Nöther Symmetry

Let us now revisit the Nöther symmetry in retrospect. The principle used is that the action is invariant under infinitesimal transformation of configuration space variables which corresponds to Nöther symmetry in spatially homogeneous and isotropic background.

If the action $A = \int L(q_i, \dot{q}_i, t)dt$ is invariant under infinitesimal transformation

$$q_i \rightarrow q_i' = q_i + \epsilon \psi_i(q_k, t), \quad t \rightarrow t' = t + \epsilon \chi(q_k, t),$$

then $[\psi_i \frac{\partial L}{\partial \dot{q}_i} - (\frac{\partial L}{\partial q_i} q_i - L)\chi]$ is the constant of motion, and instantly we have the following principles:

Time translation symmetry i.e. homogeneity of time corresponds to the conservation of energy; space translation symmetry i.e. homogeneity of space corresponds to the conservation of linear momentum and rotational symmetry i.e. isotropy of space corresponds to the conservation of angular momentum.

In absence of explicit time dependence in the Lagrangian, the quantity $\psi_i \frac{\partial L}{\partial \dot{q}_i}$ is constant of motion, hence

$$\frac{d}{dt}(\psi_i \frac{\partial L}{\partial \dot{q}_i}) = \psi_i \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) + \frac{d\psi_i}{dt} \frac{\partial L}{\partial q_i} = \psi_i \frac{\partial L}{\partial q_i} + \frac{d\psi_i}{dt} \frac{\partial L}{\partial q_i} = L_X L = \theta,$$

(4)

The framework of Nöther symmetry approach to cosmology is that, if there exists a vector field $X$, for which the Lie derivative of a given point Lagrangian $L$ vanishes i.e.

$$L_X L = 0,$$

(5)

the Lagrangian admits Nöther symmetry yielding a conserved current $\Sigma$, where

$$\Sigma = i_X \theta = \psi_i \frac{\partial L}{\partial \dot{q}_i},$$

(6)

and the Cartan one-form is given by

$$\theta_i = \frac{\partial L}{\partial q_i} dq_i,$$

(7)

and

$$X = \psi_i \frac{\partial}{\partial q_i} + \frac{d\psi_i}{dt} \frac{\partial}{\partial \dot{q}_i},$$

(8)

is the infinitesimal generator of symmetry and is the lift vector belonging to the tangent space $TQ \equiv (q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$ over the configuration space $Q \equiv (q_1, ..., q_n)$. Functions $\psi_i$ determine the symmetry in the system and are functions of configuration space variables.

The idea of the Nöther theorem, presented in this way, is to find the form of $\omega(\phi)$ and potential for which there would exist a vector field $X$ that would make $L_X L = 0$.
and hence would yield a conserved current. It is not expected in general that, in all such models there would exist certain \( \omega(\phi) \) and potential, and hence \( X \), that would make \( \mathcal{L}_X L = 0 \). Rather, while solving the set of equations \( \mathcal{L}_X L = 0 \), one might sometimes encounter inconsistencies which would imply that the Lagrangian does not admit Nöther symmetry in such models. However, it is generally believed that if certain \( \omega(\phi) \) and potential \( V(\phi) \) and hence \( X \) exist, that satisfy all the equations \( \mathcal{L}_X L = 0 \), then the Lagrangian admits Nöther symmetry.

The thesis contains six papers in which the unknown functions (e.g. \( \omega(\phi), V(\phi), f(\phi) \)) in the Lagrangian are obtained using Nöther symmetry and using Nöther current the solutions of the field equations are presented.