CHAPTER-IV

THERMOSOLUTAL CONVECTION WITH FINITE AMPLITUDE IN A ROTATING FLUID UNDER THE EFFECT OF MAGNETIC FIELD

4.1 Introduction

Thermosolutal onvection was first considered by Veronis [1] subjected to a salinity gradient produced at the interface of a layer of fluid heated from below. Sengupta and Gupta [2] generalized the problem by taking into account the rotational effect of the system. It was also pointed out by them that for disturbances of finite amplitude subcritical instability may occur revealing the fact that the system becomes unstable to infinitesimal disturbances. This work yields a conclusion that due to the magnetic influence the critical Rayleigh number decreases giving rise to the fact that the system has a destabilizing effect for the magnetic field. Further occurrence of subcritical instability is possible in this situation also for finite amplitude disturbances.

4.2 Mathematical Formulation

Taking z-axis as vertical a horizontal layer of fluid heated and salted from below is considered. The layer is under the effect of rotation with angular velocity $\Omega$ about z-axis and magnetic field is influencing it. The two bounding surfaces are taken as free and perfect conductors of heat and salt. For the sake of convenience two dimensional motion is

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1 The content of this published paper is enclosed in the pocket of this thesis at the end.
considered. Following Sengupta and Gupta [2], and Chandrasekhar [3] under Boussinesque approximation the guiding equations are

$$\frac{\partial q}{\partial t} + q \cdot v = -\frac{1}{\rho_m} v \cdot \mathbf{p} - 2\Omega \times q + q \cdot g(\alpha\mathbf{T} - \beta S) + v \cdot v^2 q + Q_1 v \cdot (v^2 h) \tag{4.1}$$

Where

$$q = (u, v, w), \quad h = (0, 0, h_z)$$

$$\frac{\partial T}{\partial t} + q \cdot v T - \omega = k_T v^2 T$$ \tag{4.2}

$$\frac{\partial S}{\partial t} + q \cdot v S - \omega = k_T v^2 T$$ \tag{4.3}

$$\frac{\partial h_z}{\partial t} + q \cdot v h_z = \eta v^2 h_z + H_z \frac{\partial w}{\partial z}$$ \tag{4.4}

$$v \cdot q = 0$$ \tag{4.5}

where \( \rho = \rho_m [1 - \alpha T - \beta S] \) \tag{4.6}

To non-dimensionalise the equations we introduce the dimensionless variables starred as,

$$q = \frac{k_T}{d} q^*, \quad t = \frac{d^2}{k_T} t^*, \quad (x, y, z) = d(x^*, y^*, z^*)$$

$$T = \Delta T T^*, S = \Delta S S^*, \quad \omega = \frac{p^* d^2}{\rho_m v k_T}, \quad \psi = k_T \psi^*$$

$$h_z = \Delta h_z K^*$$ \tag{4.7}

And the stream function as

$$u = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x}$$ \tag{4.8}

So that equation of continuity (4.5) is satisfied.
Utilizing (4.7) and (4.8) in (4.1)-(4.4) and eliminating \( \omega \), after dropping star we get,

\[
\left( \frac{1}{\sigma} \frac{\partial}{\partial t} - v^2 \right) \psi = -R \frac{\partial T}{\partial x} + R_s \frac{\partial S}{\partial x} + \frac{1}{\sigma} J(\psi, v^2 \psi) \tag{4.10}
\]

\[
- Q_1^* \frac{\partial^2}{\partial z \partial x} v^2 K
\]

\[
\left( \frac{1}{\sigma} \frac{\partial}{\partial t} - v^2 \right) v = -Y \frac{\partial \psi}{\partial z} + \frac{1}{\sigma} J(\psi, v) \tag{4.10}
\]

\[
\left( \frac{\partial}{\partial t} - v^2 \right) T = J(\psi, T) - \frac{\partial \psi}{\partial x} \tag{4.11}
\]

\[
\left( \frac{\partial}{\partial t} - \tau v^2 \right) S = J(\psi, S) - \frac{\partial \psi}{\partial x} \tag{4.12}
\]

And

\[
\left( \frac{\partial}{\partial t} - \eta' v^2 \right) K = HDw + J(\psi, K) \tag{4.13}
\]

Where

\[
\frac{H_z}{\Delta h_z} = H \quad \tau = \frac{k_s}{k_T} \quad \sigma = \frac{v}{k_T} \quad R = \frac{g \alpha \Delta T d^3}{v k_T},
\]

\[
R_s = \frac{g \beta \Delta S d^3}{v k_T} \quad \eta' = \frac{\eta}{k_T} \quad \text{and} \quad D = \frac{d}{dz}
\]
R and $R_s$ are thermal and solutal Rayleigh number respectively, $\sigma$ is the Prandtl number, $Y^2$ is the Taylor number.

Equations (4.9)-(4.13) are subjected to the following boundary conditions

$$\Psi = D^2 \psi = T = S = \frac{\partial v}{\partial z} = \frac{\partial K}{\partial z} = 0$$  at $z=0$ and $z=1$.

The linearised equations when the basic state is perturbed by infinitesimal disturbances become

$$\left( 1 - \frac{\partial}{\partial t} - v^2 \right) v^2 \psi = -R \frac{\partial T}{\partial x} + R_s \frac{\partial S}{\partial x}$$

$$- Q_1 \frac{\partial^2}{\partial x \partial z} v^2 K$$

$$\left( 1 - \frac{\partial}{\partial t} - v^2 \right) v = -Y \frac{\partial \psi}{\partial z}$$

$$\left( \frac{\partial}{\partial t} - v^2 \right) T = - \frac{\partial \psi}{\partial x}$$

$$\left( \frac{\partial}{\partial t} - v^2 \right) S = - \frac{\partial \psi}{\partial x}$$

$$\left( \frac{\partial}{\partial t} - \eta v^2 \right) K = - H \frac{\partial^2 \psi}{\partial x \partial z}$$

And

$$\left( \frac{\partial}{\partial t} - \eta' v^2 \right) K = - H \frac{\partial^2 \psi}{\partial x \partial z}$$

Let us assume the solutions (4.14)-(4.18) in the form

$$\psi = A e^{pt} \sin \pi x \sin \pi z$$
Substituting (4.19) in (4.14)-(4.18) and eliminating A, B, C, D, E we get for the lowest mode \( n=1 \),

\[
R_{\text{steady}} = \frac{R_s}{\tau} + \frac{\pi^4(1+\alpha')^3}{\alpha^2} + \frac{Y^2}{\alpha^2} - \frac{Q(1+\alpha^2)}{\alpha \eta'} \quad \text{(4.20)}
\]

Where \( Q = Q_1 H \)

For the marginal state being oscillatory \( p = ip_1, p_1 \) is real we get

\[
R_{ov} = \left[ \pi p_1^4 \left( \tau + 1 + 2\sigma + \eta' \right) - p_1^2 x^3 \right] \left[ (\tau + 1) \sigma^2 + 2\tau + \eta' \left\{ \tau + \sigma^2 + 2\tau \right\} \left( \tau + 1 \right) \right] + \frac{\eta' \sigma x^2 - p_1^2 \left( \sigma + 1 + \eta' \right)}{\sigma \tau \eta' x^2 - p_1^2 \left( \sigma + \tau + \eta' \right)} \}
\]

\[
+ R_s \left\{ \eta' \sigma x^2 - p_1^2 \left( \sigma + 1 + \eta' \right) \right\} \right\} \left\{ \sigma \tau \eta' x^2 - p_1^2 \left( \sigma + \tau + \eta' \right) \right\} \}
\]

\[
+ (\sigma Y^2/\alpha^2) \left\{ \eta' \tau x^2 - p_1^2 \left( 1 + \tau + \eta' \tau \right) \right\} \left\{ \sigma \tau \eta' x^2 - p_1^2 \left( \sigma + \tau + \eta' \right) \right\}
\]

\[
\frac{Q}{\alpha} = \frac{x \left\{ \sigma \tau x^2 - p_1^2 \left( \sigma + \tau + 1 \right) \right\} \right\} \left\{ \sigma \tau \eta' x^2 - p_1^2 \left( \sigma + \tau + \eta' \right) \right\} \quad \text{(4.21)}
\]

Where \( x = \pi^2 (\alpha^2 + 1) \). \quad \text{(4.22)}

Equation (4.20) gives \( R_{\text{steady}} \) if \( \alpha \) satisfies

\[
\eta' [2\pi^4 (2\alpha^2 - 1) (1 + \alpha^2)^2 - 2Y^2] + Q \pi^2 \alpha (1 - \alpha^2) = 0 \quad \text{(4.23)}
\]

Separating imaginary part of (4.21) after a bit simplification we get,
\[ p_1^4 - p_1^2 \left[ x^2 \left( \tau + \eta' + \sigma^2 + \eta' + 2\sigma (\eta' + \tau + 1) \right) \right] + \frac{\sigma}{x} \left\{ R_s \sigma^2 \pi^2 - R \alpha^2 \pi^2 + \sigma \ Y^2 \pi^2 - Q \alpha \pi^2 \right\} + x^4 \left\{ \sigma^2 (\tau + \eta' + \eta') + 2\sigma \tau \eta' \right\} + x \left\{ \sigma^2 \pi^2 R_s \sigma (\sigma + \eta' + \eta') - R \alpha^2 \pi^2 \sigma (\sigma + \eta' \sigma + \eta' \tau) + \sigma^2 Y^2 \pi^2 (\tau + \eta' + \eta' \tau) - Q \alpha \pi^2 \sigma (\sigma \tau + \sigma + \tau) \right\} = 0 \quad \text{(4.24)} \]

This is a quadratic equation in \( p_1^2 \) involving \( R \). It is clear from equation (4.24) that if \( R < 0 \), \( Q < 0 \) there are two changes of sign of (4.24) assuring existence of overstability when the discriminant of this equation is positive.

Elimination of \( R \) from (4.21) and (4.24) yields an equation of third degree in \( p_1^2 \). For the existence of overstability the parameters be such that it has a real root which must be positive. Solving that cubic equation in \( p_1^2 \) we get \( R_{ov} \) from (4.21).

4.3 Numerical Results

Figure 4.1 depicts \( \alpha \) versus \( R/10^5 \) for various values of \( \eta' \) and \( \tau \). It is clear that as \( \tau \) increases \( R_{\text{steady}} \) decreases giving rise to destabilizing effect to the system and it confirms the result obtained by Gupta and Sengupta [2]. From fig 4.2 as \( Y \) increases \( R_{\text{steady}} \) increases giving the stabilizing effect. \( Q \) has destabilizing effect to the system which is observed from fig. 4.3. Also the inference that as \( R_s \) increases \( R_{\text{steady}} \) increases for the case when salted from below can be drawn from the same figure. Fig. 4.4 shows that as \( \eta' \) increases \( R_C \) increases and takes its asymptotic value for large \( \eta' \). The fact that for large \( \eta' \) the system is practically unaffected.
It is obvious from fig. 4.5 that increase of $\tau$ decreases $R$ revealing the fact that $\tau$ has destabilizing effect to the system as found in the case of stationary convection. Fig. 4.6 ensures the oscillatory behavior of $R_c$ with $\tau$. The attractive decrease of $R$ due to large $\eta'$ or compared to the marginal state also implies the analogous effect of $\eta'$ in the instability phenomena which can be visualized from fig 4.7. From fig. 4.6 the effect of Prandtl number in the instability mechanism cannot be ignored since this parameter not only changes $R_c$ but also changes the critical wave number to a great extent.

4.4 Finite Amplitude Steady Convection

If disturbances are of finite amplitude and convection being steady $\frac{\partial}{\partial t} = 0$ in (4.9)-(4.13). Following perturbation method due to Veronis [4] we express all dependent variables in powers of the amplitude $\varepsilon$. So $\Psi$ can be taken as

$$\Psi = \varepsilon \Psi_o + \varepsilon^2 \Psi_1 + \varepsilon^3 \Psi_2 + \ldots$$

and similarly for other variables.

After substitution of the expressions for $\Psi$, $\nu$, $T$, $S$, $R$, $K$ into the governing equations and collecting the co-efficient of $\varepsilon$, $\varepsilon^2$, $\varepsilon^3$, etc. we eliminate $\nu_o$, $T_o$, $S_o$, $K_o$. From the first equation of each set and obtain for $\Psi_o$

$$\mathcal{L} \Psi_o = - [\nu^2 + y^2 + (\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} \eta') - \frac{\partial^4}{\partial x^2 \partial z^2} ] \Psi_o = 0$$

(4.25)

Where $Q = Q*$, $H$. 
Solution of (4.25) under above mentioned boundary conditions for the lowest mode i.e., n=1 are,

\[ 2 \Psi_0 = \frac{-2}{\pi \alpha} \sin \pi \alpha \sin \pi z \]

\[ \nu_0 = \frac{2Y}{\alpha \pi^2 (\alpha^2 +1)} \sin \pi \alpha \cos \pi z \]

\[ T_0 = \frac{2}{\pi^2 (\alpha^2 +1)} \cos \pi \alpha \sin \pi z \]

\[ K_0 = \frac{2H}{\eta' \pi (\alpha^2 +1)} \cos \pi \alpha \cos \pi z \]

Similarly eliminating \( \nu_i, T_1, S_1, K_1 \) from the second equation of each set and \( \nu_2, T_2, S_2, K_2 \) from the third equation of each set we derive the equations for \( \Psi_1 \) and \( \Psi_2 \) as,

\[ \mathcal{L} \Psi_1 = -R_1 \frac{\partial^2 \psi_o}{\partial x^2} - \frac{\partial}{\partial \tau} \left[ J(\psi_o, \nu_o) \right] + R_o \frac{\partial}{\partial \zeta} \left[ J(\psi_o, T_0) \right] + R_1 \frac{1}{\sigma} \frac{\partial}{\partial \tau} \left[ J(\psi_o, S_0) \right] + R_o \frac{\partial^2}{\partial \tau \partial \zeta} \left[ J(\psi_o, k_o) \right] \]

\[ \mathcal{L} \Psi_2 = -R_2 \frac{\partial^2 \psi_o}{\partial x^2} + R_1 \frac{\partial^2 \psi_o}{\partial x^2} + R_1 \frac{\partial}{\partial \tau} J(\psi_o, T_0) + R_o \frac{\partial}{\partial \tau} J(\psi_o, T_1) \]
\[
\begin{align*}
J(\psi_1, T_o) & \quad \frac{Y}{\sigma} \frac{\partial}{\partial z} \quad J(\psi_o, v_1) + J(\psi_1, v_o) \quad \frac{R_S}{\eta} \frac{\partial}{\partial x} \\
J(\psi_1, S_o) & \quad \frac{1}{\sigma} \left[J(v^2\psi_o, v^4\psi_1) + J(v^2\psi_1, v^4\psi_o)\right] + \frac{Q_1^*}{\eta^*} \frac{\partial^2}{\partial x \partial z} \\
& \quad [J(\psi_o, K_1) + J(\psi_1, K_o)] \\
\end{align*}
\]

(4.28)

Utilizing solutions (4.26) in (4.27) we get,

\[\mathcal{L} \Psi_1 = H^2 \alpha^2 R_1 \psi_0\]

Arguing like Veronis [4] is calculated so as to cancel the form of \(\psi_o\) from the right hand side of the above equation because a term of the form \(\psi_o\) will be a secular term and its presence may hamper the periodicity of the solution. Hence \(R_1 = 0\) so that \(\mathcal{L} \Psi_1 = 0\). Its solution under the boundary conditions is \(\Psi_1 = 0\). Thus one obtains

\[\begin{align*}
v_1 &= \frac{Y}{2\pi^3 \alpha^3 \sigma(\alpha^2 +1)} \sin 2\pi\alpha x \\
T_1 &= -\frac{1}{2\pi^3 (\alpha^2 +1)} \sin 2\pi z \\
S_1 &= -\frac{1}{2\alpha^2 \pi^3 (\alpha^2 +1)} \sin 2\pi z \\
K_1 &= \frac{H}{2\eta^2 \pi^2 \alpha^2 (\alpha^2 +1)} (\cos 2\pi\alpha x - \alpha^2 \cos 2\pi z) \\
\end{align*}\]

(4.29)

Utilizing all these we obtain from (4.28)
\[ L \Psi_2 = \left[ -2\pi \alpha R_2 - \frac{Y^2}{\pi \alpha^3 \sigma^2 (\alpha^2 + 1)} - \frac{\alpha R_S}{\pi \tau^3 (\alpha^2 + 1)} + \frac{\alpha R_o}{\pi (\alpha^2 + 1)} \right] \]

\[- \frac{8Q\pi \alpha}{\eta^3 (1 + \alpha^2)} \sin \pi \alpha \sin \pi z + \frac{Y^2}{\pi \alpha^3 \sigma^2 (\alpha^2 + 1)} \sin 3\pi \alpha \sin \pi z \]

\[+ \frac{\alpha R_S}{\pi \tau^3 (\alpha^2 + 1)} - \frac{\alpha R_o}{\pi (\alpha^2 + 1)} \sin 3\pi z \sin \pi \alpha \]

\[+ \frac{12Q \pi \alpha}{\eta^3 (1 + \alpha^2)} (\cos^2 \pi z + \cos^2 \pi \alpha \pi z) \]

\[ \text{......................... (4.30)} \]

Obviously the first term on the right hand side of (4.30) has the form of \( \Psi_0 \) and hence should vanish. This gives

\[ R_2 = \frac{R_o}{2\pi^2 (\alpha^2 + 1)} - \frac{R_S}{2\tau^3 \pi^2 (\alpha^2 + 1)} - \frac{Y^2}{2\sigma^2 \pi^2 \alpha^4 (\alpha^2 + 1)} + \frac{4Q}{\eta^3 (1 + \alpha^2)} \]

\[ \text{................. (4.31)} \]

Where \( R_o \) is the critical Rayleigh number at the onset of stationary convection with respect to infinitesimal disturbances. This reveals the fact that the system becomes unstable to finite amplitude steady disturbances before it becomes unstable to disturbances of infinitesimal disturbances.

Due to the presence of magnetic field a reverse effect is taking place leading to the conclusion that \( \eta' \) has destabilizing effect to the system.

From (4.31) it can be remarked that a stable salinity gradient and rotation reinforce each other in causing subcritical instability whereas the magnetic field has reverse behaviour to the instability phenomena.
REFERENCES


Fig. 4.1
\(\alpha\) versus R/10^5 marginal stability curve when Y = 10^4, \(R_S = 100\) and Q = 10 for different \(\tau\).
Fig. 4.2

$\alpha$ versus $R/10^5$ marginal curve when $R_S = 100$, $\tau = 0.01$, $Q = 10$, $\eta' = 10$ for different $Y$. 
Fig. 4.3

$R_s/10^3$ versus $R_c/10^5$ marginal curve when $Y = 10^4$, $\tau = 0.01$, $\eta = 0.1$ for different $Q$. 
Fig. 4.4

Log $\eta'$ versus $R_c/10^5$ marginal stability curve when $Y = 1000, \tau = 0.01, Q = 10, R_s = 100$
R/10^5 versus α oscillatory curve when \( R_S = 100, \sigma = 0.4, Q = 10, Y = 10^4, \eta' = 0.1 \) for different \( \tau \)
Fig. 4.6

$R_c/10^5$ versus $\tau$ curve when $R_s = 100$, $\sigma = 0.1$, $Q = 10$, $Y = 10^4$, $\eta' = 0.1$
Fig. 4.7

$R/10^5$ versus $\alpha$ curve when $R_s = 100$, $\sigma = 0.1$, $Q = 10$, $Y = 10^4$, $\tau = 0.07$, for different $\eta'$
Fig. 4.8

$R_c/10^2$ versus $\sigma$ curve when $R_s = 100$, $Q = 10$, $Y = 10^2$, $\eta' = 0.1$, $\tau = 0.07$