Chapter 3
Structure Connection in a Generalised Co-symplectic Manifold

3.1 Introduction


Quarter Symmetric Non-metric connection in a generalized co-symplectic manifolds was studied by S. Yadav and D. L. Suthar [73]. In this chapter, we have studied the structure connection in generalised co-symplectic manifold and deduced some interesting results. We have shown that a generalised co-symplectic manifold equipped with structure connection is completely integrable. We also find a condition for a generalised co-symplectic manifold
to be a quasi-Sasakian manifold.

3.2 Preliminaries

An odd dimensional differentiable manifold $M_n$, $n = 2m + 1$, of differentiability class $C^\infty$, is said to be an almost contact manifold ([50], [51], [52]), if there exist a vector valued linear function $\phi$, a 1-form $\eta$, the associated vector field $\xi$ satisfying for arbitrary vector field $X$

\[(a) \quad \overline{X} + X = \eta(X)\xi, \quad (b) \quad \phi(\xi) = 0, \quad (3.2.1)\]

where

\[\overline{X} \overset{\text{def}}{=} \phi X.\]

In view of (3.2.1)(a) and (3.2.1)(b), we find

\[(a) \quad \eta(\xi) = 1, \quad (b) \quad \eta(\overline{X}) = 0, \quad (3.2.2)\]

An almost contact manifold in which a Riemannian metric tensor $g$ of type $(0,2)$ satisfying

\[g(\overline{X}, Y) = g(X, Y) - \eta(X)\eta(Y), \quad (3.2.3)\]

\[g(X, \xi) = \eta(X), \quad (3.2.4)\]

for arbitrary vector fields $X$ and $Y$, is called an almost contact metric manifold and the structure $(\phi, \eta, \xi, g)$ is called an almost contact metric structure to $M_n$.

If we put

\[F(X, Y) = g(\overline{X}, Y), \quad (3.2.5)\]
3.2. PRELIMINARIES

then, we obtain
\[ F(X, Y) = F(\overline{X}, \overline{Y}), \] (3.2.6)
\[ F(X, Y) = g(\overline{X}, Y) = -g(X, \overline{Y}) = -F(Y, X). \] (3.2.7)

An almost contact metric manifold satisfying
\[ (\nabla_X F)(Y, Z) = \eta(Y)(\nabla_X \eta)(\overline{Z}) - \eta(Z)(\nabla_X \eta)(\overline{Y}), \] (3.2.8)
\[ (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 0, \] (3.2.9)

and
\[ (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) \]
\[ + \eta(X)[(\nabla_Y \eta)(\overline{Z}) - (\nabla_Z \eta)(\overline{Y})] + \eta(Y)[(\nabla_Z \eta)(\overline{X}) - (\nabla_X \eta)(\overline{Z})] \]
\[ - (\nabla_X \eta)(\overline{Z}) + \eta(Z)[(\nabla_X \eta)(\overline{Y}) - (\nabla_Y \eta)(\overline{X})] = 0, \] (3.2.10)

for arbitrary vector fields \( X, Y, Z \); are respectively called generalised co-symplectic, quasi-Sasakian and generalised quasi-Sasakian manifolds [38].

Here \( \nabla \) denotes the Riemannian connection with respect to the Riemannian metric \( g \). If on any manifold, \( \xi \) satisfies
\[ (a) \quad (\nabla_X \eta)(\overline{Y}) = -(\nabla_{\overline{X}} \eta)(Y) = (\nabla_Y \eta)(\overline{X}), \] (3.2.11)
\[ (b) \quad (\nabla_X \eta)(Y) = (\nabla_{\overline{X}} \eta)(\overline{Y}) = -(\nabla_Y \eta)(X), \]
\[ (c) \quad (\nabla_{\xi} \phi) = 0, \]

then \( \xi \) is said to be of the first class and the manifold is said to be of the first class [38].
CHAPTER 3. STRUCTURE CONNECTION IN A GENERALISED....

If on an almost contact metric manifold $\xi$ satisfies

\begin{align*}
(a) \quad (\nabla_X \eta)(Y) &= (\nabla_X \eta)(Y) = -(\nabla_Y \eta)(X), \quad \Leftrightarrow \quad (3.2.12) \\
(b) \quad (\nabla_X \eta)(Y) &= -(\nabla_Y \eta)(Y) = -(\nabla_Y \eta)(X), \\
(c) \quad (\nabla_\xi \phi) &= 0,
\end{align*}

then $\xi$ is said to be of the second class and the manifold is said to be of the second class [38].

A generalised co-symplectic manifold of first class is called normal quasi-Sasakian manifold. The necessary and sufficient condition that a quasi-Sasakian manifold to be normal is [38]

\begin{align*}
(\nabla_X F)(Y, Z) &= \eta(Y)(\nabla_Z \eta)(X) + \eta(Z)(\nabla_X \eta)(Y). \quad (3.2.13)
\end{align*}

The Nijenhuis tensor $\tilde{N}$ in generalised co-symplectic manifold is given by

\begin{align*}
(a) \quad N(X, Y) &= (\nabla_X \phi)(Y) - (\nabla_Y \phi)(X) - (\nabla_X \phi)(Y) + (\nabla_Y \phi)(X), \\
(b) \quad \tilde{N}(X, Y, Z) &= (\nabla_X F)(Y, Z) - (\nabla_Y F)(X, Z) + (\nabla_X F)(Y, Z) - (\nabla_Y F)(X, Z).
\end{align*}

Where $\tilde{N}(X, Y, Z) = g(N(X, Y), Z)$.

3.3 Structure connection in a generalised co-symplectic manifold

Let $M_n$ be an $n$–dimensional generalised co-symplectic manifold equipped with the Levi-Civita connection $\nabla$, then a structure connection $E$ in $M_n$ is
3.3. STRUCTURE CONNECTION IN A GENERALISED CO-SYMPLECTIC MANIFOLD

defined by [60]

\[ E_XY = \nabla_X Y - \eta(X)Y - \overline{X}\eta(Y) - \overline{Y}\eta(X) + g(\overline{X}, Y)\xi. \]  (3.3.1)

whose torsion tensor \( S \) is given by

\[ S(X, Y) = \eta(Y)X - \eta(X)Y + 2F(X, Y)\xi, \]  (3.3.2)

and Riemannian metric \( g \) satisfies

\[ (E_X g)(Y, Z) = 2\eta(X)g(Y, Z), \]  (3.3.3)

for arbitrary vector fields \( X, Y \) and \( Z \).

Put

\[ E_X Y = \nabla_X Y + P(X, Y), \]  (3.3.4)

where \( P \) is a tensor field of type \((1,2)\), then we have

(a) \[ P(X, Y) = -\eta(X)Y - \overline{X}\eta(Y) - \overline{Y}\eta(X) + g(\overline{X}, Y)\xi, \]  (3.3.5)

(b) \[ \tilde{P}(X, Y, Z) = \eta(Z)F(X, Y) - \eta(X)F(Y, Z) + \eta(Y)F(Z, X) - \eta(X)g(Y, Z), \]

(c) \[ S(X, Y) = P(X, Y) - P(Y, X), \]

(d) \[ \tilde{S}(X, Y, Z) = \eta(Y)g(Z, X) - \eta(X)g(Y, Z) + 2\eta(Z)g(\overline{X}, Y), \]

(e) \[ (E_X \eta)(Y) = (\nabla_X \eta)(Y) + \eta(X)\eta(Y) - g(\overline{X}, Y), \]

\[ 81 \]
where

\[ \tilde{S}(X, Y, Z) \overset{\text{def}}{=} g(S(X, Y), Z). \]

\[ \tilde{P}(X, Y, Z) \overset{\text{def}}{=} g(P(X, Y), Z). \]

Now, we have

\[ X(F(Y, Z)) = (\nabla_X F)(Y, Z) + F(\nabla_Y Y, Z) + F(Y, \nabla_Z Z) \]
\[ = (E_X F)(Y, Z) + F(E_X Y, Z) + F(Y, E_X Z). \]

Using (3.3.1) in last expression, we have

\[ (E_X F)(Y, Z) = (\nabla_X F)(Y, Z) + 2\eta(X)F(Y, Z) - \eta(Y)g(X, Z) + \eta(Z)g(X, Y). \] (3.3.6)

In the almost contact metric manifold with structure connection \( E \) it can be seen that [40]

\[ \begin{align*}
(a) & \quad \tilde{P}(\overline{X}, Y, Z) = \tilde{P}(X, Y, \overline{Z}) - \tilde{P}(X, \overline{Y}, Z), \\
(b) & \quad \tilde{P}(X, Y, Z) = 0 = \tilde{S}(X, Y, Z), \\
(c) & \quad (E_X \eta)(\overline{Y}) = (\nabla_X \eta)(\overline{Y}) - g(X, \overline{Y}), \\
(d) & \quad (E_X \phi)(Y) = (\nabla_X \phi)(Y) - \eta(Y)X + (X, Y)\zeta.
\end{align*} \] (3.3.7)

The Nijenhuis tensor \( N \) in terms of structure connection \( E \) is given by

\[ \begin{align*}
(a) & \quad N_E(X, Y) = (E_X \phi)(Y) - (E_Y \phi)(X) - \frac{1}{2}(E_X \phi)Y + (E_Y \phi)X, \\
(b) & \quad N_E(X, Y, Z) = (E_X F)(Y, Z) - (E_Y F)(X, Z) + (E_X F)(Y, \overline{Z}) - (E_Y F)(X, \overline{Z}).
\end{align*} \] (3.3.8)
where $N_{E}(X, Y, Z) = g(N_{E}(X, Y), Z)$.

**Theorem 3.1.** An almost contact metric manifold with structure connection $E$ satisfies the relation

$$\tilde{S}(\bar{X}, Y, \bar{Z}) + \tilde{S}(Y, \bar{Z}, \bar{X}) = \tilde{P}(\bar{X}, Y, Z) - \tilde{P}(X, Z, \bar{Y}). \quad (3.3.9)$$

**Proof.** Barring $X$ and $Z$ in (3.3.5)(d) and using (3.2.2)(b) and (3.2.3), we have

$$\tilde{S}(\bar{X}, Y, \bar{Z}) = \eta(Y)g(\bar{X}, \bar{Z}), \quad (3.3.10)$$

and

$$\tilde{S}(Y, \bar{Z}, \bar{X}) = -\eta(Y)g(\bar{X}, \bar{Z}), \quad (3.3.11)$$

From (3.2.2)(b), (3.2.3) and (3.3.5)(b), we have

$$\tilde{P}(\bar{X}, Y, Z) = -\eta(Z)g(X, \bar{Y}), \quad (3.3.12)$$

and

$$\tilde{P}(\bar{X}, Z, \bar{Y}) = \eta(Z)g(X, \bar{Y}), \quad (3.3.13)$$

In consequence of (3.3.10), (3.3.11), (3.3.12) and (3.3.13), we obtain (3.3.9).

**Theorem 3.2.** A generalised co-symplectic manifold with structure connection $E$ satisfies the relations

(a) $$(EXF)(Y, \bar{Z}) + (EYF)(Z, \bar{X}) + (EZF)(X, \bar{Y}) = \eta(X)[(\nabla_{\bar{Z}}\eta)\bar{Y} - g(\bar{Z}, \bar{Y})] + \eta(Y)[(\nabla_{\bar{X}}\eta)\bar{Z} - g(\bar{X}, \bar{Z})]$$

$$+ \eta(Z)[(\nabla_{\bar{Y}}\eta)\bar{X} - g(\bar{Y}, \bar{X})], \quad (3.3.14)$$

(b) $$(EXF)(Y, \bar{Z}) = 2\eta(X)F(Y, Z).$$
CHAPTER 3. STRUCTURE CONNECTION IN A GENERALISED...

Proof. From (3.2.8) and (3.3.6), we have

\[(E_X F)(Y, Z) = \eta(Y)[(\nabla_X \eta)(\overline{Z}) - g(X, Z)] - \eta(Z)[(\nabla_X \eta)(\overline{Y}) - g(X, Y)] + 2\eta(X)F(Y, Z),\]  

(3.3.15)

using (3.2.2)(a) in above equation, we at once obtain (3.3.14)(a).

Now, barring \(Y\) and \(Z\) in (3.3.15) and using (3.2.2)(a) and (3.2.6), we get the result (3.3.14)(b).

Theorem 3.3. If \(\xi\) is killing on generalised co-symplectic manifold with structure connection \(E\), then

\[(E_X F)(Y, Z) + (E_Y F)(Z, X) + (E_Z F)(X, Y) = N_E(X, Y, Z) + 2\eta(Z)[(\nabla_X \eta)(\overline{Y}) - g(\overline{X}, Y)].\]  

(3.3.16)

Proof. From (3.3.8)(b), we have

\[N_E(X, Y, Z) - (E_X F)(Y, \overline{Z}) - (E_Y F)(\overline{Z}, X) - (E_Z F)(X, Y) = (E_X F)(Y, Z) - (E_Y F)(X, Z) - (E_Z F)(X, Y).\]

Using (3.2.2)(b),(3.2.8) and (3.3.6) in the above equation, we have

\[N_E(X, Y, Z) - (E_X F)(Y, \overline{Z}) - (E_Y F)(\overline{Z}, X) - (E_Z F)(X, Y) = \eta(Y)[(\nabla_X \eta)(\overline{Z}) + (\nabla_{\overline{Z}} \eta)(\overline{X})] - \eta(X)[(\nabla_Y \eta)(\overline{Z}) + (\nabla_{\overline{Z}} \eta)(\overline{Y})] + \eta(Z)[(\nabla_Y \eta)(X) - (\nabla_X \eta)(\overline{Y})] + 2\eta(Z)g(\overline{X}, Y).\]

Since \(\xi\) is killing then putting \((\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = 0; (63), (22), (24))\) in the above equation, we obtain the desired result. \(\square\)

Theorem 3.4. An almost contact metric manifold admitting a structure
connection $E$ is a generalised co-symplectic manifold if

$$
(EXF)(Y,Z) = \eta(Y)[(EX\eta)(Z)] - \eta(Z)[(EX\eta)(Y)]
+ 2\eta(X)F(Y,Z).
$$

(3.3.17)

Proof. From (3.2.8) and (3.3.6), we have

$$
(EXF)(Y,Z) = \eta(Y)[(\nabla_X\eta)(\overline{Z})] - g(X,Z)
- \eta(Z)[(\nabla_X\eta)(\overline{Y})] - g(X,Y)
+ 2\eta(X)F(Y,Z).
$$

Using (3.3.7)(c) in above equation, we at once obtain (3.3.17). □

Theorem 3.5. A generalised co-symplectic manifold equipped with structure connection $E$ is completely integrable.

Proof. In view of (3.2.2)(b) and (3.3.17), (3.3.6)(b) becomes

$$
N_E(X,Y,Z) = \eta(Y)[(EX\eta)(\overline{Z}) + (EX\eta)(\overline{Z})] + \eta(Z)[(EY\eta)(\overline{X})
-(EX\eta)(\overline{Y})] - \eta(X)[(EY\eta)(\overline{Z}) + (EY\eta)(\overline{Z})]
- 2\eta(Y)g(X,Z) + 2\eta(X)g(\overline{Y},\overline{Z}).
$$

Barring $X$, $Y$ and $Z$ in the last expression and using (3.2.2)(b), we get

$$
N_E(X,Y,Z) = 0.
$$

We know that if $N(X,Y,Z) = 0$ for any manifold than it is completely integrable [23]. Hence the theorem. □

Theorem 3.6. If $\phi$ is killing, then on generalised co-symplectic manifold with structure connection $E$, we have

$$
(EX\eta)(\overline{Z}) + 2F(X,Z) = 0.
$$

(3.3.18)
Proof. Since $\phi$ is killing, therefore

$$(E_X F)(Y, Z) + (E_Y F)(X, Z) = 0. \quad (3.3.19)$$

In consequence of (3.3.17), (3.3.19) becomes

$$
\begin{align*}
\eta(X)[(E_Y \eta)(\overline{Z}) + 2g(\overline{Y}, Z)] + \eta(Y)[(E_X \eta)(\overline{Z})] \\
+ 2g(\overline{X}, Z) - \eta(Z)[(E_X \eta)(\overline{Y}) + (E_Y \eta)(\overline{X})] = 0.
\end{align*}
$$

Putting $\xi$ for $Y$ in the above equation and then using (3.2.1)(b), (3.2.2)(a), (3.2.2)(b) and (3.2.4), we obtain

$$(E_X \eta)(\overline{Z}) + \eta(X)(E_\xi \eta)(\overline{Z}) - \eta(Z)(E_\xi \eta)(\overline{X}) + 2F(X, Z) = 0. \quad (3.3.20)$$

Again putting $\xi$ for $X$ and using (3.2.1)(b), (3.2.2)(a), (3.2.2)(b) and (3.2.4), we get

$$(E_\xi \eta)(\overline{Z}) = 0, \quad (3.3.21)$$

from (3.3.20) and (3.3.21), we get the result. \qed

\textbf{Theorem 3.7.} If the generalised co-symplectic manifold is of first class with respect to the Riemannain connection $\nabla$, then it is also first class with respect to the structure connection $E$ and satisfies the equation

$$(E_X F)(Y, Z) = \eta(Y)[(E_Z \eta)(\overline{X}) + \eta(Z)[(E_X \eta)(\overline{Y})]]. \quad (3.3.22)$$

Proof. Barring $X$ and $Y$ in (3.3.5)(e) respectively and then using (3.2.2)(b) and (3.2.3), we find

$$(E_X \eta)(Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + g(X, Y), \quad (3.3.23)$$
3.3. STRUCTURE CONNECTION IN A GENERALISED CO-SYMPECTIC MANIFOLD

and

\[(E_X\eta)(Y) = (\nabla_X\eta)(Y) + \eta(X)\eta(Y) - g(X,Y),\] (3.3.24)

adding (3.3.23) and (3.3.24), we obtain

\[(E_X\eta)(Y) + (E_X\eta)(Y) = (\nabla_X\eta)(Y) + (\nabla_X\eta)(Y).\] (3.3.25)

In view of (3.2.11) and (3.3.25), we have

\[(E_X\eta)(Y) = -(E_X\eta)(Y).\] (3.3.26)

Again in similar way, we have

\[(E_X\eta)(Y) = (E_Y\eta)(X),\] (3.3.27)

from (3.3.26) and (3.3.27), we find

\[(E_X\eta)(Y) = -(E_X\eta)(Y) = (E_Y\eta)(X),\] (3.3.28)

taking covariant derivative of \(\phi Y = Y\) with respect to \(E\) and using (3.2.1)(b), (3.2.2)(b) and (3.3.1), we get

\[(E_X\phi)(Y) = (\nabla_X\phi)(Y) - \eta(Y)X + g(X,Y)\xi.\] (3.3.29)

Replacing \(X\) by \(\xi\) in (3.3.29) and using (3.2.1)(b), (3.2.2)(b), (3.2.4) and (3.2.11)(c), we have

\[(E_\xi\phi) = 0.\] (3.3.30)
CHAPTER 3. STRUCTURE CONNECTION IN A GENERALISED...

For (3.3.22), using (3.2.8), (3.3.6) and (3.3.7) (c), we obtain

\[(E_X F)(Y, Z) = \eta(Y)[(E_Z \eta)(\bar{X}) + \eta(Z)[(E_{\bar{X}} \eta)(Y)] + 2\eta(X)g(\bar{Y}, Z).\]

Barring \(X\) in above equation and using (3.2.2) (b), we obtain the desired result. \(\square\)

3.4 Structure connection \(E\) on quasi-Sasakian manifold

**Theorem 3.8.** Let \(\nabla\) be the Riemannian connection and \(E\) be a structure connection. Then an almost contact metric manifold is a generalised quasi-Sasakian manifold of the first kind if

\[
(E_X F)(Y, Z) + (E_Y F)(Z, X) + (E_Z F)(X, Y) - 2[\eta(X)F(Y, Z) + \eta(Y)F(Z, X) + \eta(Z)F(X, Y)] = 0.
\]

**Proof.** From (3.3.6), we have

\[
(E_X F)(Y, Z) = (\nabla_X F)(Y, Z) + 2\eta(X)F(Y, Z),
\]

taking covariant derivative of \(\eta(\bar{Z}) = 0\) with respect to \(\nabla\) and using (3.3.1), we obtain

\[
(E_X \eta)(\bar{Z}) = (\nabla_X \eta)(\bar{Z}) - g(\bar{X}, \bar{Z}),
\]

using (3.3.28), (3.4.2) and (3.4.3) in (3.2.10), we get the required result. \(\square\)
Theorem 3.9. A generalised co-symplectic manifold is quasi-Sasakian manifold if
\[(E_X F)(\xi, Z) = (E_Z F)(\xi, X).\] (3.4.4)

where E being a structure connection.

Proof. From (3.3.6), we have
\[
(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) \\
= (E_X F)(Y, Z) + (E_Y F)(Z, X) + (E_Z F)(X, Y) \\
- 2\eta(X)g(\overline{Y}, Z) - 2\eta(Y)g(\overline{Z}, X) - 2\eta(Z)g(\overline{X}, Y).
\]

Using (3.3.17) in above expression, we find
\[
(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) \\
= \eta(Y)[(E_X F)(\xi, Z) - (E_Z F)(\xi, X)] \\
+ \eta(Z)[(E_Y F)(\xi, X) - (E_X F)(\xi, Y)] \\
+ \eta(X)[(E_Z F)(\xi, Y) - (E_Y F)(\xi, Z)],
\] (3.4.5)

from (3.2.9) and (3.4.5), we get at once (3.4.4). \qed

Theorem 3.10. A quasi-Sasakian manifold is normal iff
\[
(E_X F)(Y, Z) = \eta(Y)(E_Z \eta)(\overline{X}) \\
+ \eta(Z)(E_X \eta)(Y) + 2\eta(X)F(Y, Z).
\] (3.4.6)

where E being structure connection.

Proof. Using (3.2.2)(b),(3.3.6) and (3.3.7)(c) in (3.2.13), we get at once (3.4.6). \qed

Theorem 3.11. On a normal quasi-Sasakian manifold with structure con-
CHAPTER 3. STRUCTURE CONNECTION IN A GENERALISED....

connection $E$ following relations holds

\[(a) \quad N_E(\overline{X}, \overline{Y}, \overline{Z}) = 0, \quad (3.4.7)\]
\[(b) \quad N_E(X, Y, Z) + N_E(Y, X, Z) = 0, \quad (3.4.7)\]
\[(c) \quad (E_X F)(Y, Z) - (E_Y F)(Z, X) + (E_Z F)(X, Y) \]
\[\quad = N_E(X, Y, Z) - \frac{1}{2}[P(\overline{X}, \overline{Y}, Z)], \quad (3.4.8)\]
\[(d) \quad (E_\xi F)(Y, Z) = 2F(Y, Z). \quad (3.4.9)\]

Proof. In view of (3.2.2)(b) and (3.4.6), (3.3.8)(b) becomes

\[N_E(X, Y, Z) = 2\eta(Z)F(X, Y) + 2\eta(X)F(Y, \overline{Z}). \quad (3.4.8)\]
\[-2\eta(Y)F(X, \overline{Z})\]

Barring $X$, $Y$ and $Z$ in the last expression and using (3.2.2)(b), we get

\[N_E(\overline{X}, \overline{Y}, \overline{Z}) = 0.\]

It shows that a normal quasi-Sasakian manifold equipped with structure connection $E$ is completely integrable.

Using (3.2.2)(b) in (3.4.8), we get (3.4.7)(b).

In consequence of (3.2.2)(b),(3.3.5)(b),(3.3.2),(3.3.26),(3.4.6) and (3.4.8), we obtain (3.4.7)(c).

From (3.2.5),(3.3.9) and (3.3.7)(d), we have

\[(E_X F)(Y, Z) = 2\eta(X)g(\overline{Y}, Z) + g((E_X \phi)(Y), Z), \quad (3.4.9)\]

replacing $X$ by $\xi$ in (3.4.9) and using (3.2.2)(a) and (3.3.30), we get (3.4.7)(d).

\[\square\]