Chapter 4

ANALYSIS OF DIRECTION OF ARRIVAL ALGORITHMS

4.1 Normalized Power Method

First the data is obtained which is arranged in the format of time series. After that on the data rectangular window is applied. The estimates for directions [7] can be performed using the following equation.

\[ P_{Normalized} = \frac{S_v^H(\theta) T \text{C} S(\theta)}{N_{\text{antenna}}} \]  

(4.1)

where

- \( S_v(\theta) = \text{Steering vector for an angle } \theta \)
- \( S_v^H(\theta) = \text{Hermitian transpose of steering vector} \)
- \( T \text{C} = \text{total correlation matrix} \)
- \( N_{\text{antenna}} = \text{Number of antenna elements} \)

Array modified is a set of steering vector \( S_0 \) in relation with various direction \( \theta \) in DOA estimation. It is usually measured at the time of array calibration \( P_B(\theta) \) is evaluated using array manifold and an estimate of the array correlation matrix. The directions of radiating sources are indicated by peaks in \( P_B(\theta) \).

4.1.1 Mathematical Analysis of Normalized Power Method

1. The steering vector \( S_v(\theta) \) for an antenna array comprising of \( N_{\text{antenna}} \) elements is calculated by using

\[ S_v(\theta) = \begin{bmatrix} 1 \\ e^{-j \frac{2\pi}{\lambda} dsin(\theta)} \\ \vdots \\ e^{-j (N_{\text{antenna}} - 1) \frac{2\pi}{\lambda} dsin(\theta)} \end{bmatrix} \]  

(4.2)
Where, \( \theta \) = angle at which signal is falling
\( \lambda \) = operating wavelength for antenna
\( d \) = distance between antenna elements

2. 
\[
d = \frac{\lambda}{2} \tag{4.3}
\]
Dipole antenna is assumed in the array, to avoid grating lobes.

3. Substituting (4.3) in (4.2) we get the following

\[
S_v(\theta) = \begin{bmatrix}
1 \\
e^{-j\frac{2\pi}{\lambda} \sin(\theta)} \\
\vdots \\
e^{-j(\text{Nantenna} - 1)\frac{2\pi}{\lambda} \sin(\theta)}
\end{bmatrix} \tag{4.4}
\]

Simplifying equation (4.4) by cancelling the value of \( \lambda \) one can obtain

\[
S_v(\theta) = \begin{bmatrix}
1 \\
e^{-j\pi \sin(\theta)} \\
\vdots \\
e^{-j\pi(\text{Nantenna} - 1)\sin(\theta)}
\end{bmatrix} \tag{4.5}
\]

4. The amplitudes of the signal is arranged in the column vector format of order \( N_{\text{users}} \times 1 \)

\[
A_v = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{N_{\text{users}}}
\end{bmatrix} \tag{4.6}
\]

5. The hermitian transpose of the amplitude vector can be computed as below

\[
A_v^H = [a_1^*, a_2^*, \ldots, a_{N_{\text{users}}}^*] \tag{4.7}
\]

6. The signal correlation matrix is computed using the following equation

\[
SC = A_vA_v^H \tag{4.8}
\]
\[ SC = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_{\text{users}}} \end{bmatrix} [a_1^*, a_2^*, \ldots, a_{N_{\text{users}}}^*] \] (4.9)

\[ SC = \begin{bmatrix} a_1 a_1^* & a_1 a_2^* & \ldots & a_1 a_{N_{\text{users}}}^* \\ a_2 a_1^* & a_2 a_2^* & \ldots & a_2 a_{N_{\text{users}}}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{N_{\text{users}}} a_1^* & a_{N_{\text{users}}} a_2^* & \ldots & a_{N_{\text{users}}} a_{N_{\text{users}}}^* \end{bmatrix} \] (4.10)

7. The detection algorithm is build by assuming the cross correlation between two different signals as zero
\[ E\{x, y^*\} = 0 \] (4.11)

By using (4.10) and (4.11) we can obtain the following
\[ SC = \begin{bmatrix} a_1 a_1^* & \ldots & \ldots & 0 \\ 0 & a_2 a_2^* & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{N_{\text{users}}} a_{N_{\text{users}}}^* \end{bmatrix} \] (4.12)

8. The array manifold vector is computed for the set of \( N_{\text{users}} \) and is the combination of steering vectors across \( N_{\text{users}} \) directions \( \{\theta_1, \theta_2, \ldots, \theta_{N_{\text{users}}}\} \)
\[ A_M = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ e^{-j\pi\sin(\theta_1)} & e^{-j\pi\sin(\theta_2)} & \ldots & e^{-j\pi\sin(\theta_{N_{\text{users}}})} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\pi(N_{\text{antenna}}-1)\sin(\theta_1)} & e^{-j\pi(N_{\text{antenna}}-1)\sin(\theta_2)} & \ldots & e^{-j\pi(N_{\text{antenna}}-1)\sin(\theta_{N_{\text{users}}})} \end{bmatrix} \] (4.13)

9. The hermitian transpose of the array manifold vector is computed by using the following equation.
\[ A_M^H = \begin{bmatrix} 1 & e^{j\pi\sin(\theta_1)} & \ldots & e^{j\pi(N_{\text{antenna}}-1)\sin(\theta_1)} \\ 1 & e^{j\pi\sin(\theta_2)} & \ldots & e^{j\pi(N_{\text{antenna}}-1)\sin(\theta_2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j\pi\sin(\theta_{N_{\text{users}}})} & \ldots & e^{j\pi(N_{\text{antenna}}-1)\sin(\theta_{N_{\text{users}}})} \end{bmatrix} \] (4.14)
10. The Noise amplitude matrix can be defined as follows

\[ N_A = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_{N_{\text{antennas}}} \end{bmatrix} \]  

(4.15)

11. The Hermitian transpose of the Noise Amplitude Matrix can be defined as below

\[ N_A^H = [n_1^*, n_2^*, \ldots, n_{N_{\text{antennas}}}^*] \]  

(4.16)

12. The Noise correlation matrix can be computed using the following equation

\[ NC = N_A N_A^H \]  

(4.17)

13. Substituting the value of \( N_A \) from equation (4.15) and \( N_A^H \) from equation (4.16) in equation (4.17)

\[ NC = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_{N_{\text{antennas}}} \end{bmatrix} \begin{bmatrix} n_1^* \\ n_2^* \\ \vdots \\ n_{N_{\text{antennas}}}^* \end{bmatrix} \]  

(4.18)

Multiplication of the above matrices will result in the following equation

\[ NC = \begin{bmatrix} n_1 n_1^* & n_1 n_2^* & \ldots & n_1 n_{N_{\text{antennas}}}^* \\ n_2 n_1^* & n_2 n_2^* & \ldots & n_2 n_{N_{\text{antennas}}}^* \\ \vdots & \vdots & \ddots & \vdots \\ n_{N_{\text{antennas}}} n_1^* & n_{N_{\text{antennas}}} n_2^* & \ldots & n_{N_{\text{antennas}}} n_{N_{\text{antennas}}}^* \end{bmatrix} \]  

(4.19)

The method considers that the noise element of one antenna does not impact the other antenna element noise. Hence cross correlation between the noise elements is zero.

\[ NC = \begin{bmatrix} n_1 n_1^* & \ldots & \ldots & 0 \\ 0 & n_2 n_2^* & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & n_{N_{\text{antennas}}} n_{N_{\text{antennas}}}^* \end{bmatrix} \]  

(4.20)
14. The noise is an Additive White Gaussian Noise with a variance of $\sigma^2$. Hence

$$n_1n_1^* = n_2n_2^* = n_3n_3^* = \ldots = n_{\text{antennas}}n_{\text{antennas}}^* \tag{4.21}$$

Using the assumption as defined in (4.21) in (4.20) one can obtain the value

$$NC = \begin{bmatrix}
\sigma^2 & \ldots & \ldots & 0 \\
0 & \sigma^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \sigma^2
\end{bmatrix} \tag{4.22}$$

Taking $\sigma^2$ out one can obtain the following

$$NC = \sigma^2 \begin{bmatrix}
1 & \ldots & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix} \tag{4.23}$$

$$NC = \sigma^2 I \tag{4.24}$$

Where

$\sigma^2 = \text{variance of noise}$

$I = \text{Identity matrix}$

15. The total correlation matrix can be computed using the following equation

$$TC = A_vS C A_v^H + \sigma^2 I \tag{4.25}$$

16. The power spectrum for the normalized power is computed using the equation defined in (4.1) by varying the value of $\theta$ between $\frac{-\pi}{2}$ to $\frac{\pi}{2}$
Summary of Normalized Power method

4.2 Maximum Entropy Method (MEM)

As per Lang and McClellan algorithm for a signal Fourier transform must be followed by entropy computation. The MEM [11] power spectrum can be computed using

\[ P_{MEM} = \frac{1}{S_v(\theta)^H M_{EC} M_{EC}^H S_v(\theta)} \]  

(4.26)
where $S_v(\theta) = \text{Steering vector from an angle } \theta$

$S_v^H(\theta) = \text{Hermitian transpose of steering vector}$

$M_{EC} = \text{column in total coorelation matrix which corresponds to maximum entropy}$

$M_{EC}^H = \text{Hermitian transpose of } M_{EC}$

### 4.2.1 Mathematical analysis of MEM method

The mathematical analysis of Maximum Entropy Method is same as normalized power method from step1 to step 16 as discussed in section 4.1. The remaining steps are as given below

17. Compute the inverse of total correlation matrix as defined in equation (4.25)

$$TC_{inv} = \frac{1}{|TC|} \text{adjoint}(TC) \quad (4.27)$$

Where $|TC| = \text{Magnitude of total correlation matrix}$

$\text{adjoint}(TC) = \text{adjoin of total correlation matrix}$

18. Pick up the column from $TC_{inv}$ which corresponds to maximum entropy

$$M_{EC} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{N_{\text{antennas}}} \end{bmatrix} \quad (4.28)$$

Where $m_i = i^{th}$ element of M which is one of value of maximum entropy column.

19. Find the Hermitian transpose of equation (4.29)

$$M_{EC}^H = [m_1^*, m_2^*, \ldots, m_{N_{\text{antennas}}}^*] \quad (4.29)$$

20. Compute the value of power spectrum for MEM method using equation (4.27) by varying the value of $\theta$ between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
4.3 Multiple Signal Classification Method (MUSIC)

MUSIC [8] method provides the estimation of the mobile user directions with lesser bias as compared to traditional methods. MUSIC assumes that the noise vectors across the antenna elements are not correlated. MUSIC requires the knowledge of number of users
to be detected before estimation. The Subspace DOA algorithms perform extra step of
Eigen value decomposition apart from regular power spectrum computation.

MUSIC method computes the Eigen values for the total correlation matrix. After
computing the Eigen values. The values which have the lesser value are taken. The Eigen
vectors are found out for those Eigen values whose magnitude is less. These Eigen vectors
are used in the power spectrum estimation.

The MUSIC method power spectrum is given by

\[ P_{MUSIC} = \frac{1}{S_v(\theta)^H NSNS^H S_v(\theta)} \]  \hspace{1cm} (4.30)

where \( S_v(\theta) \) = Steering vector from an angle \( \theta \)

\( S_v^H(\theta) \) = Hermitian transpose of steering vector

\( NS \) = Noise Subspace

\( NS^H \) = Hermitian transpose of noise subspace

4.3.1 Mathematical analysis of MUSIC method

The steps for MUSIC algorithm are same as that of normalized power method from step1
to step11 as described in the section 4.1.1

12. The Eigen values of Total Correlation matrix is found by solving the characteristic
equation given by

\[ |TC - \lambda I| = 0 \]  \hspace{1cm} (4.31)

The solution to equation gives \( N_{\text{antennas}} \) values called Eigen values \( \{\lambda_1, \lambda_2, \ldots, \lambda_{N_{\text{antennas}}}\} \)

13. Sort the Eigen Values in ascending order

14. Pick the first \( N_{\text{antennas}} - N_{\text{users}} \) values

15. The Eigen Vector for specific Eigen value \( \lambda_a \) is found by solving the equation given by

\[ TCEV_n = \lambda_a EV_n \]  \hspace{1cm} (4.32)

Where \( EV_n \) is \( N_{\text{antennas}} \times 1 \) matrix comprising of unknown variables.
Expanding equation in matrix notation, one can obtain

\[
\begin{bmatrix}
tc_{0,0} & tc_{0,1} & \cdots & tc_{0,N_{\text{antennas}}} \\
tc_{1,0} & tc_{1,1} & \cdots & tc_{1,N_{\text{antennas}}} \\
\vdots & \vdots & \ddots & \vdots \\
tc_{N_{\text{antennas}},0} & tc_{N_{\text{antennas}},1} & \cdots & tc_{N_{\text{antennas}},N_{\text{antennas}}} 
\end{bmatrix}
\begin{bmatrix}
EV_1 \\
EV_2 \\
\vdots \\
EV_{N_{\text{antennas}}}
\end{bmatrix}
= \lambda_a
\begin{bmatrix}
EV_1 \\
EV_2 \\
\vdots \\
EV_{N_{\text{antennas}}}
\end{bmatrix}
\]

(4.33)

Multiplying the matrices, a set of simultaneous equations as defined in are obtained

\[
tc_{0,0}EV_1 + tc_{0,1}EV_2 + \cdots + tc_{0,N_{\text{antennas}}}EV_{N_{\text{antennas}}} = \lambda_a EV_1
\]
\[
tc_{1,0}EV_1 + tc_{1,1}EV_2 + \cdots + tc_{1,N_{\text{antennas}}}EV_{N_{\text{antennas}}} = \lambda_a EV_2
\]
\[
\vdots
\]
\[
tc_{N_{\text{antennas}},0}EV_1 + \cdots + tc_{N_{\text{antennas}},N_{\text{antennas}}}EV_{N_{\text{antennas}}} = \lambda_a EV_{N_{\text{antennas}}}
\]

(4.34)

(4.35)

Since there are \(N_{\text{antennas}}\) Unknowns we have \(N_{\text{antennas}}\) simultaneous equations which can be solved to obtain \(EV_1, EV_2, \ldots, EV_{N_{\text{antennas}}}\). These \(N_{\text{antennas}}\) values form Eigen vector matrix

16. The Step 15 is repeated for all the noise Eigen values and we can get the Noise Sub Space

\[
NS = \begin{bmatrix}
e_{1,1} & e_{1,2} & e_{1,3} \cdots & e_{1,N_{\text{antennas}},1} \\
e_{2,1} & e_{2,2} & e_{2,3} \cdots & e_{2,N_{\text{antennas}},2} \\
\vdots & \vdots & \vdots & \vdots \\
e_{1,N_{\text{antennas}},1} & e_{2,N_{\text{antennas}},2} & e_{3,1} \cdots & e_{N_{\text{antennas}},N_{\text{antennas}}}
\end{bmatrix}
\]

(4.36)

17. The Hermitian transpose of NS is computed

18. Compute the value of power spectrum for MUSIC method using equation (4.30) by varying the value of \(\theta\) between \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\).
Summary of MUSIC method

![Image of MUSIC method diagram]

**Figure 4.3: Music Algorithm**

4.4 **QR Decomposition Method**

1. The linear operator using upper triangular matrix R derived from QR factorization of spectral matrix, or from upper triangular matrix U obtained from LU decompo-
sition. The power spectrum for the gamma Operator can be shown as below

\[ P_\Gamma = \frac{1}{a^H(\theta)E_\Gamma E_\Gamma^H a(\theta)} \]  

where \( a(\theta) = \text{Steering vector for direction } \theta \)
\( a^H(\theta) = \text{Hermitian transpose of steering } \theta \)
\( E_\Gamma = \text{decreasing eigen values for eigen space} \)
\( E_\Gamma^H = \text{Hermitian transpose for } E_\Gamma \)

2. The decreasing Eigen space is found out by applying it on the Q orthonormal matrix the following equation \( \Gamma = QR \)

where, \( Q = \text{orthonormal matrix} \)
\( R = \text{upper triangular matrix} \)

3. The orthonormal matrix is then found out by using the following matrix

\[ Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix} \]  

\( Q_{ij} = \text{partitioned Q matrix.} \)

4. The propagator matrix is given by the following equation

\[ \pi^+ = -Q_{22}Q_{12}^+ \]  

5. The Eigen vectors are obtained for \( \pi^+ \)
4.5 LUI Method - Proposed DOA Algorithm

LU Factor and Interpolation Method finds the Eigen Signal subspace and then find the LU Factor and determine the directions and then finally substitute the value in the power spectrum equation. The Linear Algebra has defined LU Factorization which is incorporated in the proposed method. The second important change is the LU Factor method performs the interpolation of antenna array so that more accuracy can be obtained in the detection
of power spectrum. The power spectrum for LU Factor DOA method can be found using the following equation.

\[ P_{\text{Spectrum}} = \frac{1}{S^H \text{LU}_f \text{LU}_f^H S} \]  

where \( S = \text{Steering vector for angle } \theta \)

\( S^H = \text{Hermitian transpose} \)

\( \text{LU}_f = \text{LU factorization} \)

\( \text{LU}_f^H = \text{Hermitian transpose of LU factorization} \)

### 4.5.1 Mathematical Analysis of LUI Method

1. The LUI Method first computes the Interpolated manifold vector which provides the delay computation of EM waves for various angles.

\[ M_{AM} = \begin{bmatrix} 1 & 1 & 1 \\
 e^{-j \frac{2\pi}{N} \sin(\theta_1)} & e^{-j \frac{2\pi}{N} \sin(\theta_2)} & e^{-j \frac{2\pi}{N} \sin(\theta_N)} \\
 e^{-j \frac{4\pi}{N} \sin(\theta_1)} & e^{-j \frac{4\pi}{N} \sin(\theta_2)} & e^{-j \frac{4\pi}{N} \sin(\theta_N)} \\
 e^{-j (N_I-1) \frac{2\pi}{N} \sin(\theta_1)} & e^{-j (N_I-1) \frac{2\pi}{N} \sin(\theta_2)} & e^{-j (N_I-1) \frac{2\pi}{N} \sin(\theta_N)} \end{bmatrix} \]  

2. Compute the Hermitian transpose of the Interpolated Manifold vector \( M_{AM}^H \).
3. The amplitude vector for \( N \) number of users is computed using the following equation with the assumption of cross correlation is zero.

\[
A_v = \begin{bmatrix}
a_1a_1^* & \ldots & \ldots & 0 \\
0 & a_2a_2^* & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & a_Na_N^*
\end{bmatrix}
\]  

(4.42)

Where \( a_i \) = \( i^{th} \) amplitude value.

4. The total correlation matrix is computed using the following equation.

\[
TC = M_AM_SM_A^H + C_{noise}
\]  

(4.43)

5. The Eigen values are found out for the total correlation matrix and then the Eigen values are grouped which are having high dimension. The roots of the following equation are found to get Eigen values.

\[
|TC - \lambda I| = 0
\]  

(4.44)

The Eigen values are arranged in descending order and then first \( N_M \) values are picked.

\( \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{N_M}\} \) The above set represents the Eigen values of signal.

6. If \( \lambda \) is the Eigen value then Eigen vector \([31]\) is found by using the equation.

\[
TCe = \lambda e
\]  

(4.45)

Where, \( e \) = matrix of unknowns of order \( N \times 1 \)

TC = Total correlation of order \( N \times N \)

\( \lambda \) = eigen value

If we expand the above equation we get,

\[
\begin{bmatrix}
tc_{11} & tc_{12} & \ldots & \ldots & tc_{1N} \\
tc_{21} & tc_{22} & \ldots & \ldots & tc_{2N} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
tc_{N1} & tc_{N2} & \ldots & \ldots & tc_{NN}
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_N
\end{bmatrix} = \lambda
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_N
\end{bmatrix}
\]  

(4.46)

The above equation can be converted into the following
There are $N$ simultaneous equations and there are $N$ unknowns which can be solved to obtain the Eigen vector for $\lambda$ Eigen values

\[
\begin{align*}
tc_{11}e_1 + tc_{12}e_2 + \cdots + tc_{1N}e_N &= \lambda e_1 \\
tc_{21}e_1 + tc_{22}e_2 + \cdots + tc_{2N}e_N &= \lambda e_2 \\
&\vdots \\
tc_{N1}e_1 + tc_{N2}e_2 + \cdots + tc_{NN}e_N &= \lambda e_N
\end{align*}
\]  \quad (4.47)

7. The above process is repeated for all the $N_M$ eigen values to obtain the signal subspace.

Consider a set $\{\lambda_1, \lambda_2, \ldots, \lambda_{N_M}\}$ which are signal Eigen values then Eigen vectors are found out for all $N_M$.

Let $Sub_{signal}$ represent the signal subspace which combines the Eigen vectors for all the Eigen values then it is given by following equation

\[
Sub_{signal} = \begin{bmatrix}
e_{11} & e_{21} & \cdots & \cdots & e_{1N_M} \\
e_{12} & e_{22} & \cdots & \cdots & e_{2N_M} \\
& \vdots & \vdots & \vdots & \vdots \\
e_{1N} & e_{2N} & \cdots & \cdots & e_{NN_M}
\end{bmatrix}
\]  \quad (4.49)

Where each column represents the Eigen vector for corresponding Eigen value.

8. Perform the LU Factorization for Signal Subspace $LU_f$

$Sub_{signal} = LU$

$L$ = lower triangular matrix

$U$ = upper triangular matrix
In $L$ matrix the diagonal elements as 1 and above diagonal all values are zero and below diagonal it has values of certain magnitude.

$$L = \begin{bmatrix}
1 & 0 & \ldots & \ldots & 0 \\
L_{21} & 1 & \ldots & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
L_{N1} & L_{N2} & \ldots & \ldots & 1
\end{bmatrix} \quad (4.50)$$

$$U = \begin{bmatrix}
u_{11} & u_{12} & \ldots & \ldots & u_{1N} \\
0 & u_{22} & \ldots & \ldots & u_{2N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \ldots & u_{NN}
\end{bmatrix} \quad (4.51)$$

9. In $U$ matrix diagonal elements and above has certain values below diagonal are all zero. For the TC matrix the subspace is found out by using steps 5,6,7,8. For the signal subspace LU decomposition is performed.

10. The power spectrum is then computed using the following equation

$$P_{Spectrum} = \frac{1}{S^H S_{\theta} L U_f L U_f^H S_{\theta}} \quad (4.52)$$

where $S =$ Steering vector for angle $\theta$

$S^H =$ Hermitian transpose

$L U_f =$ LU factorization

$L U_f^H =$ Hermitian transpose of LU factorization
4.6 Performance Characteristics of DOA Algorithms

1. RMSE Computation: The RMSE [10] value can be computed using the following equation

\[
RMSE = \sqrt{\frac{1}{IL} \sum_{i=0}^{I-1} \sum_{l=0}^{L-1} (\theta_{estimated} - \theta_{actual})^2}
\]  

(4.53)
where \( L \) = Number of runs
\( \theta_{estimated} \) = estimated direction
\( \theta_{actual} \) = actual direction
\( I \) = number of iterations

2. **Bias:** An unbiased estimate is one whose expected value equal true value of the parameter. If the expected value differs from the parameter’s true value, the estimate is said to be biased.

\[
\theta_{bias} = |\theta_{estimated} - \theta_{actual}|
\]  
(4.54)

where \( \theta_{estimated} \) = estimated direction
\( \theta_{actual} \) = actual direction

3. **Resolution:** Resolution is defined as the estimates ability to reveal the presence of sources have equal energy and nearly equal angles. If two sources are resolved it gives two distinct peaks in the spectrum. If not resolved, only one peak is observed and the better resolved would have a narrow spectral peak.