CHAPTER-VIII
Fluid Layer convection by concentration and Temperature dependent surface tension

8.1. Introduction

The hydrodynamic stability problem was governed by S. Chandrasekhar [1]. He showed the principle of exchange of stability is valid for the problem where surface tension is taken to be constant. Thermal convection was studied by pearson [6]. Thermosolutal convection was taken into account by veronis [7] without taking the buoyancy effect to the convective behaviour of the system. Sengupta and Gupta [4] though considered the problem together with rotation but did not consider even mode of solution to the problem. So in these respect those earlier works remained incomplete. The present paper follows a long line of investigation concerning the concentration and temperature dependent surface tension driven convection problem considering it in three aspects possessing three types of appropriate boundary condition and stability characteristic of different parameters e.g. τ, σ, Rs on the mechanism is studied more rigorously.

8.2. Formulation of the problem

A horizontal layer of fluid confined between the planes z = o and z = d is considered so that x axis is taken along the bottom surface and z axis is along the vertical. Considering the motion as two dimensional and letting the temperatures and concentration in the lower and upper boundary as T and T- ΔT and S and S- ΔS respectively the basic lineaised perturbed equations which being non-dimensional are given by
\[(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma}) \psi = iaR \theta - iaRs \phi \quad (8.1)\]

\[(D^2 - a^2 - p)\theta = ia\psi \quad (8.2)\]

and \[\{ \tau (D^2 - a^2) - p \} \phi = ia\psi \quad (8.3)\]

where \(D = \frac{d}{dz}\), time, velocity, distance, pressure, temperature concentration are scaled as \(k_t/d^2\), \(d/K_T\), \(d\), \(\frac{\rho_m v k_T}{d^2}\), \(\Delta T\) and \(\Delta S\)

Here \(\tau = \frac{k_S}{k_T}\) and the disturbance grows with \(e^{iaz + pt}\)

8.3. Analysis

Equation (8.1) - (8.3) to be solved under the following conditions.

(a) \(\psi = D^2 \psi = \theta = \phi = 0\) on \(z = 0\)  ..............................................(8.4)

\[D \psi = D^3 \psi = D \theta = D \phi = 0\) on \(z = 1\)

(b) \(\psi = D^2 \psi = \theta = \phi = 0\) on \(z = \pm \frac{1}{2}\) for even mode ........ (8.5)

(c) \(\psi = D^2 \psi = \theta = \phi = 0\) on \(z = \pm \frac{1}{2}\) for odd mode ........ (8.6)

Multiplying (8.1) by \(\psi^*(z)\), integrating between the limits of \(z\) and similarly multiplying (8.2) and (8.3) by \(\theta^*(z), \phi^*(z)\) respectively and then integrating between the limits, utilising either of the above three boundary conditions we can obtain after a simple manipulation by variational method.

\[
\int_0^1 \left[ |D^2 \psi|^2 + 2a^2 |D \psi|^2 + a^4 |\psi|^2 - R (|D \theta|^2 + a^2 |\theta|^2) \right] + R_{s\tau} \left( |D \phi|^2 + a^2 |\phi|^2 \right) \, dz + \frac{D}{\sigma} \int_0^1 |D \psi|^2 + a^2 \frac{D}{\sigma} \int_0^1 |\psi|^2 \\
- Rp^* \int_0^1 |\theta|^2 \, dz + R_{s\phi} \int_0^1 |\phi|^2 \, dz = 0 \quad ........... (8.7)
\]
Separating imaginary part of (8.7)

\[
\frac{D}{\sigma} \left[ \int \{ |D\psi|^2 + a^2 |\psi|^2 + R \sigma|D\theta|^2 - R_s\sigma|\phi| \} dz \right] = 0 \quad (8.8)
\]

where the range of \( z \) covers from lower to upper surface.

when \( R > 0 \), \( R_s < 0 \) the expression inside the third bracket is positive definite leading to the conclusion that \( P_i \) must be zero. So principle of exchange of stability is established for the system being heated from below and salted from above.

(8.3a) Assuming solution of (8.1) - (8.3) subject to the boundary condition (8.4) in the form

\[
\psi = A e^{i\alpha + pt} \sin \{(2n + 1) \frac{\pi}{2} z \} \quad (8.9)
\]

\[
\theta = B e^{i\alpha + pt} \sin \{(2n + 1) \frac{\pi}{2} z \} \quad (8.10)
\]

\[
\varphi = D e^{i\alpha + pt} \sin \{(2n + 1) \frac{\pi}{2} z \} \quad (8.11)
\]

substituting (8.9)-(8.11) in (8.1)-(8.3) and eliminating \( A, B \) and \( D \) we obtain,

\[
(k_n^2 + a^2) (k_n^2 + a^2 + \frac{p}{\sigma}) = \frac{Ra^2}{k_n^2 + a^2 + p} - \frac{R_s a^2}{\tau (k_n^2 + a^2) + p} \quad (8.12)
\]

where \( k_n = (2n + 1) \frac{\pi}{2} \)

for steady case considering the lowest mode \( n = 1 \)

\[
R_{st} = \frac{R_s}{\tau} + \frac{(k_n^2 + a^2)^3}{a^2} \quad (8.13)
\]

From (13) \( R_{critical} \) will be obtained for \( a = 3.332162 \) and given by \( \frac{R_s}{\tau} + 3328.651283 \) when the marginal state is oscillatory, \( p = ip_1 \), \( p_1 \) being real.
Separating real and imaginary parts of (8.12)

\[
R_{\text{overstability}} = \frac{1}{\sigma a^2} \left[ k_1^2 \{ \sigma k_1^2 - p_1^2 \} \right] + R_s \frac{\tau^2 k_1^2 + p_1^2}{\tau^2 k_1^2 + p_1^2}
\]  
(8.14)

together with

\[
p_1^2 = R_s a^2 \frac{(1 - \tau)}{(1 + \sigma) k_1} - \tau^2 k_1^2
\]  
(8.15)

(8.3b) Thanking solution of (8.1) - (8.3) subject to the boundary condition (8.5) for even mode.

\[
\psi = A_e \cos (2n + 1) \pi z
\]  
(8.16)

\[
\theta = B_e \cos (2n + 1) \pi z
\]  
(8.17)

\[
\phi = D_e \cos (2n + 1) \pi z
\]  
(8.18)

Proceeding analogously as (8.3a) we get the same expression for \( R_{\text{stationary}} \), \( R_{\text{ov}} \) and \( p_1^2 \) only \( k_n = (2n + 1)\pi \) should be taken.

(8.3c) solution of (8.1) - (8.3) subject to the boundary condition (8.6) for odd mode can be taken as

\[
\psi = A_o \sin 2n\pi z
\]  
(8.19)

\[
\theta = B_o \sin 2n\pi z
\]  
(8.20)

and

\[
\phi = D_o \sin 2n\pi z
\]  
(8.21)

Replacing \( k_n \) by \( 2n\pi \) we get some expression of \( R_{\text{st}}, R_{\text{ov}} \) and \( p_1^2 \) as obtained in (8.13) - (8.15).

8.4. Finite amplitude steady convection

For finite amplitude steady convection \( p \) is necessarily zero and \( \psi \) has the expansion

\[
\psi = \varepsilon \psi_o + \varepsilon^2 \psi_1 + \varepsilon^3 \psi_2 + \ldots.
\]  
(8.22)
\[ R = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots \quad (8.23) \]

\( \theta, \phi \) has expansion like \( \psi \) given in (8.22).

Substituting in (8.1) - (8.3) and collecting coefficients of \( \varepsilon, \varepsilon^2, \ldots \) we get,

\[ L \psi_0 = - \left[ \nabla^6 + \left( \frac{R_0}{\tau} - R_0 \right) \frac{\partial^2}{\partial x^2} \right] \psi_0 = 0 \quad (8.24) \]

Similarly eliminating \( \theta_1, \phi_1 \) from the 2nd equation and \( \theta_2, \phi_2 \) from the third equation of each set we derive the equation for \( \psi_1 \) and \( \psi_2 \) as

\[ L \psi_1 = - R_1 \frac{\partial^2 \psi_0}{\partial x^2} \quad (8.25) \]
\[ L \psi_2 = - R_2 \frac{\partial^2 \psi_0}{\partial x^2} - R_1 \frac{\partial^2 \psi_1}{\partial x^2} \quad (8.26) \]

The solution to the linearised stability problem satisfying boundary condition,

\[ \begin{align*}
\psi_0 &= D^2 \psi_0 = \theta_0 = \phi_0 = 0 \text{ on } z = 0 \\
D \psi_0 &= D^3 \psi_0 = D\theta_0 = D\phi_0 = 0 \text{ on } z = 1 
\end{align*} \quad (8.27) \]

\[ \psi_0 = - \frac{1}{ia} e^{iax} \sin \frac{3\pi}{2} z \quad (8.28) \]

\[ \theta_0 = \frac{1}{k_1^2 + a^2} e^{iax} \sin \frac{3\pi}{2} z \quad (8.29) \]

and \( \phi_0 = \frac{1}{\tau(k_1^2 + a^2)} e^{iax} \sin \frac{3\pi}{2} z \quad (8.30) \)

corresponding to the lowest mode \( n = 1 \).

Taking \( \psi = B_1 e^{iax} \sin (2n+1) \frac{\pi}{2} z \quad (8.31) \)

we obtain from (8.24),

\[ R_o = \frac{R_0}{\tau} + \frac{(k_1^2 + a^2)^3}{a^2} \]
which is the same expression obtained for $R_{st}$ in §13.

\[ L\psi_1 = -\pi^2 a^2 R_1\psi_0 \quad (8.32) \]

Following Veronis, to remove the presence of $\psi_0$ on R.H.S. we demand $R_1 = 0$,

\[ L\psi_1 = 0 \text{ with b.c (8.4) for } \psi_1 \text{ and yields } \psi_1 = 0 \text{ is the solution.} \]

From equation (8.26),

\[ L\psi_2 = -R_2 \frac{\partial^2 \psi_0}{\partial x^2} - R_1 \frac{\partial^2 \psi_1}{\partial x^2} = -R_2 \frac{\partial^2 \psi_0}{\partial x^2} \quad (8.33) \]

Arguing as above $R_2 = 0$

So $R_1 = R_2 = ...... = 0$ \quad (8.34)
8.5. **Numerical results**

Fig. 8.1 explains the fact that critical Rayleigh number is small for overstability in compared to the case of marginal state. $\tau$ has the destabilizing effect to the system for most of the fluid $\tau \approx 1$, since increase of $\tau$ decreases $R_c$, critical Rayleigh number giving rise to instability mechanism. Fig. 8.3 points out the first results in case of even mode solution. In such solution for small $\tau$ the above property holds good but for large $\tau$ the opposite behaviour is observed from Fig. 8.4. For odd mode solution and for the long wave for which $a \to 0$, oscillatory instability does not exit which supports the theoretical result [5] and can be visualised from figure 8.5. The effect of $\tau$ to the system is analogous to the same obtained for even mode solution only change of wave number is occurred which can be realised from Fig. 8.6. It is well expected from figure 8.7 that $R_s$ versus $R_c/10^3$ will be a straight line of gradient depending upon the other parameters. It is interestingly seen that odd mode solution has the more destabilising effect than the even mode one. In the case of stationary state same behaviour is observed from Fig. 8.8 only the change found is that slope of the curve (straight line) increases for $\sigma=1, \tau = .01$ as compared to the marginal state for the same parameters.
Reference


Variation of $R/10^3$ with $a$ for $R_g = 100$, $t = 0.01$. Solid curve for stationary and dotted curve for oscillatory when $a = 1$. 

Fig. 8.1
Plots $R_e/10^3$ versus $\tau$, $R_s = 100$. Solid curve for stationary and dotted curve for oscillatory when $\sigma = 0.3$. 

Fig. 8.2
Fig. 8.3

Variation of $R/10^3$ with $a$ for even mode solution when $R_s = 100$, $\tau = 0.01$. —— curve for stationary and dotted curve for oscillatory when $\sigma = 1$. 
Fig. 8.4
\( \tau \) versus \( R_c/10^3 \) when \( R_s = 100 \) for even mode solution. —— curve for stationary and dotted curve for oscillatory when \( \sigma = 0.3 \).
Fig. 8.5
Variation of $R/10^3$ with $\alpha$ for odd mode solution when $R_S = 100$, $\tau = 0.01$. ——— curve for stationary and dotted curve for oscillatory when $\sigma = 1$. 
Fig. 8.6
Variation of $R_C/10^3$ with $\tau$ for odd mode solution when $R_S = 100$. ——— curve for stationary and dotted curve for oscillatory when $\sigma = 0.3$. 
Fig. 8.7
Rs versus $R_C/10^3$ stationary curve when $\tau = 0.01$, $\sigma = 1$. —— curve for different lower and upper boundary conditions, XXXXX-for even mode and --- for odd mode.
Fig. 8.8
Rs versus $R_C/10^3$ oscillatory curve when $\tau = 0.01, \sigma = 1$. ——— curve for different two boundaries, xxxxxx for even mode solution and ——— for odd mode solution.