Introduction

Cosmology, the science of cosmos, deals with the large scale structure of the universe. The problem of dynamics of the universe may be considered to be that of a smoothed out continuous distribution of matter interacting solely through gravitation. In the Newtonian scheme, the solution is not unique unless one has suitable boundary conditions which is plainly lacking in case of the universe [1]. A way of escape from the situation is to introduce some symmetry condition in place of boundary conditions. The most common idea is to consider the matter distribution to be homogeneous. This idea brings a fresh difficulty. Because of homogeneity the pressure gradient force vanishes and thus there is apparently no way to balance the gravitational interaction and to obtain an acceleration-free motion.

With the geometrization of Physics, as is the case in general relativity [2,3], one may overcome the above problem of boundary conditions with the assumption of spatial homogeneity. The most natural choice of the universe is the spatially homogeneous and isotropic universe, in which a dynamical equilibrium is maintained through the gravitational interaction and the kinetic energy of expansion. The spatially homogeneous and isotropic model of the universe based on the cosmological principle, which asserts that all fundamental observers at a given cosmic time notice the same large scale features of the universe and do not see any preferred direction, is not only convincing from the philosophical point of view, but also from observations. The Standard Cosmology [4,5,6,7], based on the cosmological principle and the Einstein's field equations, is well accepted from the following observational evidences:

Systematic study of spectral lines reveals recession of galaxies from each other and galaxies at a distance greater than 21.6 Mpc move away with a velocity proportional to the distance i.e. \( V = HD \). The expansion of the universe is isotropic (and homogeneous) and extrapolating the recession of galaxies backward in time we would find a singular origin of the universe and the age of the universe is \( H^{-1} \) since the singular big-bang epoch in the past.

One of the clearest pieces of evidence of the big-bang model is the cosmic microwave background radiation (CMBR), discovered by Penzias and Wilson (1967). The CMBR is the remnant radiation of the early hot universe. The important characteristics of CMBR are its isotropy and its Planckian spectrum corresponding to a temperature of 2.7 K. In the standard cosmology, at the early stage, matter and radiation were in thermal equilibrium and at high temperature no neutral atom existed, the electrons strongly interacted with photons so that one had blackbody radiation throughout the universe. As the universe cooled down due to expansion, at about certain temperature \( \sim 4000 K \) (recombination era) matter and radiation were decoupled which is also called the last scattering surface. The radiation gas subsequently cooled down by expansion and now it cooled down to 2.7 K.

The recession of galaxies and CMBR provide us with information on the evolution of the universe up to the recombination era (\( \sim 10^5 \) years) after the big-bang.
Light element abundances allow us to go back in time \( \sim 1 - 100 \text{ sec.} \) in the Standard Cosmology. In different stars within the same galaxy particle constituents vary, however there is a sufficient uniformity to permit one to talk of an universal abundance of lighter elements. There exist universal values \( \frac{n_e}{n} \simeq 0.3 \) and \( \frac{D}{H} \simeq 10^{-5} \) (by weight). The stellar nucleosynthesis could not explain this value, but provide solution from the primordial nucleosynthesis process in the standard big-bang model.

The Standard Cosmology is very successful in its predictions of the Hubble expansion law, the CMBR and the abundance of light elements. However these successes do not probe the behaviour of the model all the way back to the very early universe. There are problems in the Standard Cosmology. Some of the important problems [7,8,9] are the singularity problem, horizon problem, flatness problem, structure formation problem etc. All these problems involve times when the universe was much less than a second old. Some of the problems are connected with very special conditions that are required as the universe first emerged from the big-bang. On the other hand, some problems are conceptual and require more general attention on the theory of gravitation or quantum theory of gravitation. Let us explain a few of these problems.

I. Horizon Problem: [10] Let a light signal starting at \( t = 0 \) (i.e. at the instant of big-bang) from the origin travels upto the coordinate \( r \) in time \( t \), then with Friedmann metric

\[
ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],
\]

we have for photons

\[
\int_0^r \frac{dr}{\sqrt{1 - kr^2}} = \int_0^t \frac{dt}{a}.
\]

In the standard model as \( t \to 0, a \sim t^{\frac{1}{3}} \) (for radiation dominated era) and hence the righthandside of equation (2) converges to a finite value for any finite \( t \) which means that at any finite time light can travel only upto a finite distance. Thus at the early epochs each particle was causally connected to a limited number contained within the regions connected by light signals. This is referred to as the existence of horizons. The CMBR that we are receiving right now may be considered to be coming from the last scattering surface at redshift \( Z \simeq 1000 \). The microwave radiation should therefore bear the imprint of these last scattering surface; isotropy of the microwave radiation thus indicating a homogeneity over the last scattering surface. However in the standard model, causal connection was not established over this surface when the microwave radiation left it. This puzzle of explaining why the universe appears to be uniform over a distance large compared with horizon is called horizon problem.

II. The Flatness Problem: [11] The flatness problem is related to the observational fact that the present energy density of the universe \( \rho \) is within one or atmost two orders of magnitude different from the critical energy density \( \rho_c \) (by critical energy density we mean the energy density for a flat universe) and the ratio
could be effectively just equal to unity in the very early universe. The value \( \Omega = 1 \) represents an unstable equilibrium under time evolution. If \( \Omega \) is ever exactly equal to one, it will remain exactly equal to one forever. But if \( \Omega \) differs slightly from one, a few instants after the big-bang, then this deviation from one would grow rapidly with time. The standard model offers no explanation of why \( \Omega \) began so close to one, but merely assumes this as an initial condition.

One may resolve the horizon problem and the flatness problem by introducing inflationary models [12,13,14,15,16]. The idea of inflationary universe involves the following scenario:

1. As the universe cools down, at a certain stage, there is a spontaneous symmetry breakdown, so that the symmetric state (even if it may be a local minimum of the energy) does not correspond to the global minimum.

2. The potential function may be so flat that the transition from the symmetric state to the state of minimum energy takes a long time. During this period the vacuum energy stress-tensor simulates a cosmological term.

3. The dominating cosmological term results in an exponential expansion whereby the horizon are enormously increased in size and the curvature is reduced almost to zero i.e. we have in effect a solution to the horizon and flatness problem.

The inflationary scenario, first proposed by Guth [12], is a first order phase transition, in which the universe became trapped in metastable false vacuum state. According to the model the energy density of the universe became dominated by the vacuum energy density and resulted an exponential expansion. The phase transition proceeded by quantum tunneling to the true vacuum state, which led to the nucleation of bubbles of a true vacuum. The regions of the universe in the false vacuum state were expanding exponentially, the expansion was more effective than the decay to the true vacuum state, bubbles of true vacuum never percolated, and the energy density in the expanding walls of the true vacuum bubbles could not be converted to radiation through bubble-wall collisions. This problem is referred to as the graceful exit problem.

To rescue from the graceful exit problem several inflationary models viz. new inflationary model [13,14], chaotic inflationary model [14] were proposed, but all these models require fine tuning of the effective potential parameters. La and Steinhardt [17,18,19] received the idea of a first order phase transition in the theories in which the gravitational constant may vary, such as Jordan-Brans-Dicke theories [20,21,22,23]. This model is called extended inflation. In such theories the scale factor of the universe is a power law rather than exponential during inflation (in Brans-Dicke theory with parameter \( \omega \), the scale factor increases as \( t^{\omega+\frac{1}{2}} \) at late times). Then the exponential bubble nucleation rate will eventually overcome the power law expansion rate and the universe will exit from inflationary phase. The extended inflationary model may solve many of the fine tuning problems associated with new and chaotic inflation. The inflationary model based on the Jordan-Brans-Dicke theory is not without problems due to the existence of large vacuum bubbles at the end of inflation. Large bubbles will not have enough time to thermalize be-
fore recombination and hence will lead to distortions in the CMBR. Assuming these distortions to be sufficiently small, the Brans-Dicke parameter $\omega$ is then restricted to be less than 50 and possibly much smaller, which is in conflict with the lower bound, $\omega > 500$, from time-delay experiments. Several models have been proposed to resolve the problem of bubble thermalization, for instance, allowing $\omega$ to vary with time [24] (models of this type are called hyperextended inflation) or by incorporating the scenario into either an induced-gravity model [25,26] or a Kaluza-Klein model [27] or by modifying the gravitational couplings of the inflation [28]. These models suppress large bubbles in the inflationary epoch and hence allow sufficient thermalization of the bubbles before decoupling to remove microwave-background constraints.

The scalar-tensor theory of gravity is the simplest generalization of general theory of gravity in which the gravitational interaction is mediated by long range scalar field $\phi$ in addition to the usual tensor field $g_{\mu\nu}$ present in Einstein's theory. The strength of the coupling between the scalar field and gravity, in general, is determined by the coupling function $\omega(\phi)$. The Brans-Dicke theory is a special case of the general scalar-tensor theory of gravity in which $\omega$ is constant and its value is constrained by the classical test of general relativity. Bergmann, Wagoner and Nordtvedt [22,23,29] generalized the scalar-tensor theory in which the scalar field has a dynamical coupling with gravity and/or an arbitrary self interaction. Further, the recent unification schemes [30] of fundamental interaction based on supergravity or superstrings naturally associate a long range scalar partner to the usual tensor field present in Einstein's gravity. In the weak energy limit different unification schemes reduce gravity theories having a non-minimal coupling between the scalar field $\phi$ with curvature $R$ of the geometry.

The main objective of this thesis is to study the cosmological solutions in the general scalar-tensor theories of gravity where $\phi$ is nonminimally coupled with Ricci curvature, with an arbitrary coupling function $\omega(\phi)$ and/or with an arbitrary dependence of the gravitational constant $G(\phi)$ based on symmetries of the configuration space variables through which dynamics are described. We consider cosmological solutions in the spatially homogeneous and isotropic background. In finding cosmological solutions in the scalar-tensor gravity theories usually we assume functional form of the potential of the matter (scalar) field and the coupling function $\omega(\phi)$. However, apart from geometric symmetries in spacetime, there may be symmetry in the configuration space depending on the nature of the potential and $\omega(\phi)$. This symmetry in the configuration space is the Nöther symmetry. In general the nature of potential and coupling function are not known, though they are prescribed in an adhoc way in some cases. In absence of any knowledge in the functional form of potential and $\omega(\phi)$ we may impose symmetry in configuration space to find the above unknown functions. Besides this there is another motivation behind adopting Nöther Symmetry in cosmology. Because of the nonlinear nature of the gravitational theory, it is very difficult and sometimes impossible to find the exact solutions of the field equations without considering some simplified assumptions, like the equa-
tion of state or other assumptions. In the cosmological context, the Hamiltonian is constrained to zero giving rise to the constraint equation which is actually the first integral of the combination of the field equations. If we could find a general technique which would give, besides the constraint equation, another first integral of a different combination of the field equations, then it might help us to find the exact solutions of the field equations for a wider class of metric and for different types of matter field without invoking unphysical assumptions. Such an equation can be obtained if we restrict the Lagrangian to have Nöther symmetry and by requiring the Lagrangian to have Nöther symmetry we can have a conserved current which is nothing but the first integral of a different combination of the field equations. Once this is found we can associate a cyclic co-ordinate for the Lagrangian that would help us to find the exact solutions of the field equations.

So far there is no unique way to find the functional form of $\omega(\phi)$ in the general scalar-tensor theories. We propose a way to find $\omega(\phi)$ and potential and/or gravitational constant $G(\phi)$ from the Nöther symmetry of the Lagrangian of the theories following the approach given by de Ritis et al [31]. The principle used by de Ritis et al [31,32] in finding the unknown functions, e.g. the potential function and the coupling function, in the Lagrangian is that the action is invariant under infinitesimal transformation of configuration space variables which corresponds to Nöther symmetry in the spatially homogeneous and isotropic background. Mathematically, if there exists a vector field $X$, for which the Lie derivative of a given point Lagrangian $L$ vanishes i.e. $\mathcal{L}_X L = 0$, the Lagrangian admits Nöther symmetry yielding a conserved current $\Sigma$. The vector field $X$ generates the symmetry. Using this principle we find the unknown functions appearing in the Lagrangian and then present the solution of the field equations which are briefly discussed below:

In Paper-I we consider the scalar-tensor theory without any self interaction of the scalar field $\phi$ and devoid of any other matter fields and present the gravitational coupling function $\omega(\phi)$ as allowed by the Nöther symmetry. We also obtain some exact solutions of the field equations in the spatially homogeneous and isotropic background thereby showing that the attractor mechanism is not effective enough to reduce the theory to Einstein's theory. It is observed that, asymptotically the scalar-tensor theory goes over to Einstein's theory with a finite value of $\omega$ supporting the recent claims [33,34,35] in this direction. It is also shown in this paper that the Brans-Dicke action with variable $\omega$ can be reduced to usual form of induced gravity theory except the term containing the generic potential of the scalar field.

To generalise the previous work with the inclusion of matter fields, in Paper-II, we consider the scalar-tensor theory with another scalar field $\chi$ (apart from $\phi$ field), minimally coupled with geometry, to study the behaviour of the potential $V(\chi)$ of the scalar field $\chi$ in spatially homogeneous and isotropic background as allowed by the Nöther symmetry. It has been observed that in spatially homogeneous and isotropic background Nöther symmetry admits only a constant value of $\omega$ and a potential proportional to $\chi^2$. In finding the solution of the field equations in this case too much mathematical complicacy arises. To understand the early universe
scenario, we consider a vacuum dominated universe i.e. the theory in which the Lagrangian density $L_m$ of the matter field $\chi$ contains only the cosmological constant term $\Lambda$ instead of $\chi$ field. This choice is only possible if the vacuum energy density dominates over the kinetic energy of $\chi$ field. Then we consider another case in which $L_m$ is replaced by a massless scalar field $\chi$ along with a cosmological constant term $\Lambda$. For such cases it is shown that Nöther symmetry admits $\omega$ as a function of $\phi$ and it is singular at some value of $\phi$. It is to be noted that the form of $\omega(\phi)$ obtained by Barker [37] (based on a special choice in the Nordtvedt [29] parameter) is different from us, though both have the same singular nature of $\omega(\phi)$. For those cases the exact solutions are presented which are the inflationary solutions and are exponentially expanding in the asymptotic region unlike the power law expansion in the hyperextended inflation. So with that coupling function $\omega(\phi)$ the graceful exit problem has no solution. An attractive feature of the solutions is that the scalar field $\phi$ approaches continuously towards a constant value $\phi_c$ and $\omega(\phi) \to \infty$ as $t \to \infty$ i.e. the scalar-tensor gravity theory is indistinguishable from Einstein's theory for large $t$ and $\phi = \phi_c$ is an attractor of the equations of motion.

In the previous works no self interaction of the scalar field $\chi$ is considered. So in Paper-III we consider the scalar-tensor theory in which a nontrivial potential $V(\phi)$ of the scalar field is introduced in the scalar-tensor theory [29]. The presence of the potential $V(\phi)$ may give rise to phase transition in the early universe. The potential $V(\phi)$ may be identified as an effective or variable cosmological constant. The forms of $\omega(\phi)$ and $V(\phi)$ are obtained from Nöther symmetry and are grouped in three different classes belonging to different physical systems. In order to justify the potential and the coupling function, we present the exact solutions of the field equations in all the cases. The basic feature is that in all the cases the solutions are inflationary solutions and are expanding exponentially in the asymptotic region, though the nature of expansions are different depending on the initial behaviour. So the graceful exit problem has no solution even in this case. From the exact solutions it is confirmed that when $\omega$ is time-dependent, the scalar field $\phi$ approaches towards a constant value $\phi_c$ and $\omega(\phi) \to \infty$ as $t \to \infty$ i.e. the scalar-tensor gravity theory is indistinguishable from the Einstein's theory for large $t$ and $\phi = \phi_c$ is an attractor of the equations of motion. Further, it is observed that in a special case the action reduces to the induced gravity action.

Paper-IV is an extension of Paper-III in which we present the solution of the field equations for two different cases, namely, for $\phi^2$-potential and $\phi^4$-potential, obtained from Nöther symmetry, which were not discussed in the previous paper. These types of potentials are of importance for studying the possibility of inflation in the early universe. It is shown that for both the cases the solutions are exponentially expanding asymptotically and both reduce to Einstein's theory without any graceful exit from inflation. Further, it is shown that, for $\phi^2$-potential, the theory, in the asymptotic limit, goes over to GTR with finite value of $\omega$ i.e. the final state of the universe in the scalar-tensor theory is distinguishable from Einstein's theory supporting the earlier works in this direction [33,34,35]. The advantage of our work is
that we determine $\omega(\phi)$ from symmetry rather than setting it to be a constant \([33,34]\) or having an adhoc choice \([35]\) for $\omega(\phi)$. It is also important to note that, for $\phi^2$-potential, the effective Newtonian gravitational constant is found to be determined by the effective potential. However, for $\phi^4$-potential we present the solutions for large $t$ and $\omega(\phi) \to \infty$ as $t \to \infty$ i.e. the theory reduces to Einstein's theory asymptotically and $\phi(t \to \infty)$ is an attractor of the equations of motion.

In the induced gravity unification scheme \([25]\), the action of general relativity appears as an effective action induced, in the weak energy limit, by the quantum properties of the vacuum state of the matter fields. Induced gravity bears a strong resemblance to Brans-Dicke theory though the motivations behind the two are rather different. While Brans-Dicke theory is an attempt to incorporate Mach's principle in general relativity, the idea of induced gravity is based on the observation in gauge theories that dimensional coupling constants, which arise in a low energy effective theory, can be expressed in terms of vacuum expectation values of scalar fields. The important common feature is that the gravitational constant is a function of the scalar field $\phi$. So in Paper-V we consider the general induced gravity action with arbitrary dimensional coupling function $f(\phi)$ and potential $V(\phi)$. The Newtonian gravitational constant is proportional to $\frac{1}{f(\phi)}$. By requiring the Nöther symmetry it is shown that in some special cases the coupling function $f(\phi)$ can be expressed in closed form which are previously discussed by Capozziello and de Ritis \([38,39,40]\). In their work they obtained the second order differential equation for the dimensional coupling function $f(\phi)$ and then considered some special solutions of that equation. However, in our work we present the exact first integral of the second order differential equation for $f(\phi)$, obtained from Nöther symmetry, in the closed form which is new in the literature concerning induced gravity theories. However, in general $f(\phi)$ contains an infinite number of terms. The effective strength of gravitational interaction can be expressed as an infinite series and the different order of terms are important at the different regime of evolution of the early universe. Then we present the solutions of the field equations for different forms of $f(\phi)$ which reveal that some forms of $f(\phi)$, obtained from Nöther symmetry, lead to unphysical solutions. In some cases we have physically acceptable solutions. In one case we have exponential expansion of the universe and asymptotically $f(\phi)$ and hence the effective Newtonian gravitational constant approaches a constant value.

In the scalar-tensor theory the dimensionless coupling function $\omega$ is a dynamical quantity and the model may also contain a nontrivial potential term $V(\phi)$. In this theory one may expect an explanation of the variation of $\omega$ from the early universe to the present universe. Further the unification schemes \([30]\) of fundamental interactions viz. the supergravity or superstrings or induced gravity unification schemes \([25]\) naturally associate a long range scalar partner to the usual tensor field of gravity. These unification schemes reduce to the gravity theories having a nonminimal coupling between the scalar field $\phi$ with curvature $R$ of the geometry at the weak energy limit. The dimensional coupling function $f(\phi)$ in the induced gravity theory can be expressed as the vacuum expectation value of the effective potential at the
weak energy limit. To accommodate all these features in general, in Paper-VI, we postulate an action with arbitrary dimensional coupling function $f(\phi)$, dimensionless coupling function $\omega(\phi)$ and potential function $V(\phi)$. The inverse of $f(\phi)$ gives the effective strength of gravitational interaction. By requiring Noether symmetry the forms of $f(\phi)$, $\omega(\phi)$ and $V(\phi)$ are presented for some special cases. Also with a particular choice of $f(\phi)$ and $V(\phi)$ we present $\omega(\phi)$ as allowed by Noether symmetry. The solutions of the field equations reveal that we have exponential expansion of the universe and the scalar field $\phi$ and hence $f(\phi)$ increases with the expansion of the universe resulting in a decrease in the effective Newtonian gravitational constant. We thereby conclude that the asymptotic freedom for gravity is recovered i.e. gravity reduces to a sort of average field with respect to the other interactions [36]. This is also to be noted that the theory asymptotically goes over to GTR for finite value of the coupling function $\omega$, so the final state of the universe is distinguishable from Einstein's gravity supporting the recent claims [33,34,35]. Further, in one case the effective Newtonian gravitational constant $G_N$, in the asymptotic limit, is shown to be determined by the effective potential. Also the coupling function $\omega(\phi)$ in the asymptotic limit can be expressed in terms of $G_N$ and finally in terms of the Noether constant of motion $\Sigma$ arising out of Noether symmetry. It is important to note that if we choose $\Sigma = 0$, we have $\omega \to \infty$ as $t \to \infty$; on the other hand if $\Sigma \neq 0$, $\omega$ is finite as $t \to \infty$. There are thus two kinds of solutions: one distinguishable ($\omega \neq \infty$ as $t \to \infty$) and the other indistinguishable ($\omega \to \infty$ as $t \to \infty$) from Einstein's gravity.