Chapter 5

COMMON FIXED POINT AND BEST APPROXIMATIONS

5.1 INTRODUCTION

Brosowski (1969) proved that if $T$ is linear and non-expansive self map of $K$ with $p \in F(T)$ and $Q_K(p)$ is a nonempty compact and convex set, then $T$ has a fixed point in $Q_K(p)$, where $K$ is a nonempty subset of normed linear space $X$, $F(T)$ is the set of point of $T$ and $Q_K(p) = \{x \in K : \|x - p\| = d(p, K)\}$ is the set of best approximants to a point $p \in X$. Subrahmanyan (1977) replaced the requirement that $Q_K(p)$ is nonempty by the assumption that $K$ is a finite-dimensional subspace of $X$. Singh (1979) proved the result of Brosowski (1969) by relaxing the linearity of the operator $T$ and the convexity of $Q_K(p)$. Singh (1979) observed that the non-expansiveness of $T$ on $Q_K(p) \cap \{p\}$ is enough for his earlier result. Hicks and Humphries (1982) noted that Singh’s result remains true if $T(K) \subset K$ is replaced by $T(\partial K) \subset K$, where $\partial K$ is the boundary of $K$ in $K$. Simoluk (1981) proved Subrahmanyan’s result by assuming that $T$ is linear and $T(K)$ is relatively compact instead of $M$ is finite-dimensional. Habiniak (1989) removed the linearity of $T$ from Smoluk’s result. Sahba, Khan and

The following results will be used in the sequel.

**Definition 5.1** Let $K$ be a subset of a Banach space $E$, and let $F = \{f_\alpha/\alpha \in K\}$ be family of maps from $[0,1]$ into $K$ having property that for each $\alpha \in K$, we have $f_\alpha(1) = \alpha$, such a family $F$ is said to be

(i) contractive, if there exists a function $\phi : (0,1) \to (0,1)$ such that for $\alpha, \beta \in K$ and for all $t \in (0,1)$ we have $d(f_\alpha(t), f_\beta(t)) \leq \phi(t)d(\alpha, \beta)$.

(ii) jointly continuous, if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in $K$, implies $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in $K$.

**Definition 5.2** Let $K$ be a nonempty subset of metric space $X$, $T$ and $S$ are self-mappings of $K$. Then

(i) $T$ is $S$-non-expansive (non-expansive with respect to $S$) if

$$d(T(x), T(y)) \leq d(S(x), S(y)) \text{ for all } x, y \in K.$$  

(ii) $T$ is asymptotically $S$-non-expansive (asymptotically non-expansive with respect to $S$) if there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$,
$k_n \to 1$ as $n \to \infty$ such that

$$d(T^n(x), T^n(y)) \leq k_n d(S(x), S(y)) \text{ for all } x, y \in K.$$

(iii) $T$ is asymptotically regular if for all $x \in K$,

$$d(T^n(x), T^{n+1}(y)) \to 0 \text{ as } n \to \infty.$$

(iv) $K$ is q-starshaped if there exists a point $q \in K$ such that $(1 - \lambda)q + \lambda x \in K$ for all $x \in K$.

If $S$ is the identity mapping in (i) and (ii), then $T$ is non-expansive and $T$ is asymptotically non-expansive.

The following example shows that the class of asymptotically $S$-non-expansive mapping is wider than the class of $S$-non-expansive mappings in normed linear spaces.

**Example 5.1** Let $X = \ell_2$ with usual norm and let $K$ be a closed unit ball in $X$. Define a self-mapping $T$ of $K$ by

$$T(x) = (0, x_1^2, A_2x_2, A_3x_3, ...) \text{ for all } x = (x_1, x_2, ...) \in K,$$

where $\{A_i\}$ is a sequence of real numbers such that $0 < A_i < 1 \ \forall i$, and $\Pi_{i=1}^{\infty} A_i = \frac{1}{2}$.

Define a self-mapping $S$ of $K$ by

$$S(y) = (0, y_1, y_2, ...) \ \forall \ y = (y_1, y_2, ...) \in K.$$

Then $T$ is asymptotically $S$-non-expansive but not $S$-non-expansive.

In the following example, the given mapping is asymptotically non-expansive and asymptotically regular.
Example 5.2 Let $X = \ell_2$ with usual norm and let $K$ be a closed unit ball in $X$. Define a self-mapping $T$ by $T(x) = (x_2, x_3, \ldots)$ for all $x = (x_1, x_2, \ldots) \in K$, then $T$ is asymptotically non-expansive and asymptotically regular.

Theorem 5.1 (Al-Thagafi(1996)). Let $K$ be a nonempty closed subset of metric space $X$ and let $T, S$ be self-mappings of $K$ with $T(K) \subset S(K)$. If $\overline{T(K)}$ is complete, $S$ is continuous, $T$ and $S$ are commuting and $T$ is $S-$contraction, then $T$ and $S$ have a unique common fixed point in $K$.

Theorem 5.2 (P.Vijayaraju and M.Marudai(2004)) Let $K$ be a nonempty closed subset of a normed linear space $X$, $T$ and $S$ be self-mappings of $K$ with $T(K) \subset S(K)$, and $q \in F(S)$. If $K$ is $q$-starshaped, $\overline{T(K)}$ is complete bounded, $T$ is demicompact, $S$ is continuous and affine with respect to $q$, $T$ and $S$ are commuting and $T$ is $S-$non-expansive, then $T$ and $S$ have a common fixed point in $K$.

Theorem 5.3 (P.Vijayaraju and M.Marudai(2004)) Let $K$ be a nonempty closed subset of a normed linear space $X$, $T$ and $S$ be self-mappings of $K$ with $T(K) \subset S(K)$ and $q \in F(S)$. If $K$ is $q$-starshaped, $\overline{T(K)}$ is compact, $S$ is continuous and affine with respect to $q$, $T$ and $S$ are commuting and $T$ is asymptotically $S$-non-expansive and asymptotically regular, then $T$ and $S$ have a common fixed point in $K$.

In this chapter, let $F(T, S)$ denote the set of common fixed points of $T$ and $S$ and $F(S)$ the set of fixed points of $S$. 
In this Chapter using the concept of contractive jointly continuous family of mappings we prove some common fixed point theorems for S-non-expansive and asymptotically S-non-expansive self-mappings which are not necessarily linear defined on non-star-shaped domain. These result are extensions of results of P.Vijayaraju and M.Marudai (2004), Al-Thangafi (1996), Habiniak (1989), Sahab and Khan (1987) and Dotson (1972). In section 3, as a consequences we also obtain some applications of fixed points to best approximation.

5.2 FIXED POINTS FOR PAIR OF MAPPINGS

**Theorem 5.4** Let \( K \) be a nonempty closed subset of a metric space \((X, d)\). \( T \) and \( S \) be self-mappings of \( K \) with \( T(K) \subset S(K) \). Suppose \( K \) has a contractive jointly continuous family \( \mathcal{I} = \{ f_x / x \in K \} \) such that \( T(f_x(\alpha)) = f_{Tx}(\alpha) \) for all \( x \in K \) and \( \alpha \in [0, 1] \). If \( \overline{T(K)} \) is compact, \( S \) is continuous, \( S \) and \( T \) are commuting and \( T \) is \( S \) non-expansive, then \( T \) and \( S \) have a common fixed point in \( K \).

**Proof.** For each \( n \in \mathbb{N} \), let \( \{k_n\} \) be a sequence of real numbers in \([1, \infty)\) with \( k_n \geq k_{n+1}, k_n \to 1 \) as \( n \to \infty \). Define \( T_n : K \to K \) as \( T_n(x) = f_{Tx}(k_n) \), for each \( x \in K \).

Since \( S \) and \( T \) are commuting,

\[
T_n(Sx) = f_{TSx}(k_n) = f_{STx}(k_n) = ST_n(x)
\]
Hence $T_n$ and $S$ are commuting.

Since $T(K) \subset S(K)$,

Therefore, $T_n(K) \subset S(K)$.

Since $f$ is contractive and $T$ is $S$ non-expansive, we have

$$d(T_n x, T_n y) = d(f_{T x}(k_n), f_{T y}(k_n))$$
$$\leq \phi(k_n) d(T x, T y)$$
$$\leq \phi(k_n) d(S x, S y) \forall x, y \in D.$$ 

Therefore, $T_n$ is $S$ contraction.

Since $T_n(x) = T(f_x(k_n)) \in T(K)$ and since $T(K)$ is compact and $T_n(K)$ is compact. Hence by Al-Thagafi Theorem, exists $x_n \in K$ such that $x_n \in F(T_n, S)$ for each $n$.

Since $\{T x_n\}$ is a sequence in compact set $\overline{T(K)}$, $\exists$ a subsequence $\{T x_{n_i}\} \to x_0$ $\in \overline{T(K)}$.

Since $x_{n_i} = T_n(x_{n_i}) = f_{T x_{n_i}}(k_{n_i}) \to f_{x_0}(1) = x_0$, the continuity of $S$ and $T$ implies that $x_0 \in F(S,T)$. □

**Remark 5.1** Suppose $D$ is a starshaped(with $q$ as a star center) subset of a normed space $X$. Define $f_q(t) = (1 - t)x + t q, x \in D, t \in [0,1]$. Then $\exists = \{f_x/x \in K\}$ is a contractive jointly continuous family with $\phi(t) = t$. Thus the class of subsets of $D$ with the property of contractive and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.
Hence by this remark, the previous theorem is an extension of results of P. Vijayaraju and M. Marudai (2004), Al-Thagafi (1996), Habiniak (1989), Sahab and Khan (1987) and Dotson (1972).

**Theorem 5.5** Let $K$ be a nonempty closed subset of a metric space $(X, d)$. $T$ and $S$ be self-mappings of $K$ with $T(K) \subset S(K)$. Suppose $K$ has a contractive jointly continuous family $\mathcal{F} = \{f_x \mid x \in K\}$ such that $T(f_x(\alpha)) = f_T x(\alpha)$ for all $x \in K$ and $\alpha \in [0, 1]$. If $\overline{T(K)}$ is compact, $S$ is continuous, $S$ and $T$ are commuting and $T$ is asymptotically $S$-non-expansive and asymptotically regular, then $T$ and $S$ have a common fixed point.

**Proof.** For each $n \in N$, let $\{k_n\}$ be a sequence of real numbers in $[1, \infty)$ such that $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$. Define $T_n : K \to K$ as

$$T_n (x) = f_{T^n x}(k_n), \text{ for each } x \in K.$$ 

Since $S$ and $T$ are commuting

$$T_n (Sx) = f_{T^n Sx}(k_n) = f_{S T^n x}(k_n) = S T_n (x).$$

Hence $T_n$ and $S$ are commuting. Since $T(K) \subset S(K)$, $T_n(K) \subset S(K)$. Since $f$ is
contractive and $T$ is asymptotically $S$-non-expansive, we have

\[
d(T_n x, T_n y) = d(f_{T^n x}(k_n), f_{T^n y}(k_n))
\leq \phi(k_n) d(T^n x, T^n y)
\leq \phi(k_n) d(Sx, Sy) \quad \forall \ x, y \in D.
\]

\[\therefore \ T_n \text{ is } S \text{ contraction.}\]

Since $T_n(x) = T(f_x(k_n)) \in T(K)$ and since $\overline{T(K)}$ is compact and $\overline{T_n(K)}$ is compact.

Hence by Al-Thagafi Theorem, $\exists x_n \in K$ such that $x_n \in F(T_n, S)$ for each $n$.

Since $\{T^n x_n\}$ is a sequence in a compact set $\overline{T(K)}$, $\exists$ a subsequence $\{T^{n_j} x_{n_j}\} \to x_0$ $\in \overline{T(K)}$. Hence $x_{n_j} = T_n(x_{n_j}) = f_{T^{n_j} x_{n_j}}(k_{n_j}) \to f_{x_0}(1) = x_0$. Since $S$ is continuous and since $x_{n_j} = S(x_{n_j}), S(x_0) = x_0 \ x_0 \in F(x_0)$. Since $T$ is asymptotically $S$-non-expansive and since $S$ is continuous,

\[
d(d(T^{n_j} x_{n_j}, T^{n_j} x_0) \leq k_{n_j} d(S(x_{n_j}), S(x_0)) \to 0 \text{ as } j \to \infty.
\]

Hence $T^{n_j} x_0 \to x_0$ as $j \to \infty$.

Also Since $T$ is asymptotically $S$-non-expansive and since $T$ is asymptotically
regular,

\[ d(x_0, Tx_0) \leq d(x_0, T^{m_j}x_0) + d(T^{m_j}x_0, T^{m_j+1}x_{n_0}) + d(T^{m_j+1}x_{n_0}, Tx_0) \]

\[ \leq d(x_0, T^{m_j}x_0) + d(T^{m_j}x_0, T^{m_j+1}x_{n_0}) + k_1 d(S(T^{m_j}x_0), S(x_0)) \]

\[ \leq d(x_0, T^{m_j}x_0) + d(T^{m_j}x_0, T^{m_j+1}x_{n_0}) + k_1 d(T^{m_j}S(x_0), S(x_0)) \]

\[ \leq d(x_0, T^{m_j}x_0) + d(T^{m_j}x_0, T^{m_j+1}x_{n_0}) + k_1 d(T^{m_j}x_0, x_0) \]

\[ \leq (1 + k_1)d(x_0, T^{m_j}x_0) + d(T^{m_j}x_0, T^{m_j+1}x_{n_0}) \to 0 \text{ as } j \to \infty. \]

Hence \( x_0 \in F(S, T) \).

5.3 FURTHER APPLICATIONS TO BEST APPROXIMATIONS

Let \( D = P_C(x_0) \cap D^S_C(x_0) \), where \( D^S_C(x_0) = \{ x \in C : S(x) \in P_C(x_0) \} \).

The following theorem is an application to best approximation which is an extension of Vijayaraju P. and Marudai M(2004), Al-Thangafi (1996), Sahab and Khan(1987), Habiniak(1989) and Hicks and Humphries(1982).

**Theorem 5.6** Let \( C \) be a nonempty subset of a metric space \( (X, d) \). Let \( T \) and \( S \) be self-mappings of \( C \). Let \( p \in F(T, S) \). Suppose that \( P_C(p) \) is nonempty and approximately compact and there is a contractive, jointly continuous family of functions associated with \( P_C(p) \). If \( D \) is closed, \( S(D) = D \), \( S^2 = S \) \( S \) is continuous, \( S \) and \( T \) are commuting and \( T \) is \( S \) non-expansive on \( D \cup \{ p \} \), then \( T \) and \( S \) have a common fixed point in \( P_C(p) \).
Proof. Let \( x \in D \). Since \( T \) is \( S \) non-expansive and since \( S(x) \in PC(p) \)

\[
d(T(x), p) = d(T(x), T(p)) \leq d(S(x), S(p)) = d(S(x), P) = d(p, C).
\]

Hence \( T(x) \in PC(p) \).

\[
d(S(T(x)), p) = d(TS((x)), T(p)) \leq d(S^2(x), S(p)) = d(S(x), P) = d(p, C).
\]

Hence \( T(x) \in D^S_C(p) \). Therefore \( T(x) \in D = S(D) \).

Hence \( T(D) \subset S(D) \). Since \( PC(p) \) is compact and since \( D \) is closed subset of \( PC(p) \), \( D \) is compact. Hence by theorem, \( S \) and \( T \) have a common fixed point in \( D \) and hence in \( PC(p) \).

\[\square\]

**Corollary 5.7** Let \( C \) be a nonempty subset of a metric space \((X, d)\) and let \( T \) be a non-expansive. Let \( T(C) \subset C \) and \( p \in F(T) \). Suppose that \( PC(p) \) is nonempty and approximately compact and there is a contractive, jointly continuous family of functions associated with \( PC(p) \). Then \( T \) has a fixed point, which is a best approximation to \( p \) in \( PC(p) \).

**Proof.** The proof of the corollary follows immediately by putting \( S = I \), the identity mapping in the previous theorem.  

\[\square\]