Chapter 2

Fuzzy Sliding Mode Control for uncertain nonlinear systems

2.1 Introduction

The inherent nonlinear nature of physical systems can have much richer and more complex behaviors. The system may have natural or artificial nonlinearities. Apart from these nonlinearities, there are other plant uncertainties such as changes in plant parameters, inaccurate identification, and uncertainties arising from different sources. Because of these characteristics the uncertain nonlinear systems are difficult to control.

SMC is one of the most effective nonlinear robust control approaches since it provides the system dynamics with an invariance property to uncertainties once the system dynamics is controlled in the sliding mode. It also possess many advantages such as insensitivity to parameter variations, external disturbance rejection, and fast dynamic response. However, there is undesirable chattering in the control effort and bounds on the uncertainties are required in the design of SMC. The uncertainties usually includes unmodelled dynamics, parameter variation and external disturbances. If the actual bounds of the uncertainties exceeds the assumed values designed in the controller the stability of the system is not guaranteed [48]. The SMC was also successfully used to stabilize the systems with nonlinear inputs [49–53] and non smooth nonlinearities as function of input structure [54–56]. In [57] a robust SMC scheme was proposed for a class of uncertain MIMO nonlinear systems with the unknown external disturbance, the system uncertainty, and the backlash-like hysteresis. Using radial basis function (RBF) NN, the system uncertainty was approximated. The SMC suffers from high frequency chattering which is observed in control input and state dynamics of the system. A number of methods were proposed to overcome the chattering problem, one of the approach was approximating the signum function by a high gain saturation function
and the other technique was based on the observer design to reduce the chattering [58].

The other approach was to design a higher order SMC, which retains the property of
robustness and eliminates chattering [59,60] and in [61], uses a PI type sliding surface,
whose gains are tuned by adaptive SMC to achieve chattering free performance.

These approaches, however, provide no guarantee of convergence to the sliding mode
and involve a trade-off between chattering and robustness. The reduced chattering can
be achieved by combining the attractive features of SMC and FLC without sacrificing
its robust performance. The integration of these methods overcomes the weakness of
one method over the other so that, the guaranteed stability and robust control per­
cformance can be achieved, such a controller is called as FSMC. The FSMC preserves
robustness characteristics of SMC, as the rule base of FLC is designed based on con­
ventional SMC.

Hence, from an implementation perspective, FSMC is used for systems or plants hav­
ing uncertainty and nonlinearities. Due to fast computing facilities, implementation of
FSMC has motivated the research in the area of nonlinear control systems. Among the
developed control strategies for robust nonlinear control, FSMC plays an important
role because it not only stabilizes nonlinear uncertain system but also provides the
capability of disturbance rejection and insensitivity to parameter variations.

In this Chapter, a FSMC design for uncertain nonlinear systems is proposed. In the
proposed method, the design of controller starts by defining a sliding surface for the
given system. The crisp sliding surface is used as input to the FLC and the rule base
is developed on the basis of SMC to get FSMC as crisp control output. The obtained
controller is applied to an uncertain nonlinear system. The stability analysis is given to
prove its convergence. Using two examples of uncertain nonlinear systems, the simu­
lation results are compared with SMC method.

2.2 Fuzzy Sliding Mode Control

In this section, basic design methods of SMC, FLC and FSMC for a given system are
discussed.

2.2.1 Introduction to sliding mode control

This section presents brief introduction of SMC theory, which will be used in subse­
quent chapters. The SMC has attracted interest of many researchers because of its
simplicity in design, invariance to system dynamics characteristics and external per­
turbation. SMC can be used in variety of operational modes, such as, regulation,
trajectory tracking control, model following and as observer. It also reduces the complexity of feedback control design by enabling the decoupling of overall system motion into independent partial components of lower dimensions. The design of SMC has two steps as shown in Figure (2.1).

1. Design of stable sliding surface.

2. Design of control law to force state dynamics of the system onto the chosen sliding surface in finite time.

The sliding surface is optimally designed to address all constraints and required specifications.

The basic notion of SMC is discussed here by considering an uncertain nonlinear system represented in controllable canonical form as shown below:

\[
\begin{align*}
\dot{x}_i &= x_{i+1} \quad 1 \leq i \leq n-1 \\
\dot{x}_n &= f(x) + B(x) u(t) + d(t) \\
y &= x_1.
\end{align*}
\]

where, \( x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n \) is the state vector, \( f(x) \) and \( B(x) \) \( \in \mathbb{R}^n \rightarrow \mathbb{R} \) are nonlinear functions not known exactly, but can be written as \( f(x) = \hat{f}(x) + \Delta f(x) \) and \( B(x) = \hat{B}(x) + \Delta B(x) \), where, \( \hat{f}(x) \) and \( \hat{B}(x) \) are nominal functions, and \( \Delta f(x) \), \( \Delta B(x) \) are uncertain parts, \( u(t) \) and \( d(t) \) are the control input and external disturbance respectively.

To proceed for the design of SMC for the uncertain nonlinear system (2.1) the following assumption is required.

**Assumption 2.2.1** The uncertain functions are bounded by,

\[
\begin{align*}
|\Delta f(x)| &\leq F, \\
|\Delta B(x)| &\leq B
\end{align*}
\]
and

\[ d(t) < D_{\text{max}} \in \mathbb{R}. \]

The design of SMC begins by defining a sliding surface for a given system as,

\[ s = C_1 x_1 + C_2 x_2 + \cdots + C_{n-1} x_{n-1} + C_n x_n = 0 \quad (2.2) \]

The sliding surface coefficients \( C_i, \quad i = 1, 2, \ldots, n \) are chosen such that the surface is strictly Hurwitz [62].

After a proper sliding surface is obtained, the next step is to design a control \( u \) such that any state \( x \) outside the sliding surface \( s = 0 \) is driven to reach the surface or manifold in finite time. Taking time derivative of (2.2), with \( C_n = 1 \) to get,

\[ \dot{s} = C_1 x_2 + C_2 x_3 + \cdots + C_{n-1} x_n + f(x)(t) + B(x)u(t) + d(t) = 0 \]

Once the system is in sliding mode, it will satisfy the following condition,

\[ s = 0 \quad \text{and} \quad \dot{s} = 0 \quad (2.3) \]

The reaching condition is defined as,

\[ s \dot{s} < 0. \quad (2.4) \]

Solving for \( u_{eq} \), using equivalent control method by considering disturbances and uncertainties as zero, to get,

\[ u_{eq} = -\frac{1}{B(x)} \left[ \sum_{i=1}^{n-1} C_i x_{i+1} + f(x) \right] \quad (2.5) \]

where \( B(x) \) must be nonzero function.

The equivalent control \( u_{eq} \) is designed to satisfy the inequality (2.4), such that, the state trajectories are driven and attracted towards sliding surface and made sliding on it.

To satisfy the reaching condition (2.4), \( \dot{s} \) is chosen as,

\[ \dot{s} = -K \, \text{sign}(s) \quad \forall \ t \quad K > 0 \quad (2.6) \]

The reaching time can be obtained by integrating (2.6) with respect to time, to get,

\[ t_r = \frac{|s(0)|}{K}. \]

\( t_r \) is the time required for the state vector to reach \( s = 0 \). The sufficient condition for this behavior is to choose a Lyapunov function \( L(x) \) for system and force the control to meet the Lyapunov stability criterion as shown,

\[ L(x) = \frac{1}{2} s^2 \quad (2.7) \]
the derivative of (2.7) must be negative definite to ensure the stability. The basic SMC law has the form,

\[ u = u_{eq} + u_n \]  

(2.8)

where, \( u_n \) is discontinuous control law it is also called as reaching law is expressed as,

\[ u_n = -K \text{ sign}(s) \]  

(2.9)

where,

\[ K > \frac{F + D_{\text{max}}}{B}. \]

The \( K \) will ensure stable operation of the system [63]. The essential draw back of conventional SMC is high frequency chattering in control input \( u \) due to the signum function \( K \text{ sign}(s) \) in (2.9). This can be avoided by selecting discontinuous control law as

\[ u_n = -K \text{ sat}\left(\frac{s}{\Phi}\right), \]

where \( \text{sat}(\cdot) \) is saturation function, will satisfy the reaching condition (2.5) and \( \Phi \) is boundary layer introduced around sliding surface \( s = 0 \).

The modified SMC law becomes,

\[ u = u_{eq} - K \text{ sat}\left(\frac{s}{\Phi}\right) \]  

(2.10)

Theorem 2.2.1 For the uncertain nonlinear system (2.1) satisfying the assumption 2.2.1 if the SMC law is chosen as (2.8). Then, the system trajectories globally exponentially converge to the sliding surface (2.2).

Proof 2.2.1 Consider a Lyapunov function candidate as (2.7) and taking time derivative of it we get,

\[ \dot{L}(x) = s \dot{s} = s \left[ C_1 x_2 + \cdots + C_n \dot{x}_n \right] = s \left[ C_1 x_2 + \cdots + C_n[f(x)(t) + B(x)u(t) + d(t)] \right] \]  

(2.11)

\[ \dot{L}(x) \leq |sC_nB(x)| \left[ (C_nB(x))^{-1}C_nf(x) \right] + sC_nB(x)(u_{eq} + u_n) + |sC_nB(x)| \]  

(2.12)

\[ \leq |sC_nB(x)| \left[ (C_nB(x))^{-1}C_nf(x) \right] + s(u_{eq} + u_n) + |s| \]

\[ \leq |s| \left[ (C_nB(x))^{-1}C_nf(x) \right] - K |s| \]

\[ \dot{L}(x) \leq |s| \left[ (C_nB(x))^{-1}C_nf(x) \right] - K |s| < 0. \]

To satisfy above inequality \( K \) should be chosen as \( K > |(C_nB(x))^{-1}C_nf(x)|, \dot{L}(x) < 0 \).

From Theorem (2.2.1), stability of the system for the controller (2.8) is proved. The system is asymptotically stable once its dynamics are on sliding surface (2.2).
The modified SMC (2.10) shows robust performance towards uncertainties and external disturbances. The chattering can be made negligible by wider boundary layer, which deteriorates precise control due to limited control bandwidth. To overcome such limitations and to improve the performance of the system, SMC is combined with FLC to develop an alternative called as FSMC.

2.2.2 Fuzzy logic control

There are many events occur around us can be classified as certain and uncertain. The class of uncertain phenomenon can be further classified as random and fuzzy. In random models, because of uncertainty, the relation between system dynamics are hard to put in equation form. Using fuzzy models relation between system dynamics can be expressed using words or linguistic terms. The fuzzy logic was introduced by Lotfi A. Zadeh [64] in 1965, since then many researchers have developed and applied fuzzy logic for various systems [65]. FLC is one of the computational intelligent control approach has been a subject of intense research in the past two to three decades. The most attractive feature of FLC is, its ability to convert a linguistic qualitative control strategy into quantitative control actions. Because of this, the FLC was used to control ill-modeled and complex nonlinear systems. The basic structure of a FLC is shown in Figure (2.2), which comprises several principle components: a normalization factor set of input variables, the fuzzification unit, a knowledge base unit, the decision making logic, the defuzzification unit and a normalization factor of output variables. In this section, single input single output FLC is considered.

The input is multiplied by its normalization factor $K_{in}$ to get normalized variable, which is input to the fuzzification unit and output is multiplied with $K_f$ to get normalized output value. The details of each unit is addressed as follows:

![Figure 2.2: General structure of fuzzy logic controller.](image-url)
• Fuzzification unit

1. Acquires the values of the input variable.
2. Converts input data into fuzzy sets labeled with linguistic value.

The fuzzification unit converts crisp input information into linguistic variables like high, low, medium etc., that the decision making logic can easily activate and apply for knowledge base unit.

• Knowledge base unit. The knowledge base unit contains human experts description of control in linguistic form represented by a set of If-Then rules. It has two sub units as data base and rule base.

1. The data base has the information of linguistic variables used to define the linguistic control rules and data manipulation in FLC.
2. The rule base characterizes the experts control knowledge by means of a set of linguistic control rules.

• Decision making logic.

The decision making logic also called as inference engine emulates ability of experts decision making and knowledge application for best control of system. It is the kernel of the FLC. The fuzzy control actions are infer by employing the fuzzy implications and the rules of inference. The decision making logic determines the output value from available input data using the knowledge base.

• Defuzzification unit.

The defuzzification unit converts the decision of inference engine into crisp value for system control.

1. Defuzzification unit does the scaling and mapping by converting the range of output variables into corresponding universe of discourse.
2. The defuzzification unit gives out the non-fuzzy control output from inferred fuzzy control action.

The two most famous types of fuzzy controllers are Mamdani and Takagi-Sugeno type controller. Both the controllers have same control structure but they differ in way the linguistic variable rules are represented.
2.2.3 Fuzzy sliding mode control for uncertain nonlinear systems

In this section, we will discuss the general design procedure of FSMC. As we know to minimize the chattering problem FLC is designed based on SMC. This extension will guarantee asymptotic stability of FLC \[66\]. The FSMC has a Single-input-Single-output (SISO) system with input as switching surface \( s \) and output after defuzzification is a crisp control value \( u_{fn} \).

In FSMC design, the FLC is designed by defining its input as crisp sliding surface \( s \), which is fuzzified on the universe of discourse of \( s \). The fuzzy sets of a sliding surface are defined by triangular form and trapezoidal form of membership functions. The triangular form membership function is defined as,

\[
\mu_F(s) = \begin{cases} 
0, & \text{for } s < a \\
\frac{(s-a)}{(b-a)}, & \text{for } a \leq s < b \\
\frac{(c-s)}{(c-b)}, & \text{for } b \leq s \leq c \\
0, & \text{for } s > c
\end{cases}
\]

where, \( a \), \( b \) are end points and \( c \) is high point on universe of discourse of a fuzzy set.

Similarly the trapezoidal form of membership function is defined as,

\[
\mu_T(s) = \begin{cases} 
0, & \text{for } s < a \\
\frac{(s-a)}{(b-a)}, & \text{for } a \leq s < b \\
1, & \text{for } b \leq s < c \\
\frac{(d-s)}{(d-c)}, & \text{for } c \leq s \leq d \\
0, & \text{for } s > d
\end{cases}
\]

where, \( d \) is high point on universe of discourse of a fuzzy set.

The linguistic variables for the elements of input membership functions \( s \) are \{ PB, PM, ZE, NM, and NB \}. They are positive big, positive medium, zero, negative medium and negative big respectively.

The output membership function is a singleton function, whose fuzzy sets are defined on the normalized universe of discourse \( u_i = u_{eq} \pm K \). The singleton membership function is given as,

\[
\mu_H(u) = \begin{cases} 
1, & \text{for } u = u_i \\
0, & \text{otherwise}
\end{cases}
\]

where, \( u_i \) is an element of an universe of discourse.
The linguistic variables for the elements of output membership functions are, \{ Bigger, Big, Medium, Small, and Smaller \}.

The rule base is derived based on SMC with boundary layer. Each rule will connect the input sliding surface to the output control value depending upon the distance of the state to the sliding surface \( s = 0 \).

The general fuzzy rule for the FSMC is defined as

\[ R_i : \text{if } s_i \text{ is } F^n_{s_i} \text{ then } u_i \text{ is } F^n_{u_i}. \]

where, \( F^n_{s_i} \) and \( F^n_{u_i} \) are input and output fuzzy sets. The fuzzy rules partition the phase plane into two semi-planes separated by a sliding surface.

The compositional rule of inference, which determine the influence of antecedent part of the fuzzy rule on consequent part of the fuzzy rule is defined as,

\[ H_{s_i}(u_i) = \sup_{x \in s_i} \left[ \min \left[ \mu_{F^n_{s_i}}(s_i), \mu_{F^n_{u_i}}(u_i) \right] \right]. \]

where, \( \sup_{x \in s_i} \) is for maximum operation. The crisp output value is extracted from fuzzy output by use of center of area defuzzification principle given as,

\[ u_{fn}(s_i, u_i) = \frac{\sum_i u_{fn} \mu(s_i, u_i, u_{fn})}{\sum_i \mu(s_i, u_i, u_{fn})} \]

(2.14)

to get output of FSMC. The FSMC provides a mapping from the crisp function \( s \) to a control output \( u_{fn} \). The fuzzy values of the FSMC are opposite in sign on the separated planes similar to the modified SMC. The magnitude of the fuzzy control signal of a specific rule is proportional to the states away from the sliding surface \( s = 0 \). Thus, the control action always tries to keep the state vector on the sliding surface by decreasing the distance to the sliding surface. The analytical form [67] of FSMC output is written as

\[ u_{fn} = -K_f(|s|)\text{sign}(s) \]

The total FSMC law has the form,

\[ u = u_{eq} - K \text{sat}(s/\Phi) \]

(2.15)

where, \( K = K_f(|s|) \). The (2.15) has similar structure that of modified SMC obtained by introducing a boundary layer across the surface. In FSMC, the boundary layer across \( s = 0 \) is divided into sublayers as shown in Figure (2.3). When the state trajectory reaches the outer boundary layer and moves towards origin, at each layer different SMC control signal is generated. Thus, the control (2.15) can be considered as modified SMC with nonlinear gain at each layer.

The stability of system (2.1) for the control (2.15) can be proved as in Theorem 2.2.1.
2.2.4 Design steps for fuzzy sliding mode control

1. Design the sliding surface \( s \) for the given system.

2. Derive equivalent control \( u_{eq} \) using (2.3).

3. Design input and output fuzzy membership functions on the universe of discourse of \( s \) and \( u \) as shown in Figure (2.4).

(a) The input fuzzy sets are obtained by fuzzyfying the universe of \( s \) as

\[
\left[ -\frac{\Phi}{2}, -\frac{\Phi}{4}, 0, +\frac{\Phi}{4}, +\frac{\Phi}{2} \right]
\]
(b) The output fuzzy sets are obtained by fuzzyfying the universe of \( u_{eq} \pm K \) as
\[
\left[ u_{eq} - \frac{K}{2}, \; u_{eq} - \frac{K}{4}, \; u_{eq}, \; u_{eq} + \frac{K}{2}, \; u_{eq} + \frac{K}{4} \right].
\]

4. Define the inference engine as (2.13).

5. Define the rule base as,
   
   (a) if \( s \) is NB then \( u_{fn} \) is Bigger.
   
   (b) if \( s \) is NM then \( u_{fn} \) is Big.
   
   (c) if \( s \) is ZE then \( u_{fn} \) is Medium.
   
   (d) if \( s \) is PM then \( u_{fn} \) is Small.
   
   (e) if \( s \) is PB then \( u_{fn} \) is Smaller.

6. Obtain \( u_{fn} \) using (2.14) and total control law using (2.15)

2.3 Simulation studies

In this section, two simulation examples are considered to validate the proposed FSMC. The performance of proposed controller is compared with SMC to show the effectiveness of proposed controller. The simulation is carried out using MatLab.

2.3.1 Example 1

Consider a uncertain nonlinear system [68] described by the state equations,
\[
\dot{x}(t) = f(x) + B(x) u + d(t)
\]  
(2.16)

\[
f(x) = \begin{bmatrix}
0 \\
1 \\
0.5x_2 + 0.2 \cos(x_1)\sqrt{x_1^2 + x_2^2} - 0.5 \sin(x_2)
\end{bmatrix}, \quad B(x) = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad \text{and} \; d(t) = 0.3.
\]

with initial conditions, \( x_1(0) = 1 \) and \( x_2(0) = 3 \).

The sliding surface is selected as,
\[
s = 2x_1 + x_2.
\]

The equivalent control law is obtained using (2.5) as,
\[
u_{eq} = -\left[x_1 + 0.5x_2 + 0.2 \cos(x_1)\sqrt{x_1^2 + x_2^2} - 0.5 \sin(x_2)\right]
\]
The $u_{fn}$ term is obtained by following FSMC design procedure discussed in section 2.2.4.

The dynamic behavior of state variable $x_1(t)$, control input $u$ and sliding surface obtained using FSMC are shown in Figures (2.5-2.7).

From Figures (2.5-2.7) it can be seen that, the proposed control method was successful in stabilizing the system with smaller overshoot as compare to modified SMC and minimizes the chattering and show faster reaching time with similar control effort.

\begin{equation}
\dot{x}_1 = x_2 \\
\dot{x}_2 = f(x) + B(x) u
\end{equation} (2.17)
The sliding surface for the system is defined as,

\[ s = c_1 x_1 + x_2 \]  

(2.18)

Equivalent control is,

\[ u_{eq} = -c_1 \dot{x}_2 - f(x) \]

\[ B(x) \]

The term is obtained by following the FSMC design procedure discussed in section 2.24.

The simulation is carried out for \( c_1 = 2 \), \( \Phi = 3 \) and \( K = 5 \), the initial values as \( x_1(0) = -0.0542 \) and \( x_2(0) = 0 \).

The dynamic behavior of states variable, control input and sliding surface are obtained using conventional SMC SMC with boundary layer also called as modified SMC and FSMC are shown in Figures (2.8-2.10). From Figures (2.8-2.10) it can be seen that, the proposed control method was successful in minimizing the chattering in control input and system dynamics as similar to modified SMC.

Results

The performances of SMC and FSMC are tested on two examples by MatLab simulation. Figures (2.5-2.7) and (2.8-2.10) shows the comparative results of SMC and FSMC.
for both systems. The modified SMC and FSMC methods successfully minimize the chattering present in conventional SMC and shows smoother responses.

### 2.4 Conclusion

In this Chapter, the design procedure of SMC and FSMC for uncertain nonlinear systems are explained. The control structure of SMC has two terms as continuous and discontinuous control laws. The chattering due to discontinuous control law was
minimized by designing a SMC with boundary layer around sliding surface. The FSMC was designed to minimize the chattering in control input and system dynamics due to discontinuous control law of conventional SMC. The FSMC was designed by considering sliding surface as input and rule base is designed based on SMC control law. The performance of proposed method was tested on two nonlinear systems. The asymptotic stability of nonlinear control system was achieved using these controllers. The FSMC was successful in reducing the chattering as similar to SMC with boundary layer. It is concluded that, the FSMC developed in this Chapter can be applied very well to nonlinear systems.