CHAPTER - I
INTRODUCTION

This work deals with graph labeling. All the graphs considered here are finite and undirected. The terms not defined here are used in the sense of Harary [10].

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

By a \((p, q)\) graph \(G\), we mean a graph \(G = (V, E)\) with \(|V| = p\) and \(|E| = q\).

Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967 [19]. Rosa introduced a function \(f\) from a set of vertices in a graph \(G\) to the set of integers \(\{0, 1, 2, \ldots, q\}\) so that each edge \(xy\) is assigned the label \(|f(x) - f(y)|\), all labels are distinct. Rosa called this labeling \(\beta\)-valuation. Independently, Golomb [8] studied the same type of labeling and called as graceful labeling.

Graceful labeling originated as a means of attacking the conjecture of Ringel [18] that \(K_{2n+1}\) can be decomposed into \(2n+1\) subgraphs that are all isomorphic to a given tree with \(n\) edges. Those graphs that have some sort of regularity of structure are said to be graceful. Sheppard [21] has shown that there are exactly \(q!\) gracefully labeled graphs with \(q\) edges. Rosa [19] suggested three possible reasons why a graph fails to be graceful.
1) $G$ has “too many vertices” and “not enough edges”

2) $G$ has “too many edges”

3) $G$ has “wrong parity”

Labeled graphs serve as useful models for a broad range of applications such as X-ray crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly, interesting applications of graph labeling can be found in [1, 2, 3, 4, 5, 9, 20].

In the recent years, dozens of graph labeling techniques such as $\alpha$-labeling, $k$ and $(k, d)$-graceful, $k$-equitable, skolem graceful, odd graceful and graceful like labeling have been studied over 1000 papers [7].

In 1990, Antimagic graphs was introduced by Harrifield and Ringel [11]. A graph with $q$ edges is said to be antimagic if its edges can be labeled with 1, 2, ..., $q$ such that the sums of the labels of the edges incident to each vertex are distinct. This notion of antimagic labeling was extended to hypergraphs by Sonntag [24].

In 1985, Lo [17] introduced the edge – graceful graph which is a dual notion of graceful labeling. A graph $G(V, E)$ is said to be edge – graceful if there exists a bijection $f$ from $E$ to \{1, 2, ..., $q$\} such that the induced mapping $f^+$ from $V$ to \{0, 1, 2, ..., $p - 1$\} given by $f^+(x) = (\sum f(xy))(mod p)$ taken over all edges $xy$ is a bijection.

Every edge – graceful graph is found to be antimagic and Lo [17] found a necessary condition for a graph with $p$ vertices and $q$ edges to be edge – graceful as $q(q+1) = \frac{p(p+1)}{2} (mod p)$.
Lee [14] noted that this necessary condition extends to any multigraph with $p$ vertices and $q$ edges. He conjectured that any connected simple $(p,q)$ graph with $q(q + 1) \equiv \frac{p(p - 1)}{2} \pmod{p}$ is edge–graceful and found that this condition is sufficient for the edge–gracefulness of connected graphs. He also [15] conjectured that all trees of odd order are edge–graceful.

Small [23] has proved that spiders in which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same, are edge–graceful. Keene and Simoson [12] proved that all spiders are edge–graceful if the spider has odd order with exactly three end vertices. Cabaniss, Low, and Mitchem [6] have shown that regular spiders of odd order are edge–graceful. Lee, Seah, and Wang [16] gave a complete characterization of edge–graceful $p_n^k$ graphs. Shiu, Lam, and Cheng [22] proved that the composition of the path $P_3$ and any null graph of odd order is edge–graceful.

Lo proved that all odd cycles are edge–graceful. Wilson and Riskin [25] proved that the Cartesian product of any number of odd cycles is edge–graceful.

This thesis emerged with the growing interest in the notion of edge–graceful graphs and its noteworthy conjectures.

By the necessary condition of edge–gracefulness of a graph one can verify that even cycles, and paths of even length are not edge–graceful. But whether trees of odd order are edge–graceful is still open.

Motivated by the notion of edge–graceful graphs and Lo’s conjecture, we define a new type of labeling called **strong edge–graceful labeling** by
relaxing its range through which we can get strong edge – graceful labeling of even order trees.

A \((p, q)\) graph \(G\) is said to have **strong edge – graceful labeling** if there exists an injection \(f\) from the edge set to \(\left\{1, 2, \ldots, \left\lfloor \frac{3q}{2} \right\rfloor \right\}\) so that the induced mapping \(f^+\) from the vertex set to \(\{0, 1, \ldots, 2p - 1\}\) defined by \(f^+(x) = \sum \{f(xy) \mid xy \in E(G)\} \pmod{2p}\) are distinct. A graph \(G\) is said to be **strong edge – graceful** if it admits a strong edge – graceful labeling.

In this thesis, we investigate strong edge – graceful labeling (SEGL) of some graphs. In some situations, we also present edge – graceful labeling of some trees towards attempting to the Lo’s Conjecture.

We now present chapter-wise summary.

**Chapter I** briefly introduces the thesis.

**Chapter II** provides the fundamental concepts of graph theory and definitions of graphs which are needed for the rest of this thesis.

**Chapter III** defines a new type of labeling called strong edge – graceful labeling by relaxing the range of edge – graceful labeling. Here, we discuss the strong edge – graceful labeling of the graphs \(F_n, C_n, P_n, C_n^+, K_{1,n}, TW(n), SP(1^3, m)\) and \(P_m \odot nK_1\).

**Chapter IV** deals with the edge – graceful labeling schemes of the bistar trees and some specific families of trees such as \(\langle K_{1,n} : K_{1,m} \rangle\), \(Y_n\), \(FC(1^m, K_{1,n})\), \(mG_n\) and \(N\langle K_{1,n} : K_{1,n} \rangle\).

**Chapter V** deals with the strong edge – graceful labeling of the graphs \(C_m \odot K_{1,n}, M(P_n), Fl(n), C_3^{(t)}\) and butterfly graphs.
Chapter VI deals with the strong edge – graceful labeling schemes of the graphs $Fl_n$, $P_n^+$, $M_n$, $W_n$, $T_n$ and $B_{n,m}$.

Chapter VII provides the strong edge – graceful labeling schemes of the graphs $K_2 \odot C_n$, $C_n \oplus P_1$, $G(n)$, $SF(n)$ and $S_n$.

Chapter VIII discusses the strong edge – graceful labeling schemes of some disconnected graphs such as disjoint unions of paths, cycles and star graphs.

Finally, the thesis ends with bibliography.