$ and $S$ are weakly compatible, then $T$ and $S$ have a unique common fixed point.

The following result ascertains existence of unique common fixed point of weakly compatible mappings in normal cone metric spaces.

Theorem 1.48 Let $(X, d)$ be a cone metric space, and $P$ be a normal cone with normal constant $K$. Suppose the mappings $T, S : X \to X$ satisfy the contractive condition
\[d(Tx, Sy) \leq r[d(Tx, Sy) + d(Ty, Sx) + d(Tx, Sx) + d(Ty, Sy)], \quad r \in [0, 1/4).\]
for all $x, y \in X$, where $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$. If $T(X) \subseteq S(X)$ and $S(X)$ is complete subspace of $X$, then $T$ and $S$ have a unique coincidence point in $X$. Moreover, if $T$ and $S$ are weakly compatible, then $T$ and $S$ have a unique common fixed point.

The next theorem provides existence of unique fixed point of contractive mappings in cone metric space with $w$-distance, which is the generalization of the theorems of H.Lakzian, F.Arabyani(2009).
Theorem 1.49 Let $(X,d)$ be a complete cone metric space with $w$-distance $p$. Let $P$ be a normal cone on $X$. A mapping $T : X \to X$ satisfy the contractive condition

$$p(Tx,Ty) \leq ap(x,Tx) + bp(y,Ty) \quad \text{for all } x,y \in X, \quad a + b < 1, \quad a,b \in [0,1).$$

Then $T$ has a unique fixed point in $X$. For each $x \in X$, the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

Theorem 1.50 Let $(X,d)$ be a complete cone metric space with $w$-distance $p$. Let $P$ be a normal cone on $X$. Suppose a mapping $T : X \to X$ satisfy the contractive condition

$$p(Tx,Ty) \leq ru \quad \text{where } u \in \{p(x,y), p(x,Tx), p(y,Ty)\}$$

for all $x,y \in X$ and $r \in [0,1)$. Then $T$ has a unique fixed point in $X$.

The following corollary is a special case of the of H.Lakzian et al.,(2009) theorem.

Corollary 1.51 Let $(X,d)$ be a complete cone metric space with $w$-distance $p$. Let $P$ be a normal cone on $X$. Suppose a mapping $T : X \to X$ satisfy the contractive condition

$$p(Tx,Ty) \leq kp(x,y),$$

for all $x,y \in X$, where $k \in [0,1)$ is a constant. Then $T$ has a unique fixed point in $X$. For each $x \in X$, the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

The following theorems ascertain existence of unique fixed point of the contractive mappings in the setting of $P$ is a normal cone with normal constat $K$ in the above theorem.
1.6. AN OUTLINE OF THE RESULTS OF THE THESIS

**Theorem 1.52** Let \((X, d)\) be a cone metric space with \(w\)-distance \(p\). Let \(P \subset X\) and \(P\) is a regular cone, and let \(S\) be the class of functions \(\alpha : \mathbb{R}^+ \to [0, 1)\) satisfies
\[
\alpha(t_n) \to 1 \implies t_n \to 0
\]
and \(T : X \to X\) satisfies the contractive condition
\[
p(Tx, Ty) \leq \alpha(p(x, y))p(x, y)
\]
for all \(x, y \in X\). Then \(T\) has a unique fixed point in \(X\).

In chapter 4, results on fixed points for multivalued mappings and common fixed points of pair of multivalued mappings in the setting of cone metric space are established which are extension of results of Seong Hoon Cho and Mi Sun Kim (2009) and Bose and Mukherjee (1977). And introduce the cone convex metric space, as a consequence, some fixed point theorems in cone convex metric space are established. And study the partial cone metric spaces and proved some fixed point theorems in partial cone convex metric space are established.

**Theorem 1.53** Let \((X, d)\) be a complete cone metric space and the mapping \(T : X \to \text{CB}(X)\) be a multivalued map satisfying for each \(x, y \in X\),
\[
H(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]
\]
for all \(x, y \in X\), and \(a + b < \frac{1}{2}\), \(a, b \in [0, \frac{1}{2})\). Then \(T\) has a fixed point in \(X\).

**Corollary 1.54** Let \((X, d)\) be a complete cone metric space and the mapping \(T : X \to \text{CB}(X)\) be a multivalued map satisfies condition
\[
H(Tx, Ty) \leq a(d(Tx, y) + d(x, Ty))
\]
for all \(x, y \in X\), where \(a \in [0, \frac{1}{2})\) is a constant. Then \(T\) has a fixed point in \(X\).
The next theorems ascertains existence of unique fixed point of multivalued map in cone metric spaces.

**Theorem 1.55** Let \((X, d)\) be a complete cone metric space and \(T : X \to \text{CB}(X)\) be a multivalued map satisfy the condition

\[
H(Tx, Ty) \leq ru \quad \text{where} \quad u \in \{d(x, y), d(x, Tx), d(y, Ty)\}
\]

for all \(x, y \in X\) and \(r \in [0, 1)\). Then \(T\) has a fixed point in \(X\).

**Corollary 1.56** Let \((X, d)\) be a complete cone metric space and the mapping \(T : X \to \text{CB}(X)\) be a multivalued mapping satisfy the condition

\[
H(Tx, Ty) \leq kd(x, y)
\]

for all \(x, y \in X\) where \(k \in [0, 1)\) is a constant. Then \(T\) has a fixed point in \(X\).

The following theorems ascertain existence of unique fixed point of multivalued mapping in cone metric spaces in the setting of \(P\) is a normal cone with normal constat \(K\) in the above theorem.

**Theorem 1.57** Let \((X, d)\) be a complete cone metric space, and \(P\) a normal cone with normal constant \(K\). Suppose the mapping \(T : X \to \text{CB}(X)\) be multivalued mapping satisfying the condition

\[
H(Tx, Ty) \leq ru \quad \text{where} \quad u \in \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\}
\]

for all \(x, y \in X\), and \(r \in [0, 1)\). Then \(T\) has a fixed point in \(X\).

**Corollary 1.58** Let \((X, d)\) be a complete cone metric space, and \(P\) a normal cone with normal constant \(K\). Suppose the mapping \(T : X \to \text{CB}(X)\) be multivalued
mapping satisfies the condition
\[
H(Tx, Ty) \leq ru \quad \text{where} \quad u \in \{d(x, y), d(x, Tx), d(y, Ty)\}
\]
for all \(x, y \in X\), and \(r \in [0, 1)\). Then \(T\) has a fixed point in \(X\).

The following theorems ascertains existence of unique fixed point of multivalued mapping in cone metric spaces with a map \(\alpha\).

**Theorem 1.59** Let \((X, d)\) be a cone metric space and let \(S\) be the class of functions \(\alpha : \mathbb{R}^+ \to [0, 1)\) satisfies \(\alpha(t_n) \to 1\) implies \(t_n \to 0\) and \(T\) be a multivalued map on \(X\) with \(Tx\) is nonempty closed subset of \(X\) and a regular cone, for each \(x \in X\)
\[
H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)
\]
for each \(x, y \in X\). Then \(T\) has a unique fixed point in \(X\).

A generalization of the result of Bose and Mukherjee (1977) for contractive type mappings in complete cone metric space

**Theorem 1.60** Let \((X, d)\) be a complete cone metric space and let the mapping \(T_1, T_2 : X \to C(X)\) satisfying the following conditions
(i) for each \(x \in X\), \(T_1(x), T_2(x) \in CB(X)\),
(ii) \(H(T_1(x), T_2(y)) \leq \alpha_1d(x, T_1(x)) + \alpha_2d(y, T_2(y)) + \alpha_3d(y, T_1(x)) + \alpha_4d(x, T_2(y)) + \alpha_5d(x, y)\) where \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\) are non negative real numbers and \(\Sigma_{i=1}^5 \alpha_i < 1\) and \(\alpha_1 = \alpha_2 \) or \(\alpha_3 = \alpha_4\). Then there exists \(p \in X\) such that \(p \in T_1(x) \cap T_2(x)\).

The preceding theorem subsumes the the following result of Bose and Mukherjee (1977).
Corollary 1.61 Let \((X, d)\) be a complete cone metric space and \(T : X \rightarrow C(X)\) be a map, satisfying the following conditions

(i) for each \(x \in X\), \(T(x), T(y) \in CB(X)\),

(ii) \(H(T_1(x), T_2(y)) \leq qd(x, y)\) for some \(q \in [0, 1)\).

Then there exists an element \(p \in X\) such that \(p \in T(p)\).

Corollary 1.62 Let \(X\) be a complete cone metric space and \(T_1, T_2 : X \rightarrow C(X)\) be two multi-valued mappings satisfying the following conditions. For any \(x, y \in X\),

\[H(T_1(x), T_2(y)) \leq \alpha_1 d(x, T_1(x)) + \alpha_2 d(y, T_2(y)) + \alpha_3 d(x, T_1(x)) + \alpha_4 d(x, T_2(y)) + \alpha_5 d(x, y)\]

where \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\) are non negative real numbers and \(\sum_{i=1}^{5} \alpha_i < 1\) and \(\alpha_1 = \alpha_2\) or \(\alpha_3 = \alpha_4\). Then there exists \(x \in X\) such that \(x \in T_1(x)\) and \(x \in T_2(x)\).

Definition 1.13 Let \((X, d)\) be a cone metric space, and \(I = [0, 1]\). A continuous mapping \(R : X \times X \times I \rightarrow X\) is said to be a convex structure on \(X\) if for each \((x, y, \lambda) \in X \times X \times I\) and \(u \in X\),

\[d(u, R(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)\]

A space \(X\) together with a convex structure \(R\) is called a cone convex metric space.

The following theorems, provides existence of common fixed point of pair of mappings in Cone convex metric spaces

Theorem 1.63 Let \(E\) be a nonempty closed subset of a complete cone convex metric space \(X\) and let \(S, T : X \rightarrow X\) be self mappings satisfying,

\[d(Sx, Ty) \leq k[d(x, y) + d(x, Ty) + d(y, Sx)]\]
for all \( x, y \in E \) and \( 0 < k < 1 \). Suppose that \( \{x_n\} \) associated with \( S \) and \( T \) is defined by

(i) \( x_0 \in E \),

(ii) \( x_{n+1} = R(Ty_n, x_n, \alpha_n), \ n = 0, 1, 2, ... \),

(iii) \( y_n = R(Sx_n, x_n, \beta_n), \ n = 0, 1, 2, ... \),

where \( 0 \leq \alpha_n, \beta_n \leq 1 \) and \( \{\alpha_n\} \) is bounded away from zero. If \( \{x_n\} \) converges to some point \( p \in E \), then \( p \) is the common fixed point of \( T \) and \( S \).

**Corollary 1.64** Let \( E \) be a nonempty closed subset of a complete cone convex metric space \( X \) and let \( T : X \to X \) be self-map satisfying,

\[
d(Tx, Ty) \leq k[d(x, y) + d(x, Ty) + d(y, Tx)]
\]

for all \( x, y \in E \) and \( 0 < k < 1 \). The sequence \( \{x_n\} \) is defined by \( x_0 \in E, x_{n+1} = R(Ty_n, x_n, \alpha_n), y_n = R(Tx_n, x_n, \beta_n), n = 0, 1, 2, ... \), where \( 0 \leq \alpha_n, \beta_n \leq 1 \) and \( \{\alpha_n\} \) is bounded away from zero, converges to a fixed point of \( T \).

The following theorem ascertains existence of a unique fixed point of a self-map in complete cone convex metric spaces

**Theorem 1.65** Let \( E \) be a nonempty closed subset of a complete cone convex metric space \( X \), and let \( T : X \to X \) be a self-map satisfying

\[
d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]
\]

for all \( x, y \in E \), and \( a + b < \frac{1}{k}, \ a, b \in [0, \frac{1}{4}) \). The sequence \( \{x_n\} \) is defined by
(i) \( x_0 \in E, \)

(ii) \( x_{n+1} = R(Ty_n, x_n, \alpha_n), \quad n = 0, 1, 2, \ldots, \)

(iii) \( y_n = R(Tx_n, x_n, \beta_n), \quad n = 0, 1, 2, \ldots, \)

where \( 0 \leq \alpha_n, \beta_n \leq 1 \) and \( \{\alpha_n\} \) is bounded away from zero. If \( \{x_n\} \) converges to some point \( p \in E \), then \( p \) is the fixed point of \( T \).

The following theorem ascertains existence of a common fixed point of quasi-non-expansive mappings in complete cone convex metric space

**Theorem 1.66** Let \( E \) be a nonempty closed subset of a complete cone convex metric space \( X \) and let \( S, T : X \to X \) be quasi-non-expansive mappings. Suppose that \( \{x_n\} \) associated with \( S \) and \( T \) is defined by

(i) \( x_0 \in E, \)

(ii) \( x_{n+1} = R(Ty_n, x_n, \alpha_n), \quad n = 0, 1, 2, \ldots, \)

(iii) \( y_n = R(Tx_n, x_n, \beta_n), \quad n = 0, 1, 2, \ldots, \)

where \( 0 \leq \alpha_n, \beta_n \leq 1 \) and \( \{\alpha_n\} \) is bounded away from zero. If \( \{x_n\} \) converges to some point \( p \in E \), then \( p \) is the common fixed point of \( T \) and \( S \).

**Corollary 1.67** Let \( E \) be a nonempty closed subset of a complete cone convex metric space \( X \) and let \( T : X \to X \) be a quasi-non-expansive mapping. The sequence \( \{x_n\} \) is defined by \( x_0 \in E, \)

\[ x_{n+1} = R(Ty_n, x_n, \alpha_n), \quad y_n = R(Tx_n, x_n, \beta_n), \]
\[ n = 0, 1, 2, ..., \text{ where } 0 \leq \alpha_n, \beta_n \leq 1, \text{ and } \{x_n\} \subset E. \text{ Then } \{x_n\} \text{ converges to a unique fixed point of } T. \]

The following theorems ascertains existence of unique fixed point of contraction mappings in a partial cone convex metric space.

**Theorem 1.68** Let \((X, p)\) be a complete partial cone metric space, \(P\) be a normal cone with normal constant \(K\). Suppose that the mapping \(T : X \to X\) satisfy the contractive condition

\[ p(Tx, Ty) \leq ap(x, Tx) + bp(y, Ty) \]

for all \(x, y \in X\), and \(a + b < 1, a, b \in [0, 1)\). Then \(T\) has a unique fixed point in \(X\). For each \(x \in X\), the iterative sequence \(\{T^n x\}_{n \geq 1}\) converges to the fixed point.

**Corollary 1.69** Let \((X, p)\) be a complete partial cone metric space, \(P\) be a normal cone with normal constant \(K\). Suppose that the mapping \(T : X \to X\) satisfy the contractive condition

\[ p(Tx, Ty) \leq k[p(x, Tx) + p(y, Ty)] \]

for all \(x, y \in X\), and \(k \in (0, 1/2)\). Then \(T\) has a unique fixed point in \(X\). For each \(x \in X\), the iterative sequence \(\{T^n x\}_{n \geq 1}\) converges to the fixed point.

**Theorem 1.70** Let \((X, p)\) be a complete partial cone metric space, \(P\) be a normal cone on \(X\). Suppose a mapping \(T : X \to X\) satisfy the contractive condition

\[ p(Tx, Ty) \leq r \max\{p(x, y), p(x, Tx), p(y, Ty)\} \]

for all \(x, y \in X\) and \(r \in [0, 1)\). Then \(T\) has a unique fixed point in \(X\).
Theorem 1.71 Let \((X,p)\) be a complete partial cone metric space, \(P\) be a normal cone on \(X\). Suppose a mapping \(T, S : X \to X\) satisfy the contractive condition

\[ p(Tx, Sy) \leq a[p(x, Tx) + p(y, Sy)] \]

for all \(x, y \in X\) and \(a \in (0, 1/2)\). Then \(T\) and \(S\) have a unique common fixed point in \(X\).

In Chapter 5, in section 2, using the concept of contractive jointly continuous family of mappings we prove common fixed point theorems for \(S\)-non-expansive and asymptotically \(S\)-non-expansive mappings in the setting of metric spaces which are not necessarily linear defined on non-star-shaped and non-convex domain, which are extension of results of Vijayaraju and Marudai(2004), Al-Thagafi (1996), Sahab and Khan(1987), Habiniak(1989), Hicks and Humphries(1982) and Dotson(1972). And as a consequence, in section 3, we establish some applications to best approximation.

The following two theorems are generalizations of theorem of Vijayaraju and Marudai(2004), Al-Thagafi(1996).

Theorem 1.72 Let \(K\) be a nonempty closed subset of a metric space \((X,d)\). \(T\) and \(S\) be self-mappings of \(K\) with \(T(K) \subset S(K)\). Suppose \(K\) has a contractive jointly continuous family \(\mathcal{S} = \{f_x/x \in K\}\) such that \(T(f_x(\alpha)) = f_{Tx}(\alpha)\) for all \(x \in K\) and \(\alpha \in [0,1]\). If \(\overline{T(K)}\) is compact, \(S\) is continuous, \(S\) and \(T\) are commuting and \(T\) is \(S\) non-expansive, then \(T\) and \(S\) have a common fixed point in \(K\).

Theorem 1.73 Let \(K\) be a nonempty closed subset of a metric space \((X,d)\). \(T\) and
S be self-mappings of K with \( T(K) \subset S(K) \). Suppose K has a contractive jointly continuous family \( \mathcal{F} = \{ f_x / x \in K \} \) such that \( T(f_x(\alpha)) = f_Tx(\alpha) \) for all \( x \in K \) and \( \alpha \in [0, 1] \). If \( T(K) \) is compact, \( S \) is continuous, \( S \) and \( T \) are commuting and \( T \) is asymptotically \( S \)-non-expansive and asymptotically regular, then \( T \) and \( S \) have a common fixed point.

The following theorem is an application to best approximation which is an extension of Sahab and Khan (1987), Habiniak (1989) and Hicks and Humphries (1982).

**Theorem 1.74** Let \( C \) be a nonempty subset of a metric space \( (X, d) \). Let \( T \) and \( S \) be self-mappings of \( C \). Let \( p \in F(T, S) \). Suppose that \( P_C(p) \) is nonempty and approximately compact and there is a contractive, jointly continuous family of functions associated with \( P_C(p) \). If \( D \) is closed, \( S(D) = D \), \( S^2 = S \), \( S \) is continuous, \( S \) and \( T \) are commuting and \( T \) is \( S \)-non-expansive on \( D \cup \{p\} \), then \( T \) and \( S \) have a common fixed point in \( P_C(p) \).

**Corollary 1.75** Let \( C \) be a nonempty subset of a metric space \( (X, d) \) and let \( T \) be a non-expansive. Let \( T(C) \subset C \) and \( p \in F(T) \). Suppose that \( P_C(p) \) is nonempty and approximately compact and there is a contractive, jointly continuous family of functions associated with \( P_C(p) \). Then \( T \) has a fixed point, which is a best approximation to \( p \) in \( P_C(p) \).