Chapter 3

REPRESENTATION THEORY $SL(2,\mathbb{C})$ AND
GENERATING RELATIONS FOR THE POLYNOMIAL
SET $R_n(\beta;\gamma;x,y)$
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3.1 INTRODUCTION

Hypergeometric polynomials in at least one variables emerge as often as possible in a wide variety of problems in applied mathematics, theoretical physics, engineering sciences, operations research and statistics. It is then obvious that a detailed study of the analytical behaviour of such polynomials will be of great importance. Sharma [93] has obtained certain generating functions for modified Laguerre polynomials by Lie group-theoretic methods. Group-theoretic discussions of generating functions had been done independently by Chatterjee and Chakrabarty [21], Chongdar [22], Das [24], Ghosh Bandana [44], Shahwan [91], Shukla [96], Subuhi Khan [107] and many other researchers. The details of these methods are based on the approaches given by Miller [69], Srivastava and Manocha [103], Weisner [118], [120] etc.

To discuss the representation theory $SL(2, \mathbb{C})$ in detail, this Chapter has been divided into two sections.

SECTION-A

In this section, our aim is to discuss the representation $SL(2, \mathbb{C})$ {a complex local Lie group} and use it to derive the generating relations for the polynomial set $R_n(\beta; \gamma; x, y)$ with respect to a suitable basis. The suitable $J$-operators are considered which generate a 3-dimensional Lie algebra of generalized Lie derivatives of a
multiplier representation of \( \text{SL}(2, \mathbb{C}) \) of \( \mathfrak{sl}(2, \mathbb{C}) \). This gives many known and unknown results as its applications.

### 3.2 DEFINITION OF THE REPRESENTATION THEORY OF \( \text{SL}(2, \mathbb{C}) \)

Let \( \mathfrak{sl}(2, \mathbb{C}) \) be the Lie algebra of a three dimensional complex local Lie group \( \text{SL}(2, \mathbb{C}) \), a multiplicative \( 2 \times 2 \) matrix group with elements.

\[
\text{SL}(2, \mathbb{C}) \equiv \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \mathbb{C} \right\}
\]

such that determinant of the matrix \( g = 1 \).

A basis for \( \mathfrak{sl}(2, \mathbb{C}) \) is provided by the matrices

\[
j^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad j^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad j^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}
\]

with commutation relations

\[
[j^3, j^+] = j^+, \quad [j^3, j^-] = -j^-, \quad [j^+, j^-] = 2j^3.
\]

We need the following observations ([103], P. 323):

**Observation I.** Let \( L \left( x, y, \frac{d}{dy}, n \right) \) be a linear differential operator containing parameter \( n \). Assuming that \( L \) is a polynomial in \( n \), we construct a partial differential operator

\[
L \left( x, y, \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \right) \text{by substituting } \frac{\partial}{\partial y} \text{ for } \frac{d}{dy} \text{ and } z \frac{\partial}{\partial z} \text{ for } n. \text{ The function }
\]

\[
z = z^n v_n(x, y) \text{ is a solution of } L \left( x, y, \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \right) z = 0, \text{ if and only if }
\]

\[
u_n(x, y) \text{ is a solution of } L \left( x, y, \frac{d}{dy}, n \right) u = 0.
\]
Observation II. Let $G(y, z)$ have a convergent expansion of the form

$$\sum_{n} g_n(y) z^n,$$

where $n$ is not necessarily a non-negative integer.

If $L\left(x, y, \frac{\partial}{\partial y}, z \frac{\partial}{\partial z}\right) G(y, z) = 0$ then within the region of convergence of the series (3.2.5), $u = g_n(y)$ is a solution of (3.2.4). In particular, if $G(y, z)$ is regular at $y = 0$ then $u = g_n(y)$ is also regular at $y = 0$.

3.3 REPRESENTATION OF $SL(2, \mathbb{C})$ AND GENERATING RELATIONS

The operator functional notation form of the differential equation (2.7.5) is

$$L\left(x, y, \frac{d}{dy}, n\right)$$

$$= y^2(x - y)^2 D^2 - y(x - y) \left\{(3n + \gamma - \beta - 1)x - (2n + \gamma - 2)y\right\} D$$

$$+ n\left[(2n + \gamma - \beta)x^2 - (3n + 2\gamma - 1)xy + (n + \gamma - 1)y^2\right] = 0,$$

where $D = \frac{d}{dy}$.

Now, replacing $\frac{d}{dy}$ by $\frac{\partial}{\partial y}$, $n$ by $z \frac{\partial}{\partial z}$ and $R_n(\beta; \gamma; x, y)$ by $f(x, y, z)$ in (3.3.1), we get the partial differential equation

$$\left\{ y^2(x - y)^2 \frac{\partial^2}{\partial y^2} + y(x - y)(2y - 3x)z \frac{\partial^2}{\partial z \partial y} ight.$$

$$- y(x - y)[(\gamma - \beta - 1)x - (\gamma - 2)y] \frac{\partial}{\partial y} + (2x - y)(x - y)z^2 \frac{\partial^2}{\partial z^2}.$$
\begin{align*}
+[(\gamma - \beta + 2)x^2 - 2(\gamma + 1)xy + \gamma y^2]z \frac{\partial}{\partial z} \right] f(x, y, z) = 0.
\end{align*}

Thus

\begin{align*}
(3.3.3) \quad L = L \left(x, y, \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \right) = y^2(x - y)^2 \frac{\partial^2}{\partial y^2} + y(x - y)(2y - 3x)z \frac{\partial^2}{\partial z \partial y} \\
- y(x - y)[(\gamma - \beta - 1)x - (\gamma - 2)y] \frac{\partial}{\partial y} + (2x - y)(x - y)z^2 \frac{\partial^2}{\partial z^2} \\
+ [(\gamma - \beta + 2)x^2 - 2(\gamma + 1)xy + \gamma y^2]z \frac{\partial}{\partial z}.
\end{align*}

In view of the Observation I, we conclude that

\begin{align*}
f(x, y, z) = z^n R_n(\beta; \gamma; x, y)
\end{align*}

is a solution of (3.3.2).

Let us introduce the first order linearly independent differential operators \( J^3, J^- \) and \( J^+ \) each of the form

\begin{align*}
(3.3.4) \quad A_1(x, y, z) \frac{\partial}{\partial y} + A_2(x, y, z) \frac{\partial}{\partial z} + A_3(x, y, z)
\end{align*}

such that

\begin{align*}
J^3[z^n R_n(\beta; \gamma; x, y)] = a_n z^n R_n(\beta; \gamma; x, y)
\end{align*}

\begin{align*}
(3.3.5) \quad J^-[z^n R_n(\beta; \gamma; x, y)] = b_n z^{n-1} R_{n-1}(\beta; \gamma; x, y)
\end{align*}

\begin{align*}
J^+[z^n R_n(\beta; \gamma; x, y)] = c_n z^{n+1} R_{n+1}(\beta; \gamma; x, y),
\end{align*}

where \( a_n, b_n, c_n \) are functions in \( n \) and free of \( x, y, z \) but not of \( \beta \) and \( \gamma \). On the other hand, each \( A_1, A_2, A_3 \) are expressions in \( x \) and \( y \) which is free of \( n \) but not of \( \beta \) and \( \gamma \).

Now, from (3.3.4), we can write

\begin{align*}
J^- = A_1(x, y, z) \frac{\partial}{\partial y} + A_2(x, y, z) \frac{\partial}{\partial z} + A_3(x, y, z).
\end{align*}
Thus we have

\[ J^{-} [z^{n} R_{n} (\beta; \gamma; x, y)] = \left[ A_{1}(x, y, z) \partial \frac{\partial}{\partial y} + A_{2}(x, y, z) \partial \frac{\partial}{\partial z} + A_{3}(x, y, z) \right] [z^{n} R_{n} (\beta; \gamma; x, y)] \]

\[ = A_{1}(x, y, z) z^{n} \left[ \frac{1}{y(x-y)} [nx R_{n} (\beta; \gamma; x, y) - ny^{2} R_{n-1} (\beta; \gamma; x, y)] \right] \]

\[ + A_{2}(x, y, z) n z^{n-1} \partial \frac{\partial}{\partial z} [R_{n} (\beta; \gamma; x, y)] \]

\[ + A_{3}(x, y, z) z^{n} R_{n} (\beta; \gamma; x, y). \] [using (2.5.2)]

In order to make the coefficient of \( z^{n-1} R_{n} (\beta; \gamma; x, y) \) independent of \( x, y \) and \( z \), let us substitute \( A_{1} = y^{-1} (x-y) z^{-1} \) so that

\[ J^{-} [z^{n} R_{n} (\beta; \gamma; x, y)] = -nz^{n-1} R_{n-1} (\beta; \gamma; x, y) + [nx y^{-1} + A_{2} n z^{-1} + A_{3}] z^{n} R_{n} (\beta; \gamma; x, y). \]

Further for making the coefficient of \( z^{n-1} R_{n} (\beta; \gamma; x, y) \) equal to zero, let us suppose that

\[ A_{2} = -xy^{-2} \] and \( A_{3} = 0 \) so that \( J^{-} = y^{-1} (x-y) z^{-1} \partial \frac{\partial}{\partial y} - xy^{-2} \partial \frac{\partial}{\partial z} \).

Finally, we get

\[ J^{-} [z^{n} R_{n} (\beta; \gamma; x, y)] = -nz^{n-1} R_{n-1} (\beta; \gamma; x, y). \]

Similarly, from (3.3.4), we can write

\[ J^{+} = A_{1}(x, y, z) \partial \frac{\partial}{\partial y} + A_{2}(x, y, z) \partial \frac{\partial}{\partial z} + A_{3}(x, y, z). \]

We have

\[ J^{+} [z^{n} R_{n} (\beta; \gamma; x, y)] = \left[ A_{1}(x, y, z) \partial \frac{\partial}{\partial y} + A_{2}(x, y, z) \partial \frac{\partial}{\partial z} + A_{3}(x, y, z) \right] [z^{n} R_{n} (\beta; \gamma; x, y)] \]

\[ = - (\gamma + n) y^{-2} z^{n-1} A_{1} z^{n-1} R_{n+1} (\beta; \gamma; x, y) \]

\[ + y^{-1} (x - y)^{-1} [(2n + \gamma - \beta) x - (n + 2n) y] R_{n} (\beta; \gamma; x, y) A_{2} z^{n} \]

\[ + A_{2} n z^{n-1} R_{n} (\beta; \gamma; x, y) + A_{3} z^{n} R_{n} (\beta; \gamma; x, y). \] [using (2.5.8)]
Now, for making the coefficient of $z^{n+1}R_{n+1}(\beta;\gamma;x,y)$ independent of $x$, $y$, and $z$ the coefficient of $z^{n+1}R_{n+1}(\beta;\gamma;x,y)$ equal to zero, we suppose that

$$A_1 = -y^2z, \quad A_2 = 2yz^2, \quad A_3 = yz(x-y)^{-1}\{(\gamma - \beta)x - yy\}.$$ 

Hence, we have

$$J^+ = -y^2z \frac{\partial}{\partial y} + 2yz^2 \frac{\partial}{\partial z} + yz(x-y)^{-1}\{(\gamma - \beta)x - bx\}$$

and

$$J^+[z^nR_n(\beta;\gamma;x,y)] = (\gamma + n)z^{n+1}R_{n+1}(\beta;\gamma;x,y).$$

Therefore,

$$[J^+, J^-] = J^+J^-u - J^-J^+u = \gamma u + 2z \frac{\partial u}{\partial z} = 2\left[\frac{\gamma}{2} + z \frac{\partial}{\partial z}\right]u = 2J^3u,$$

where

$$J^3 = \frac{\gamma}{2} + z \frac{\partial}{\partial z}.$$ 

Thus, we have the following operators:

$$J^3 = \frac{\gamma}{2} + z \frac{\partial}{\partial z}$$

(3.3.6)

$$J^- = y^{-1}(x-y)z^{-1} \frac{\partial}{\partial y} - xy^{-2} \frac{\partial}{\partial z}$$

$$J^+ = -y^2z \frac{\partial}{\partial y} + 2yz^2 \frac{\partial}{\partial z} + yz(x-y)^{-1}\{(\gamma - \beta)x - bx\}.$$ 

Clearly, these operators $J^3, J^+$ and $J^-$ obey the commutation relations:

(3.3.7) $[J^3, J^+] = J^+, \quad [J^3, J^-] = -J^-, \quad [J^+, J^-] = 2J^3.$

According to Theorem 2.3 [69], it concludes that the $J$-operators (3.3.6) generate a 3-dimensional Lie algebra isomorphic to $sl(2, \mathbb{C})$.

Now, to determine the multiplier representation of $SL(2, \mathbb{C})$, let us initially find the actions of $\exp(a'J^+), \exp(b'J^-)$ and $\exp(c'J^3)$ on $f \in \mathfrak{F}$ where $\mathfrak{F}$ is the complex
vector space of all expressions of $x$, $y$ and $z$ analytic in some neighbourhood of the point $(x_0, y_0) = (0, 0)$.

To obtain $\exp(a' J^+)f$, $\exp(b' J^-)f$ and $\exp(c' J^3)f$, let us integrate the equations [103]

\[
\begin{align*}
\frac{d}{da'} x(a') &= 0, \\
\frac{d}{da'} y(a') &= 0, \\
\frac{d}{da'} z(a') &= 2y(a')z^2(a'),
\end{align*}
\]

\[
\frac{d}{da'} v(a') = v(a')y(a')z(a')[x(a') - y(a')]^{-1} \{y[a(x(a') - y(a')] - \beta x(a')},
\]

(3.3.8) \[
\begin{align*}
\frac{d}{db'} x(b') &= 0, \\
\frac{d}{db'} y(b') &= [y(b')][z(b')][x(b') - y(b')], \\
\frac{d}{db'} z(b') &= -x(b')y^2(b'), \\
\frac{d}{dc'} v(b') &= 0, \\
\frac{d}{dc'} x(c') &= 0,
\end{align*}
\]

\[
\frac{d}{dc'} y(c') = 0, \\
\frac{d}{dc'} z(c') = y(c'), \\
\frac{d}{dc'} v(c') = \frac{\gamma}{2}v(c')
\]

with initial conditions $x(0) = x_0, y(0) = y_0, z(0) = z_0$ and $v(0) = 1$, where $v$ is multiplier of the representation.

Thus, if $f \in \mathfrak{F}$ is analytic in the neighbourhood of $(x_0, y_0, z_0)$, then the values of the multiplier representations of $\exp(a' J^+)f$, $\exp(b' J^-)f$ and $\exp(c' J^3)f$ are given by

\[
[T \exp(a' J^+)f](x_0, y_0, z_0) = [\exp(a' J^+)f](x_0, y_0, z_0)
\]

\[
= (x_0 - y_0)\beta (1 - y_0z_0a')^\beta (x_0 - y_0 + y_0^2z_0a')^{-\beta}
\]

\[
f \left( x_0, y_0, 1 - y_0z_0a', z_0 (1 - y_0z_0a')^{-2} \right),
\]

(3.3.9) \[
\left| y_0 \right| < 1, \left| z_0 (1 - y_0z_0a') \right| < 1.
\]

\[
[T \exp(b' J^-)f](x_0, y_0, z_0) = [\exp(b' J^-)f](x_0, y_0, z_0)
\]
\[ f \left( x_0, \frac{b'(x_0 - y_0) + y_0^2 z_0}{y_0 z_0}, \frac{y_0 z_0 (-b' + y_0 z_0)}{b' \{y_0^2 z_0 + x_0 - y_0\}} \right), \quad \left| y_0 \right| < 1. \]

\[ [T \exp(c' J^3) f](x_0, y_0, z_0) = [\exp(c' J^3) f](x_0, y_0, z_0) \]
\[ = \exp \left( \frac{\gamma c'}{2} \right) f \left( x_0, y_0, z_0 e^{-c'} \right). \]

Now, in the neighbourhood of the identity, every \( g \in \text{SL}(2, \mathbb{C}) \) can be expressed as
\[ g = \exp(a' J^+) \exp(b' J^-) \exp(c' J^3). \]

From which the operator \( T(g) \) acting on \( f \in \mathfrak{g} \) is given by

\[ [T(g)f](x_0, y_0, z_0) = [T(\exp(a' J^+) \exp(b' J^-) \exp(c' J^3)) f](x_0, y_0, z_0) \]
\[ = [T(\exp(a' J^+)) T(\exp(b' J^-)) T(\exp(c' J^3)) f](x_0, y_0, z_0) \]
\[ = \frac{(x_0 - y_0)^\beta (x_0 - y_0 + y_0^2 z_0 a')^{-\beta}}{(1 - y_0 z_0 a')^{\gamma - \beta}} \exp \left( \frac{\gamma c'}{2} \right) f \left( x_0, \xi_1, \eta_1 e^{-c'} \right), \]

where
\[ \xi_1 = \frac{(1 - y_0 z_0 a') \{ y_0^2 z_0 - b' y_0 (x_0 - y_0 z_0 a') + b' x_0 \}}{y_0 z_0}, \]
\[ \eta_1 = \frac{\{ y_0 z_0 \{ y_0 z_0 (1 - y_0 z_0 a') b' \} \}}{(1 - y_0 z_0 a')^{\gamma} \{ y_0^2 z_0 + x_0 - y_0 + y_0^2 z_0 a' \} b'}. \]

Further by setting \( a' = (-b) / d, \ b' = -cd, \ e^{c/2} = \frac{1}{d} \) and \( ad = 1 + bc \), we have

\[ [T(g)f](x_0, y_0, z_0) \]
\[ = (x_0 - y_0)^\beta \left[ \frac{dx_0}{d + by_0 z_0} - y_0 \right]^{-\beta} f \left( x_0, \xi_2, \eta_2 \right), \]

where
\[ \xi_2 = \frac{(d + y_0 z_0 b) \{ (y_0 - x_0) c + y_0^2 z_0 a \}}{y_0 z_0}, \]
\[ \eta_2 = \frac{\{ (d + y_0 z_0 b) \{ (y_0 - x_0) c + y_0^2 z_0 a \} \}}{(d + y_0 z_0 b)^{\gamma} \{ (y_0 - x_0) c + y_0^2 z_0 a \} b'}. \]
\[ \eta_2 = \frac{y_0 z_0 \{ y_0 z_0 a + c \}}{(d + y_0 z_0 b)^2 \{ (y_0 - x_0) c + y_0 z_0 a \}} \left| \frac{by_0 z_0}{d} \right| < 1, \quad \left| \frac{c}{ay_0 z_0} \right| < 1. \]

Here \( g \) lies in a sufficiently small neighbourhood of the identity element \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in SL_2. \)

Equation (3.3.11) defines a local multiplier representation of \( SL(2, \mathbb{C}). \) The Casimir operator ((69), P. 32) is given by

(3.3.12) \[ C = C_{i,0} = J^+ J^- + J^3 J^3 - J^3 = \frac{\gamma}{4} (\gamma - 2) \quad \text{[using (3.3.2)]} \]

Now \[ [C, J^-]u = CJ^- u - J^- Cu \]

so that

\[ CJ^- u = \left[ \frac{\gamma (\gamma - 2)}{4} \right] \left[ y^{-1} z^{-1} (x - y) \frac{\partial u}{\partial y} - xy^{-2} \frac{\partial u}{\partial z} \right] \]

\[ = \frac{\gamma}{4} (\gamma - 2) y^{-1} z^{-1} (x - y) \frac{\partial u}{\partial y} - \frac{\gamma}{4} (\gamma - 2) xy^{-2} \frac{\partial u}{\partial z} \]

and

\[ J^- Cu = \left[ y^{-1} z^{-1} (x - y) \frac{\partial u}{\partial y} - xy^{-2} \frac{\partial u}{\partial z} \right] \left[ \frac{\gamma (\gamma - 2)}{4} \right] \]

\[ = \frac{\gamma}{4} (\gamma - 2) y^{-1} z^{-1} (x - y) \frac{\partial u}{\partial y} - \frac{\gamma}{4} (\gamma - 2) xy^{-2} \frac{\partial u}{\partial z}. \]

Therefore, \[ [C, J^-] u = [0]u. \]

Clearly \[ [C, J^-] = [C, J^+] = [C, J^3] = 0. \]

Thus, it is verified that \( C \) commutes with \( J^-, J^+, J^3. \)

Equation (3.3.12) enables us to write (3.3.2) as

(3.3.13) \[ C f(x, y, z) = \frac{\gamma}{4} (\gamma - 2) f(x, y, z). \]

Now, let us consider some of the particular cases:
**Case 1:** When $f(x, y, z)$ is a common eigen function of $C$ and $J^3$. The simultaneous equations

\[(3.3.14) \quad C f(x, y, z) = \frac{\gamma}{4} (\gamma - 2) f(x, y, z)\]

and

\[(3.3.15) \quad J^3 f(x, y, z) = \left(v + \frac{1}{2} \gamma \right) f(x, y, z)\]

admit a solution $f(x, y, z) = z^\gamma R_v(\beta; \gamma; x, y)$.

Thus, (3.3.11) takes the form

\[(3.3.16) \quad [T(g)f](x, y, z) = (x - y)^\beta (d + byz)^{-\gamma} \left[ \frac{dx}{d + byz} - y \right]^\beta \left( \frac{c + yza}{d + yzb} \right)^v z^F_1 \left[ -v, \beta; \gamma; \left(1 - \frac{(d + yzb)((y - x)c + y^2za)}{xyz} \right)^{-1} \right] \]

satisfying the relation

\[(3.3.17) \quad C[T(g)f](x, y, z) = \frac{\gamma}{2} \left(\frac{\gamma}{2} - 1\right) [T(g)f](x, y, z).\]

Let $v$ is not an integer, then (3.3.16) has a Laurent’s series expansion

\[(3.3.18) \quad [T(g)f](x, y, z) = \sum_{n=0}^{\infty} j_n(g)R_{v+n}(\beta; \gamma; x, y)z^{v+n} \]

\[= \sum_{n=0}^{\infty} j_n(g)y^{v+n}R_{v+n}(\beta; \gamma; x, y)z^{v+n} z^F_1 \left[ -(v + n), \beta; \gamma; \frac{x}{x - y} \right] z^{v+n} \]

\[= \sum_{n=0}^{\infty} j_n(g)y^{v+n}(-y)^{-\beta} (x - y)^\beta z^F_1 \left[ -(v + n), \beta; \gamma; \frac{x}{y} \right] z^{v+n} ([10]; P. 208) \]

(or)

\[(3.3.19) \quad (d + byz)^{-\gamma} \left[ \frac{dx}{d + byz} - y \right]^\beta \left( \frac{c + yza}{d + yzb} \right)^v \]
\[ _2F_1(-v, \beta; \gamma; \left(1 - \frac{(d + yz b)\{(y - x)c + y^2za\}}{xyz}\right)^{-1}) \]
\[ = \sum_{n=-\infty}^{\infty} j_n(g)y^{v+n}(-y)^{-\beta} _2F_1(-v+n, \beta; \gamma; \frac{x}{y})^{-}z^{v+n}. \]

To determine \( j_n(g) \), let us set \( x = 1, y = 1 \) and compare the coefficients of \( z^n \),

(3.3.20) \[ j_n(g) = \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma)_k}{k!(\beta)_k} a^{-k}(bz)^{-k} d^{\beta - v} \left(\frac{-b}{d}\right)^n \]
\[ \cdot \{\Gamma(1+n)\}^{-1} _2F_1\left(v + \gamma - \beta + n; k - v; 1 + n; \frac{bc}{ad}\right) \]
\[ \cdot \frac{\Gamma(\gamma - \beta) \Gamma(\gamma + v + n)}{\Gamma(\gamma) \Gamma(\gamma + v - \beta)}. \]

Thus the generating function (3.3.19) becomes

(3.3.21) \[ \left(1 + \frac{yz b}{d}\right)^{\beta - v} \left(1 + \frac{c}{yz a}\right) \left(d - \frac{y^2zb}{(x - y)}\right)^{-\beta} \]
\[ \cdot _2F_1\left[-v; \beta; \gamma; \left(1 - \frac{(d + yz b)\{(y - x)c + y^2za\}}{xyz}\right)^{-1}\right] \]
\[ = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma - \beta) \Gamma(\gamma + v + n)}{\Gamma(\gamma) \Gamma(\gamma + v - \beta)} \frac{(-1)^k (\gamma)_k (\beta)_k (ab z)^k}{k!(\gamma)_k} \left(\frac{-b z}{d}\right)^n \]
\[ \cdot \{\Gamma(1+n)\}^{-1} _2F_1\left(v + \gamma - \beta + n; k - v; 1 + n; \frac{bc}{ad}\right) _2F_1\left[-v - n, \beta; \gamma; \frac{x}{x - y}\right] y^n. \]

where \( \xi_2 \) is given by (3.3.11) and \( \max \left\{ \frac{|\xi_2|}{|y|}, \frac{|by z|}{d}, \frac{|c}{a y z} \right\} < 1. \)

**Deductions**

1. Let \( a = d = z = 1 \) and \( b = 0 \), then (3.3.21) gives us
\[ (3.3.22) \quad \sum_{k=0}^{n} \frac{(-n)^k}{k!} R_{n-k}(\beta; \gamma; x, y) x^k \]

\[ = [y(y-e)(y^2 - ye + xc)^{-1}] R_n(\beta; \gamma; x, y^{-1}(y^2 - ye + xc)). \]

2. Let \( a = d = z = 1 \) and \( c = 0 \), then (3.3.21) gives us

\[ (3.3.23) \quad \sum_{p=0}^{n} \frac{(k + \gamma)^p}{p!} R_{k+p}(\beta; \gamma; x, y) b^p \]

\[ = (x - y)^\beta (1 - yb)^{-\gamma + \beta - 2k} [x - y(1 - yb)]^{-\beta} R_k(\beta; \gamma; x, y(1 - yb)). \]

**APPLICATIONS**

The generating relations (3.3.22) and (3.3.23) yield the following:

1. \( \sum_{n=0}^{\infty} \frac{(-\alpha - \gamma)_n}{n!} L_{v-n}^{(a)}(x, y) y^n = (1 - cy)^v L_{v}^{(a)} \left( \frac{x}{1-cy}, y \right), \quad |c| < 1. \)

2. \( \sum_{n=0}^{\infty} \frac{(1+\lambda)_n}{n!} L_{v+n}^{(a)}(x, y) b^n = (1 - by)^{v+1} \exp \left( -bx \right) L_{v}^{(a)} \left( \frac{x}{1-by}, y \right), \quad |b| < 1. \)

3. \( \sum_{n=0}^{\infty} \frac{(-\alpha - \gamma)_n}{n!} L_{v+n}^{(a)}(x) c^n = (1 - c)^v L_{v}^{(a)} \left( \frac{x}{1-c} \right), \quad |c| < 1. \)

4. \( \sum_{n=0}^{\infty} \frac{(1+\lambda)_n}{n!} L_{v+n}^{(a)}(x) b^n = (1 - b)^{v+1} \exp \left( -bx \right) L_{v}^{(a)} \left( \frac{x}{1-b} \right), \quad |b| < 1. \)

5. \( \sum_{n=0}^{\infty} \frac{(-1-\gamma)_n}{n!} M_{v+n}(z; \gamma, \rho) c^n \]

\[ = (1 - c(1-\rho))^\rho M_{\rho} \left( z; \gamma, \left( \frac{c(1-\rho)}{c(1-\rho) - 1} \right) \right), \quad |c(1-\rho)| < 1. \]

provided \( \gamma > 0, 0 < \rho < 1, z = 0, 1, 2, \ldots \)

6. \( \sum_{n=0}^{\infty} \frac{(v+\gamma)_n}{n!} (1-\rho)^{-n} M_{v+n}(z; \gamma, \rho) b^n \)
\[ = (1 - b(1 - \rho)^{-1})^{z - \gamma - \nu} \left(1 + b(1 - \rho)^{-1}\right)^{z - \gamma - \nu} \]

\[ M_v \left( z; \gamma, \left( \frac{\rho(1 - \rho) - b}{1 - \rho - b} \right) \right), \quad \left| \frac{b}{1 - \rho} \right| < 1. \]

provided \( \gamma > 0, 0 < \rho < 1, z = 0, 1, 2, \ldots \)

7. \[
\sum_{n=0}^{\infty} \frac{(-v)_n}{n!} \left( 1 - e^{-x} \right)^n \phi_{\nu,n}(z; \lambda) e^n
\]

\[ = \left\{ 1 - c e^{-\lambda} (1 - e^{-x}) \right\}^{\nu} \phi_\nu \left( z; \log \left( \frac{c + e^{2\lambda} (1 - e^{-x})^{-1}}{c + e^{2\lambda} (1 - e^{-x})^{-1}} \right) \right), \quad \left| \frac{c(1 - e^{-\lambda})}{e^{\lambda}} \right| < 1. \]

8. \[
\sum_{n=0}^{\infty} \frac{(1 + v)_n}{n!} \left( 1 - e^{-x} \right)^n \phi_{\nu,n}(z; \lambda) b^n
= \left\{ 1 - be^{-\lambda} (1 - e^{-x})^{-1} \right\}^{z - 1}
\]

\[ \left\{ 1 + be^{2\lambda} (1 - e^{-x})^{-1} \right\}^{\nu} \phi_\nu \left( z; \log \left( \frac{e^{\lambda} + be^{2\lambda} (1 - e^{-x})^{-1}}{1 + be^{2\lambda} (1 - e^{-x})^{-1}} \right) \right), \quad \left| \frac{be^{\lambda}}{(1 - e^{-\lambda})} \right| < 1. \]

9. \[
\sum_{n=0}^{\infty} \frac{(-v)_n}{n!} (1 - p)^{-n} K_{\nu,n}(z; p, N) e^n
\]

\[ = \left\{ 1 - (1 - p)^{-1} \right\}^{\nu} K_\nu \left( z; (p - cp(1 - p)^{-1}), N \right), \quad \left| \frac{c}{1 - p} \right| < 1. \]

provided \( 0 < p < 1, z = 0, 1, 2, \ldots, N. \)

10. \[
\sum_{n=0}^{\infty} \frac{(v - N)_n}{n!} (1 - p)^n K_{\nu+n}(z; p, N) b^n
\]

\[ = (1 - b(1 - p))^{-z - v - N} \left( 1 + bp^{-1}(1 - p)^{2} \right) K_\nu \left( z; (p + b(1 - p)^{2}), N \right), \quad \left| b(1 - p) \right| < 1. \]

provided \( 0 < p < 1, z = 0, 1, 2, \ldots, N. \)

**Case 2:** When \( f(x, y, z) \) is a common eigen function of the operators \( C \) and \( J^{-} \).

Let \( f(x, y, z) \) be a solution of the simultaneous equations
(3.3.24) \[ C f(x, y, z) = \frac{\gamma}{2} \left( \frac{\gamma}{2} - 1 \right) f(x, y, z) \]

and

(3.3.25) \[ J^- f(x, y, z) = -f(x, y, z) \]

which implies

(3.3.26)
\[
\left[ -\frac{1}{y(x-y)} \right] y^2 (x-y)^2 \frac{\partial^2}{\partial y^2} - y(x-y) [(\gamma - \beta - 1)x + (2 - \gamma)y] \frac{\partial}{\partial y} \\
+ y(x-y)(2y-3x) \frac{\partial^2}{\partial y \partial z} \\
+ [(\gamma - \beta + 2)x^2 - 2(\gamma + 1)xy + \gamma y^2] z \frac{\partial}{\partial z} + (2x-y)(x-y)z^2 \frac{\partial^2}{\partial z^2} \right] f(x, y, z) = 0
\]

and

(3.3.27)
\[
\left[ y^{-1} z^{-1} (x-y) \frac{\partial}{\partial y} - xy^{-2} \frac{\partial}{\partial z} + 1 \right] f(x, y, z) = 0.
\]

Assuming the general solution of (3.3.24) and (3.3.25) in the form

(3.3.28) \[ f(x, y, z) = \exp(zy) k \left( \frac{xyz}{x-y} \right) \]

and substituting this in (3.3.24), we get

(3.3.29) \[ \left( u \frac{d^2}{du^2} + (\gamma + u) \frac{d}{du} + \beta \right) k(u) = 0 \text{ where } u = \frac{xyz}{x-y}. \]

This is Kummer’s differential equation ([78], P. 36) and its solution is

(3.3.30) \[ k(u) = \, _1F_1[\beta; \gamma; -u]. \]

Thus one solution of this system is

(3.3.31) \[ f(x, y, z) = \exp (yz) \, _1F_1 \left[ \beta; \gamma; -\frac{xyz}{x-y} \right]. \]
Now, expanding this function in powers of \( z \), we get

(3.3.32) \[ \exp(yz) \, _1F_1\left[\beta; \gamma; -\frac{xyz}{x-y}\right] = \sum_{n=0}^{\infty} R_n(\beta; \gamma; x, y) \frac{z^n}{n!}, \]

which is the generating function for \( R_n(\beta; \gamma; x, y) \).

Further

(3.3.33) \[ [T(g)f](x, y, z) = (x - y)^\beta (d + yzb)^(-\gamma) \left[ \frac{dx}{d + yzb} - y \right]^{-\beta} \exp\left( \frac{c + yza}{d + yzb} \right) \]

\[ \quad \times \, _1F_1\left[\beta; \gamma; -\frac{xyz(c + yza)}{(d + yzb)(xyz - (d + yzb)((y-x)c + y^2za))}\right] \]

satisfies the relation

(3.3.34) \[ C[T(g)f](x, y, z) = \frac{\gamma}{2} \left( \frac{\gamma}{2} - 1 \right) [T(g)f](x, y, z). \]

Since \([T(g)f](x, y, z)\) is analytic at \( z = 0 \), it can be expanded in the form

(3.3.35) \[ [T(g)f](x, y, z) = \sum_{n=0}^{\infty} s_n(g) R_n(\beta; \gamma; x, y) \frac{z^n}{n!} \]

\[ = \sum_{n=0}^{\infty} s_n(g) y^n \, _2F_1\left[-n, \beta; \gamma; -\frac{x}{x-y}\right] z^n \]

\[ = \sum_{n=0}^{\infty} s_n(g) (-y)^{-\beta} (x-y)^\beta \, _2F_1\left[-n, \beta; \gamma; \frac{x}{y}\right] z^n \quad (\text{[10]; P. 208}) \]

(or)

(3.3.36) \[ (d + yzb)^{-\gamma} \left[ \frac{dx}{d + yzb} - y \right]^{-\beta} \exp\left( \frac{c + yza}{d + yzb} \right) \]

\[ \quad \times \, _1F_1\left[\beta; \gamma; -\frac{xyz(c + yza)}{(d + yzb)(xyz - (d + yzb)((y-x)c + y^2za))}\right] \]

\[ = \sum_{n=0}^{\infty} s_n(g) (-y)^{-\beta} \, _2F_1\left[-n, \beta; \gamma; \frac{x}{y}\right] z^n. \]
To compute the coefficients of \( s_n(g) \), let us put \( x = 1 \) and \( y = 1 \) in (3.3.36) so that

\[
(3.3.37) \quad (d + zb)^{\beta-\gamma} (zb)^{-\beta} \exp \left( \frac{c + z\alpha}{d + zb} \right)_1 F_1 \left( \beta; \gamma; -\frac{c + z\alpha}{(d + zb)(z - (d + zb)z\alpha)} \right) = \sum_{n=0}^{\infty} \frac{(\gamma - \beta)_n}{(\gamma)_n} s_n(g)(-y)^{-\beta} z^n.
\]

this may be written as

\[
(3.3.38) \quad (d)^{-\gamma+\beta} (b)^{-\beta} \exp \left( \frac{c}{d} \right)(1 + \frac{b\gamma}{d})^{\beta-\gamma} \exp \left( \frac{-z}{d^2 \left(1 + \frac{b\gamma}{d} \right)} \right)_1 F_1 \left( \beta; \gamma; -\frac{-z}{d^2 \left(1 + \frac{b\gamma}{d} \right)} \right) = \sum_{n=0}^{\infty} \frac{(\gamma - \beta)_n}{(\gamma)_n} s_n(g) z^n.
\]

The familiar generating function for Laguerre polynomials ([78]; P. 213) is

\[
(3.3.39) \quad (1-t)^{-1-\alpha-c} (1-t + yt)^{-\alpha} \exp \left( \frac{-xt}{1-t} \right)_1 F_1 \left( c; 1 + \alpha; \frac{xt}{1-t(1-t + yt)} \right) = \sum_{n=0}^{\infty} \frac{(1 + \alpha - c)_n}{(1 + \alpha)_n} L_n^{(\alpha)}(x)t^n.
\]

Choose \( y = 1 \), we get

\[
(3.3.40) \quad (1-t)^{-1-\alpha-c} \exp \left( \frac{-xt}{1-t} \right)_1 F_1 \left( c; 1 + \alpha; \frac{xt}{1-t} \right) = \sum_{n=0}^{\infty} \frac{(1 + \alpha - c)_n}{(1 + \alpha)_n} L_n^{(\alpha)}(x) t^n.
\]

On comparing the coefficients of \( z^n \) in (3.3.37), with the help of the generating function for Laguerre polynomials (3.3.40), we find that

\[
(3.3.41) \quad s_n(g) = d^{\beta-\gamma} b^{-\beta} \exp \left( \frac{c}{d} \right)(\frac{-b}{d})^n L_n^{(\gamma-1)} \left( \frac{1}{bd} \right).
\]

Thus we have eventually led to the bilateral generating function:
(3.3.42) \[(x - y)^\beta \left(1 + \frac{xyzb}{d}\right)^{\beta - \gamma} \{d(x - y) - y^2 zb\}^{-\beta - \beta'} \]

\[
\exp \left(\frac{xyz}{d(d + yzb)}\right) F_1 \left[\beta; \gamma; \frac{-xyz(c + yza)}{(d + yzb)(xyz - (d + yzb)((y-x)c + y^2za))}\right]
\]

\[= \sum_{n=0}^{\infty} R_n(\beta; \gamma; x, y) L_n^{(\gamma-1)} \left(\frac{1}{bd}\right) \left(\frac{-bz}{d}\right)^n, \quad |y| < 1, \quad \left|\frac{byz}{d}\right| < 1.\]

By setting \(a = d = \frac{i}{\sqrt{w}}\), \(b = c = -\frac{i}{\sqrt{w}}\), \(i = \sqrt{-1}\), in (3.3.42) assumes the form

(3.3.43)

\[(x - y)^\beta (1 - xyz)^{\beta - \gamma} \{x - y(1 - yz)\}^{-\beta} \exp \left(\frac{-xyzw}{1 - yz}\right) F_1 \left[\beta; \gamma; \frac{xyzw}{(1 - yz)(1 - y(1 - yz))}\right]
\]

\[= \sum_{n=0}^{\infty} R_n(\beta; \gamma; x, y) L_n^{(\gamma-1)} (w) z^n, \quad |y| < 1, \quad |yz| < 1.\]

which is a bilateral generating relation.

### 3.4 APPLICATIONS

We obtain the following results as usual.

1. \[
\sum_{n=0}^{\infty} \frac{n!}{(1 + \alpha)} L_n^{(\alpha)}(y) L_n^{(\gamma-1)}(w) \left(\frac{z}{y}\right)^n = \left(1 - \frac{z}{y}\right)^{-\alpha - 1} \exp \left(-\frac{(y+w)z}{y} \right) F_1 \left[-; 1 + \alpha; \frac{ywz}{y}\left(1 - \frac{z}{y}\right)\right].
\]

2. \[(1 - \rho)^{-n} M_n(s; \gamma, \rho) L_n^{(\gamma-1)} (w) z^n = \left[1 - (1 - \rho)^{-1}\right]^{-\gamma - \gamma} \left[1 - z \rho^{-1} (1 - \rho)^{-1}\right]^{s} \exp \left(-\frac{-wz}{1 - \rho - z}\right)
\]

\[F_1 \left[-s; \gamma; \frac{wz(1 - \rho)^2}{(1 - \rho - z)(\rho^2 - \rho + z)}\right],\]

provided \(\gamma > 0, 0 < \rho < 1, s = 0, 1, 2,\ldots\)
3. \[ \sum_{n=0}^{\infty} K_n(y; p, N) L_n^{(N-1)}(w)(1-p)^n \]

\[ = \{1-(1-p)z\}^{N-y}[1+p^{-1}(1-p)^2z]^y \exp\left(\frac{-w(1-p)z}{1-(1-p)z}\right) \]

\[ _1F_1\left[-y; -N; \frac{w(1-p)z}{1-(1-p)z}\{p+(1-p)^2z\}\right], \]

provided \( 0 < p < 1, \ y = 0, 1, 2, \ldots, N. \)

4. \[ \sum_{n=0}^{\infty} \frac{(1+y)n}{n!} L_n^{(a)}(x, y, b^n) = (1-by)^{-\alpha-y-1} \exp\left(\frac{-bx}{1-by}\right) L_n^{(a)}\left(\frac{x}{1-by}, y\right), |b|<1. \]

These bilateral (bilinear) generating relations are well known for special functions deduced from the relation (3.3.43) by using the conditions §2.3.

### SECTION-B

#### 3.5 INTRODUCTION

In this section, we consider the different generalized hypergeometric function

\[ \psi_{\alpha, \beta, \gamma, m}(x) \] introduced by Khanna et al. [60] given by

\[ \psi_{\alpha, \beta, \gamma, m}(x) = \frac{\beta^n(y)}{m!} (1-x)^{-m/2} _2F_1\left[-m, \alpha; \gamma; \frac{x}{\beta}\right], \]

which is valid under the following conditions:

(i) \( \alpha, \beta \) are non-zero real numbers,

(ii) \( \gamma \) is neither zero nor a negative integer,

(iii) \( m \) is a non-negative integer,

(iv) \( \alpha, \beta, \gamma \) are independent of \( m \),

(v) \( x \) is any finite complex variable such that \(|x|<1|\).


Khanna et al. [60] derived some generating relations for the generalized hypergeometric function \( \psi_{\alpha,\beta,\gamma,m}(x) \) as a function of four parameters. Here, our interest mainly lies in the fact that to obtain a new class of generating relations for the generalized hypergeometric functions \( \psi_{\alpha,\beta,2(\nu+1),m-u-1}(x) \) by constructing a realization \( D(u,m) \) of the algebraic representation of \( SL(2,\mathbb{C}) \) (a complex special linear group) [69]. In order to derive the elements of Lie algebra, an appropriate interpretation to the index \( \gamma \) and \( m \) are given all the while. These generating functions/relations, in turn yield, as special cases, a number of linear generating functions to various important classical orthogonal polynomials. Many results obtained as special cases are known however some of them are accepted to be new in the theory of special functions. We have obtained a new class of generating relations for \( \psi_{\alpha,\beta,2(\nu+1),m-u-1}(x) \) with the help of the following two independent differential recurrence relations.

### 3.6 RECURRENCE RELATIONS

\[
\begin{align*}
\frac{d}{dx} \psi_{\alpha,\beta,2(\nu+1),m-u-1}(x) & = \frac{1}{2x(1-x)} \{2\beta(-m-u)\sqrt{1-x} \psi_{\alpha,\beta,2(\nu+1),(m-1)-u-1}(x) \\
& \quad + (m-u-1)(2-x)\psi_{\alpha,\beta,2(\nu+1),m-u-1}(x) \}
\end{align*}
\]

and

\[
\begin{align*}
\frac{d}{dx} \psi_{\alpha,\beta,2(\nu+1),m-u-1}(x) & = \frac{1}{x(\beta-x)} \left[ \beta(-m-u-1) + \alpha x + \frac{(m-u-1)x(\beta-x)}{2(1-x)} \right] \psi_{\alpha,\beta,2(\nu+1),m-u-1}(x) \\
& \quad + \frac{1}{x(\beta-x)}(m-u)\sqrt{1-x} \psi_{\alpha,\beta,2(\nu+1),(m+1)-u-1}(x).
\end{align*}
\]
It is very clear that the above two independent differential recurrence relations decide the ordinary differential equation:

\[
\begin{align*}
(3.6.3) \quad & \left( x(\beta - x) \frac{d^2}{dx^2} + \left[ 2\beta (u+1) + (m-u-\alpha - 2)x - \frac{(m-u-1)x(\beta - x)}{1-x} \right] \frac{d}{dx} \\
& \quad + \alpha (m-u-1) + \frac{(m-u-1)(m-u-3)x(\beta - x)}{4(1-x)^2} \\
& - \frac{(m-u-1)}{2(1-x)} \left[ 2\beta (u+1) + (m-u-\alpha - 2)x \right] \right) \psi_{\alpha,\beta,2(u+1),m-u-1}(x) = 0.
\end{align*}
\]

Special cases

The following special cases of \( \psi_{\alpha,\beta,\gamma,m}(x) \) have been obtained:

\[
(3.6.4) \quad \lim_{\alpha \to \infty} \psi_{\alpha,1,1,\gamma,m} \left( \frac{x}{\alpha} \right) = L_m^{(\gamma)}(x),
\]

where \( L_m^{(\gamma)}(x) \) are the Laguerre polynomials.

\[
(3.6.5) \quad \lim_{\alpha \to \infty} \psi_{\alpha,\frac{1}{2},\frac{1}{2},m} \left( \frac{x^2}{\alpha} \right) = (-1)^m H_{2m}(x) / 2^{2m} m!
\]

and

\[
(3.6.6) \quad \lim_{\alpha \to \infty} \psi_{\alpha,\frac{3}{2},\frac{3}{2},m} \left( \frac{x^2}{\alpha} \right) = (-1)^m H_{2m+1}(x) / 2^{2m+1} m! 2x,
\]

where \( H_{2m}(x) \) and \( H_{2m+1}(x) \) are the even and odd Hermite polynomials, respectively.

\[
(3.6.7) \quad \lim_{\alpha \to \infty} \psi_{\alpha,1,\mu+1/2,2,m} \left( \frac{x^2}{\alpha} \right) = (-1)^m H_{2m}^{\mu}(x) / 2^{2m} m!.
\]
and

$$\lim_{\alpha \to \infty} \psi_{\alpha, 1, (\mu+3)/2, m} \left( \frac{x^2}{\alpha} \right) = (-1)^m H'_{2m+1}(x) / 2^{2m} m! 2x,$$

where $H'_{2m}(x)$ and $H'_{2m+1}(x)$ are the generalized even and odd Hermite polynomials, respectively.

$$\psi_{-y, 1, \gamma, m}(1 - \rho^{-1}) = \frac{(\gamma)_m}{m!} \rho^{\gamma} M_m(y; \gamma, \rho)$$

provided $0 < \rho < 1$, $y = 0, 1, 2, ...$ where $M_m(y; \gamma, \rho)$ are the Meixner polynomials.

$$\psi_{-y, 1, 1, m}(1 - e^\lambda) = e^{m\lambda} \phi_m(y; \lambda),$$

where $\phi_m(y; \lambda)$ are the Gottlieb polynomials.

$$\psi_{-y, 1, -N, m}(p^{-1}) = \frac{(-N)_m}{m!} (1 - p^{-1})^{-m/2} K_m(y; p; N),$$

provided $0 < p < 1$, $y = 0, 1, 2, ..., N$, where $K_m(y; p; N)$ are the Krawtchouk polynomials.

In this section, we consider the generalized hypergeometric function $\psi_{\alpha, \beta, \gamma, m}(x)$

and construct a realization of the representation $D(u, m_\alpha)$ of $SL(2, \mathbb{C})$ such that $J^+, J^-, J^3$ and $E$ take the form of differential operators acting on $F$. Then the realization of $D(u, m_\alpha)$ is extended to a local multiplier representation $T$ of $SL(2, \mathbb{C})$ on $F$. An explicit expression for the matrix elements $A_{\alpha}(g)$ is obtained in terms of hypergeometric function. The connection between operators and special functions will
prove to be a powerful tool for deriving generating functions and identities involving hypergeometric functions. Many known and unknown special cases are also deduced.

3.7 REALIZATION OF $D(u, m_0)$ AND GENERATING FUNCTIONS

Theorem 2.3 [69] is being used to obtain generating functions with the help of the realization of $D(u, m_0)$. Clearly, $sl(2, \mathbb{C})$ is the Lie algebra of the local Lie group. Now, we consider the following first order linear independent differential operators $J^3, J^+$ and $J^-$ defined as:

$$J^3 = y \frac{\partial}{\partial y}$$

(3.7.1)

$$J^+ = \frac{xy(\beta-x)}{\sqrt{1-x}} \frac{\partial}{\partial x} + \frac{y^2}{2(1-x)^{3/2}} (-\beta + x^2) \frac{\partial}{\partial y}$$

$$+ \frac{y}{2(1-x)^{3/2}} \{(2\alpha-u-1)x^2-(\beta(u+1)+2\alpha)x+2\beta(u+1)\}$$

$$J^- = \frac{xy^{-1}\sqrt{1-x}}{\beta} \frac{\partial}{\partial x} + \frac{x-2}{2\beta\sqrt{1-x}} \frac{\partial}{\partial y} - \frac{(u+1)(x-2)y^{-1}}{2\beta\sqrt{1-x}}.$$ 

These operators satisfy the commutation relations

(3.7.2) \quad \quad [J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^3.

Clearly, these operators are the generators of $sl(2)$.

To construct a realization of $D(u, m_0)$ in terms of the operators (3.7.1), we find non-zero functions $f_m(x, y) = y^mZ_m(x)$ such that the following relations in terms of the functions $Z_m(x)$:
\[
\begin{aligned}
&\left\{ \frac{x(\beta - x)}{\sqrt{1-x}} \frac{d}{dx} + \frac{1}{2(1-x)^{3/2}} [(2\alpha - u - 1 + m)x^2] \\
&- (2\alpha + \beta(u + 1) + 3m\beta)x + 2\beta(m + u + 1) \right\} Z_m(x) = (m - u)Z_{m+1}(x)
\end{aligned}
\]

and

\[
(3.7.3) \begin{aligned}
&\left\{ \frac{x\sqrt{1-x}}{\beta} \frac{d}{dx} + \frac{(m - u - 1)(x - 2)}{2\beta\sqrt{1-x}} \right\} Z_m(x) = -(m + u)Z_{m-1}(x)
\end{aligned}
\]

are valid for all \( m \in S \).

Now, if we take \( Z_m(x) = \psi_{\alpha,\beta,2(u+1),m-u-1}(x), m \in S \), then the vectors

\[ f_m(x, y) = y^m\psi_{\alpha,\beta,2(u+1),m-u-1}(x), m \in S, \]

form a basis for a realization of the representation \( D(u, m_0) \). The realization can be extended to a local multiplier representation \( T \) of \( SL(2, \mathbb{C}) \) on \( \mathfrak{F} \).

According to Theorem 2.3 [69], the operators (3.7.1) generate a Lie algebra of generalized Lie derivatives corresponding to a local multiplier \( T \) of \( SL(2, \mathbb{C}) \) on \( \mathfrak{F} \). Now if we proceed to compute the multiplier representation of the local Lie group \( SL(2, \mathbb{C}) \),

we first compute the actions of the one-parameter groups \( \exp(a^+ J^+), \exp(b^- J^-) \) and \( \exp(c^+ J^3) \) on \( f \in \mathfrak{F} \), where \( \mathfrak{F} \) is the complex vector space of all functions of \( x \) and \( y \) analytic in some neighbourhood of the point \((x_0, y_0) = (0,0)\), are obtained by integrating the following differential equations:

\[
\frac{d}{da'} x(a') = \frac{x(a')y(a')[\beta - x(a')]}{\sqrt{1-x(a')}}.
\]
\[ \frac{d}{da'} y(a') = \frac{y^2(a')}{{2[1-x(a')]^{3/2}}} \{2\beta - 3\beta x(a') + x^2(a')\}, \]

\[ \frac{d}{da'} v(a') = v(a') \frac{y(a')}{2[1-x(a')]^{3/2}} \{(2\alpha - u - 1)x^2a' - (\beta(u+1) + 2\alpha)x(a') + 2\beta(u+1)\}, \]

(3.7.4) \[
\frac{d}{db'} x(b') = \frac{x(b')y^{-1}(b')\sqrt{1-x(b')}}{\beta},
\]

\[ \frac{d}{db'} y(b') = \frac{x(b') - 2}{2\beta \sqrt{1-x(b')}}, \]

\[ \frac{d}{db'} v(b') = -v(b') \left\{ \frac{(u+1)(x(b') - 2)y^{-1}(b')}{2\beta \sqrt{1-x(b')}} \right\}, \]

\[ \frac{d}{dc'} x(c') = 0, \quad \frac{d}{dc'} y(c') = y(c'), \quad \frac{d}{dc'} v(c') = 0 \]

with initial conditions \( x(0) = x_0, \ y(0) = y_0 \) and \( v(0) = 1 \), where \( v \) is multiplier of the representation. Therefore, if \( f \in \mathfrak{S} \) is analytic in a neighbourhood of of \( (x_0, y_0) \) then the values of the multiplier representation of \( \exp(a'J^+)f, \exp(b'J^-)f \) and \( \exp(c'J^3)f \) are given by

(3.7.5) \[ [T(\exp(a'J^+))f](x_0, y_0) = \frac{(1-x_0)^{u+1}[\sqrt{(1-x_0) - \beta a'y_0}]^{u-1}}{[\sqrt{(1-x_0) - a'y_0(\beta - x_0)}]^{u}} \left( \frac{\sqrt{(1-x_0) - a'y_0(\beta - x_0)}}{(1-x_0)^{3/2} - a'y_0(\beta - x_0)} \right)^{u+1/2} \]
\begin{align*}
\cdot f & \left( \frac{x_0 \sqrt{1-x_0}}{\sqrt{1-x_0}} - a' y_0 (\beta - x_0), \frac{y_0}{\sqrt{1-x_0}} - \beta a' y_0 \left( \frac{(1-x_0)^{3/2} - a' y_0 (\beta - x_0)}{\sqrt{1-x_0}} \right) \right), \\
\left| \frac{\beta a' y_0}{\sqrt{1-x_0}} \right| & < 1, \quad \left| \frac{a'(\beta - x_0) y_0}{\sqrt{1-x_0}} \right| < 1,
\end{align*}

\begin{align*}
[T(\exp(b'J^+))f](x_0, y_0) & = \left\{ \frac{\beta^2 y_0^2 \sqrt{1-x_0}}{(\beta y_0 - b' \sqrt{1-x_0})(\beta y_0 \sqrt{1-x_0}) - b') \right\}^{1/2(u+1)} \\
\cdot f & \left( \frac{\beta x_0 y_0}{\beta y_0 - b' \sqrt{1-x_0}}, \left( \frac{(\beta y_0 \sqrt{1-x_0} - b')^2}{\beta^2 \sqrt{1-x_0}} \right)^{1/2}, \quad \left| \frac{b' \sqrt{1-x_0}}{\beta y_0} \right| < 1,
\end{align*}

\begin{align*}
[T(\exp(c'J^+))f](x_0, y_0) & = f(x_0, e^c y_0).
\end{align*}

If $SL(2, \mathbb{C})$ is given by (3.7.12), then we find that

\begin{align*}
g & = (e^{a'J^+})(e^{b'J^+})(e^{c'J^+}), \quad \text{where} \quad a' = -b/d, \quad b' = -cd, \quad e^{c/2} = d^{-1}, \quad 0 \leq \text{Im} c < 4\pi.
\end{align*}

Thus, for $|a'|, |b'|, |c'|$ sufficiently small, the operator $T(g)$ is given by

\begin{align*}
(3.7.6) \quad [T(g)f](x, y) & = [T(e^{a'J^+})T(e^{b'J^+})T(e^{c'J^+})f](x, y) \\
& \quad = \frac{(1-x)^{a+1} \left[ \sqrt{(1-x)} - \beta a' y \right]^{a-a-1}}{\left[ \sqrt{(1-x)} - a' y (\beta - x) \right]^{a}} \left( \frac{1-x}{1-\frac{a' y (\beta - x)}{(1-x)^{3/2} - a' y (\beta - x)}} \right)^{1/2(a+1)} \\
& \quad \cdot \left( \frac{\beta^2 y^2 \sqrt{1-x}}{(\beta y - b' \sqrt{1-x})(\beta y \sqrt{1-x} - b')} \right)^{1/2(a+1)} f(\xi, \eta e^{c'}),
\end{align*}

where
\[ \xi_1 = \frac{\beta xy \sqrt{1 - x}}{\sqrt{1 - x + a'y(\beta - x)}[\beta y(a'b' + 1) + b'\sqrt{1 - x}]} . \]

\[ \eta_1 = \left[ \frac{\{\beta y(a'b' + 1) + b'\sqrt{1 - x}\} + a'y(\beta - x)}{+b'[\sqrt{1 - x} - \beta a'y][\sqrt{1 - x} + a'y(\beta - x)]} \right]^{1/2} . \]

By setting \( a' = -b/d \), \( b' = -cd \), \( e^{c/2} = d^{-1} \) and using the fact that \( ad - bc = 1 \) gives us that

\[ (3.7.7) \quad [T(g)f](x, y) = \left( 1 + \frac{\beta by}{d \sqrt{1 - x}} \right)^{a - u - 1} \left( 1 + \frac{(\beta - x)by}{d \sqrt{1 - x}} \right)^{1/2} \left\{ 1 + \frac{cd \sqrt{1 - x}}{\beta y} \right\}^{\frac{1}{2}(u + 1)} \cdot \left\{ 1 + \frac{cd \sqrt{1 - x}}{\beta y} \right\}^{\frac{1}{2}(u + 1)} \cdot f(\xi_2, \eta_2 d^{-2}), \]

where

\[ \xi_2 = \frac{x}{(1 + bc) \left( 1 + \frac{(\beta - x)by}{d \sqrt{1 - x}} \right) \left( 1 + \frac{c\sqrt{1 - x}}{a\beta y} \right)} , \]

\[ \eta_2 = \left[ \frac{1 + \frac{c\sqrt{1 - x}}{a\beta y}}{(1 + bc) y \left( 1 + \frac{c\sqrt{1 - x}}{a\beta y} \right) \left( 1 + \frac{(\beta - x)by}{d(1 - x)^{3/2}} \right)} + cd \left( 1 + \frac{\beta y}{d \sqrt{1 - x}} \right) \left( 1 + \frac{(\beta - x)by}{d \sqrt{1 - x}} \right) \right]^{1/2} . \]

\[ |x| < 1, \quad \left| \frac{b\sqrt{1 - x}}{a\beta y} \right| < 1, \quad \left| \frac{\beta ay}{d \sqrt{(1 - x)}} \right| < 1. \]
For \( f \in \mathcal{F} \) and \( g \) in a small neighbourhood of the identity element \( e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) so that the expression is uniquely defined. The matrix elements of this multiplier representation with respect to the analytic basis \( f_m \) are the functions \( A_{(x,g)} \) defined by the identity

\[
(3.7.8) \quad [T(g)f_{m_0+k}](x, y) = \sum_{l=-\infty}^{\infty} A_{(x,g)}(f_{m_0+l})(x)y^{m_0+l}, k = 0, \pm 1, \pm 2, \pm 3, \ldots
\]

(or)

\[
(1+bc)^{-\frac{3}{2}(m_0+k)} a^{2(m_0+k)} \left( 1 + \frac{\beta by}{d\sqrt{1-x}} \right)^{\frac{1}{2}(a-u-m_0-k-1)} \\
\cdot \left( 1 + \frac{(\beta-x)by}{d\sqrt{1-x}} \right)^{\frac{1}{2}(u+1-m_0-k-2a)} \left( 1 + \frac{(\beta-x)ay}{d(1-x)^{3/2}} \right)^{\frac{1}{2}(a+1)} \\
\cdot \left( 1 + \frac{bd\sqrt{1-x}}{\beta y} \right)^{\frac{1}{2}(a+1)} \left( 1 + \frac{bd}{\beta y\sqrt{1-x}} \right)^{\frac{1}{2}(u+1)} \\
\cdot \left( 1 + \frac{cd}{\beta y\sqrt{1-x}} \left( 1 + \frac{\beta by}{d\sqrt{1-x}} \right)^{\frac{1}{2}(m_0+k)} \right) \\
\cdot \frac{\varphi_{\alpha,\beta, (2a+1)+1, m_0+k-a-1}(x)}{d^{3/2}} = \sum_{l=-\infty}^{\infty} A_{(x,g)}(\varphi_{\alpha,\beta, (2a+1)+1, m_0+k-a-1})(x)y^{m_0+l},
\]

which implies

\[
(3.7.9) \quad (1+bc)^{-\frac{3}{2}(2a+\mu+1)} a^{2(a+\mu+1)} \left( 1 + \frac{\beta by}{d\sqrt{1-x}} \right)^{a-\nu-\mu-1} \left( 1 + \frac{(\beta-x)by}{d\sqrt{1-x}} \right)^{\frac{1}{2}(\nu+2\alpha)}
\]
\[
\cdot \left\{ 1 + \left( \frac{\beta - x}{d} \right)^{\frac{3}{2}} \right\} \left( 1 + \frac{cd \sqrt{1-x}}{\beta y} \right) \left( 1 + \frac{cd}{\beta y \sqrt{1-x}} \right) \right\}^{-\frac{1}{4}(u+1)}
\]

\[
\cdot \left[ 1 + \frac{c}{a\beta y} \right] \left( 1 + \frac{(\beta - x)by}{d(1-x)} \right) \right\}^{-\frac{1}{2}(2v+\mu+1)}
\]

\[
\psi_{\alpha,\beta,\mu+1,\nu}(z)
\]

\[
= \sum_{l=0}^{\infty} A_{\alpha}(g) \psi_{\alpha,\beta,\mu+1,\nu}(x) y^{-l}, \quad |x| < 1, \quad |bc| < 1, \quad \left| \frac{c}{a} \right| < \left| \frac{\beta y}{\sqrt{1-x}} \right| < \left| \frac{b}{d} \right|
\]

where \( z_2 \) is given in (3.7.7).

An explicit expression for the matrix elements \( A_{\alpha}(g) \) is given by

(3.7.10) \[
A_{\alpha}(g) = \frac{(1+bc)^{-\frac{1}{2}(2v+\mu+1)} \Gamma(\mu+\nu+1)}{\Gamma(l+1)\Gamma(\nu+\mu-l+1)} \cdot \left\{ 1 + \left( \frac{\beta - x}{d} \right)^{\frac{3}{2}} \right\} \left( 1 + \frac{cd \sqrt{1-x}}{\beta y} \right) \left( 1 + \frac{cd}{\beta y \sqrt{1-x}} \right) \right\}^{-\frac{1}{4}(u+1)}
\]

By putting this value of \( A_{\alpha}(g) \) in (3.7.9), we get the generating function

(3.7.11) \[
(1+bc)^{-\frac{1}{2}(2v+\mu+1)} \left( 1 + \frac{\beta by}{d \sqrt{1-x}} \right)^{\frac{\nu-\mu-1}{2}} \left( 1 + \frac{(\beta - x)by}{d \sqrt{1-x}} \right)^{-\frac{1}{2}(2\nu+\alpha)}
\]

\[
\cdot \left\{ 1 + \left( \frac{\beta - x}{d} \right)^{\frac{3}{2}} \right\} \left( 1 + \frac{cd \sqrt{1-x}}{\beta y} \right) \left( 1 + \frac{cd}{\beta y \sqrt{1-x}} \right) \right\}^{-\frac{1}{4}(u+1)}
\]

\[
\cdot \left[ 1 + \frac{c}{a\beta y} \right] \left( 1 + \frac{(\beta - x)by}{d(1-x)^{\frac{3}{2}}} \right) + \frac{cd}{\beta y \sqrt{1-x}} \cdot \left( 1 + \frac{\beta by}{d \sqrt{1-x}} \right) \left( 1 + \frac{(\beta - x)by}{d \sqrt{1-x}} \right) \right\}^{-\frac{1}{2}(2v+\mu+1)}
\]
\[ \Psi_{\alpha,\beta,\mu+1,\nu} \begin{pmatrix} x \\ (1 + bc) \left( 1 + \frac{(\beta - x)by}{d\sqrt{1 - x}} \right) \left( 1 + \frac{c\sqrt{1 - x}}{a\beta y} \right) \end{pmatrix} \]

\[ = \sum_{l=-\infty}^{\infty} \frac{\Gamma(v + \mu + 1)}{\Gamma(v + \mu - l + 1)\Gamma(l + 1)} \left( \frac{c}{ay} \right)^l \left[ -\mu - v + l + 1, v + 1 \right] \left[ \frac{bc}{ad} \right] \Psi_{\alpha,\beta,\mu+1,v-l}(x), \]

where \( \max\left\{ |x|, |y|, |bc|, \left| \frac{\beta by}{d\sqrt{1 - x}} \right|, \left| \frac{c\sqrt{1 - x}}{a\beta y} \right| \right\} < 1, \ d = \frac{(1 + bc)}{a}. \)

This generating function is obtained under the assumption that \( v, \mu + v \in \mathbb{C} \) are not integers. When \( l + 1 \leq 0 \), \( _2F_1 \) is given by the limit

\[ (3.7.12) \quad \lim_{b \to -n} \frac{\Gamma(a, b; c; Z)}{\Gamma(b)} = \frac{(a)_{n+1}(b)_{n+1}}{(n+1)!} Z^{n+1} \left( a + n + 1, b + n + 1, n + 2; Z \right), \ n = 0, 1, 2, \ldots \]

**Deductions**

1. If \( a = d = \beta = y = 1 \) and \( b = 0 \), then equation (3.7.11) becomes

\[ (3.7.13) \quad \left( 1 - c\sqrt{1 - x} \right) \left( 1 - \frac{c}{\sqrt{1 - x}} \right)^{\frac{1}{2}} \Psi_{\alpha,1,\mu+1,\nu} \left[ \frac{x}{1 - \sqrt{1 - x}} \right] \]

\[ = \sum_{l=0}^{\infty} \frac{(-v - \mu)}{l!} \Psi_{\alpha,1,\mu+1,v-l}(x) c^l. \]

2. Taking \( a = d = \beta = y = 1 \) and \( c = 0 \), equation (3.7.11) gives us

\[ (3.7.14) \quad (1 - b\sqrt{1 - x})^{\frac{1}{2}(v+2\alpha)} \left( 1 - \frac{b}{\sqrt{1 - x}} \right)^{\alpha - \mu - v/2} \Psi_{\alpha,1,\mu+1,v} \left[ \frac{x}{1 - b\sqrt{1 - x}} \right] \]
where \(|b\sqrt{1-x}| < 1\).

### 3.8 APPLICATIONS

Further, using the conditions on \(\psi_{\alpha,\beta,\gamma,m}(x)\), as special cases mentioned earlier, we have derived the following generating relations:

1. \[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L_{\gamma-k}^{(v+k)}(x)w^k = L_{\gamma}^{(v)}(x+w).
\]

2. \[
\sum_{k=0}^{\infty} \frac{(1+v)^k}{k!} L_{\gamma+k}^{(v-k)}(x)w^k = (1+w)^{(\gamma-k)} \exp(-wx)L_{\gamma}^{(v)}(x+w), \ |w| < 1.
\]

3. \[
\sum_{k=0}^{\infty} \frac{(-v)^k}{k!} \frac{2^k}{k} H_{2v-k}^{\mu+2k}(x)w^k = H_{2v}^{\mu}(x^2 + w).
\]

4. \[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{2v+k}^{\mu+2k}(x)w^k = (1+w)^{\frac{1}{2}(\mu-1)} \exp(-wx^2)H_{2v}^{\mu}(x^2(1+w)), \ |w| < 1.
\]

5. \[
\sum_{k=0}^{\infty} \frac{(-v)^k}{k!} \frac{2^k}{k} H_{2v-k}^{\mu+2k}(x)w^k = H_{2v+1}^{\mu}(x^2 + w).
\]

6. \[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{2v-k}^{\mu-2k}(x)w^k = (1+w)^{\frac{1}{2}(\mu+1)} \exp(-wx^2)H_{2v+1}^{\mu}(x^2(1+w)), \ |w| < 1.
\]

7. \[
\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^\gamma} \frac{(y-k;\gamma+k;\rho)}{(\sqrt{\rho})^k} = M_{y;\gamma,\rho}^{-1}(1-w\sqrt{\rho}).
\]

8. \[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (1-y)^k M_{\gamma-k}(y+k;\gamma-k;\rho)(w\sqrt{\rho})^k
\]

\[
= (1+w\sqrt{\rho})^{y-1}([(1-w\sqrt{\rho}(1-\rho^{-1}))^{y-\gamma}] \cdot M_{y;\gamma,\rho}^{-1}(1-w\sqrt{\rho}(1-\rho^{-1}))],
\]

where \(|w\sqrt{\rho}(1-\rho^{-1})| < 1\), provided \(0 < \rho < 1\), \(y = 0, 1, 2, \ldots\).
9. \[
\sum_{k=0}^{\infty} \frac{(-1)^k (-y)_k}{k!} e^{\frac{\lambda}{2}(v-k)} \phi_{v-k} (y-k; \lambda) w^k \\
= \left[ 1 - we^{-\frac{\lambda}{2}} \right]^v \exp \left( \frac{\nu \lambda}{2} \right) \phi_v \left[ y; \log \left( e^{\frac{\lambda}{2}} - we^{-\frac{\lambda}{2}} \right) \right], \quad \left| we^{-\frac{\lambda}{2}} \right| < 1.
\]

10. \[
\sum_{k=0}^{\infty} \frac{(\nu+1)_k}{k!} e^{\frac{\lambda}{2}(v+k)} \phi_{v+k} (y+k; \lambda) w^k \\
= \left[ 1 - we^{-\frac{\lambda}{2}(1-e^{-\lambda})} \right]^{-v-1} \exp \left( \frac{\nu \lambda}{2} \right) \phi_v \left[ y; \log \left( e^{\frac{\lambda}{2}} - we^{-\frac{\lambda}{2}(1-e^{-\lambda})} \right) \right], \\
\left| we^{-\frac{\lambda}{2}(1-e^{-\lambda})} \right| < 1.
\]

11. \[
\sum_{k=0}^{\infty} \frac{(-y)_k}{k!(-N)_k} K_{v-k} (y-k; \theta, N-k) (w\sqrt{1-\theta^{-1}})^k \\
= K_v \left[ y; \theta(1+w\theta\sqrt{1-\theta^{-1}})^{-1}, N \right].
\]

12. \[
\sum_{k=0}^{\infty} \frac{(-1)^k (1+N)_k}{k!} K_{v+k} (y; \theta, N) \left( \frac{w}{\sqrt{1-\theta^{-1}}} \right)^k \\
= (1+w\theta\sqrt{1-\theta^{-1}})^{-N-1} \left( 1 - \frac{w\theta^{-1}}{\sqrt{1-\theta^{-1}}} \right)^{w-y-v} K_{v} \left[ y; \theta(1+w\sqrt{1-\theta^{-1}}), N \right],
\]
where \( \left| w\sqrt{1-\theta^{-1}} \right| < 1 \), provided \( 0 < \theta < 1 \), \( y = 0, 1, 2, 3, ..., N \).

These are the linear generating functions for the Laguerre, Hermite (even and odd), Meixner, Gottlieb and Krawtchouk polynomials, respectively, in which some of them are believed to be new in the theory of special functions.
CONCLUSION

In this Chapter, we have discussed the representation $SL(2, \mathbb{C})$ {a complex special linear group} and used it to derive the generating relations for $R_n(\beta; \gamma; x, y)$ and $\psi_{\alpha, \beta, 2(u+1), m-u-1}(x)$ with respect to a suitable basis. The suitable $J$-operators are considered and those have generated a three dimensional Lie algebra of generalized Lie derivatives of multiplier representation of $SL(2, \mathbb{C})$ of $sl(2, \mathbb{C})$.

We have obtained various generating functions for $R_n(\beta; \gamma; x, y)$ and $\psi_{\alpha, \beta, 2(u+1), m-u-1}(x)$ using this group theoretic method. Each generating function is followed by its applications to the classical orthogonal polynomials.