CHAPTER – 3
SOME TWO DIMENSIONAL PROBLEMS IN
MAGNETO-ELASTICITY
1. INTRODUCTION
We observe a rapid development of the coupled theory of elastic bodies. By this name, we understand an interrelation of two or more branches of phenomenological physics, so far being developed separately. As a typical example, we may mention magneto-elasticity. In this newly developed branch, problems of waves and vibrations in a homogeneous conducting elastic medium which is assumed to be situated under the influence of a constant primary magnetic field are receiving greater attention by many investigators [66, 88, 138]. Here the authors have studied the problem of moving load [58] over the boundary of a semi-infinite elastic medium under the influence of initial constant magnetic field $\overline{H} = (0,0,H_0)$.

2. BASIC EQUATIONS AND RELATIONS
The system of equations in electro-dynamics may be written as [45, 67]

$$\nabla \times \vec{E} = -\frac{\mu_0}{c} \frac{\partial \vec{h}}{\partial t}, \quad \nabla \times \vec{h} = \frac{4\pi}{c} \vec{j}, \quad \nabla \cdot \vec{h} = 0, \quad \nabla \cdot \vec{E} = 0,$$

$$\vec{j} = \lambda_0 \left[ \vec{E} + \frac{\mu_0}{c} \left( \frac{\partial \vec{u}}{\partial t} \times \overline{H} \right) \right], \quad \vec{D} = \varepsilon_0 \left[ \vec{E} + \frac{1}{\varepsilon_0} \left( \frac{\partial \vec{u}}{\partial t} \times \overline{H} \right) \right]$$

(2.1)

in which $\vec{h}$ and $\vec{E}$ are perturbations of the magnetic and electric fields respectively, $\vec{j}$ denotes the current density vector, $\overline{H}$ denotes the initial constant magnetic field, $c$ is the speed of light, $\mu_0$ and $\varepsilon_0$ are the magnetic and electric permeability respectively, $\vec{D}$ is electric induction vector, $\vec{u}$ is the displacement vector and $\lambda_0$ is the electric conductivity.

The basic equations of motion in magneto-elasticity in absence of body forces may be written as [91]

$$\sigma_{j,j} + T_{j,i,j} = \rho \ddot{u}_i$$

(2.2)

where $T_{j,i}$ denotes the Maxwell electro-magnetic stress tensor, $T_{j,i,j}$ are the Lorentz forces. The Maxwell tensor is related to the vector $\vec{h}$ in the following manner...
\[ T_{ij} = \left( \frac{\mu_0}{4\pi} \right) \left[ H_i h_j + H_j h_i - \delta_{ij} H_k h_k \right] \quad [i, j = 1, 2, 3] \] (2.3)

Eliminating the vector \( \overrightarrow{E} \) and \( \overrightarrow{j} \) and the stresses and strains and expressing the vector \( T_{ij} \) by the component of the vector \( \overrightarrow{h} \), we arrive at the system of equations [91]

\[
\nabla^2 \overrightarrow{h} - \beta \frac{\hat{J}}{\epsilon_0} = -\beta \nabla \times \left( \mu_0 \overrightarrow{h} \times \overrightarrow{H} \right), \quad \beta = 4\pi \lambda_0 \mu_0 / c^2
\]

\[
\mu \nabla^2 \overrightarrow{u} + (\lambda + \mu) \nabla \nabla \overrightarrow{u} + \left( \mu_0 / 4\pi \right) \nabla \times \left( \mu_0 \overrightarrow{h} \times \overrightarrow{H} \right) = \rho \frac{\hat{J}}{\epsilon_0}
\] (2.4)

For the elastic solid ideally conducting electricity [91]

\[
[\beta = \infty, \overrightarrow{h} = \nabla \times \left( \mu_0 \overrightarrow{h} \times \overrightarrow{H} \right)]
\]

We get

\[
\mu \nabla^2 \overrightarrow{u} + (\lambda + \mu) \nabla \nabla \overrightarrow{u} + \left( \mu_0 / 4\pi \right) \nabla \times \left( \mu_0 \overrightarrow{h} \times \overrightarrow{H} \right) = \rho \frac{\hat{J}}{\epsilon_0}
\] (2.5)

If the magnetic field \( \overrightarrow{H} \) is of the form \( \overrightarrow{H} = (0, 0, H_0) \) equation (2.5) reduces to

\[
\mu \nabla^2 \overrightarrow{u} + (\lambda + \mu + \alpha_0^2 \rho) \nabla \nabla \overrightarrow{u} = \rho \frac{\hat{J}}{\epsilon_0}
\] (2.6)

Where

\[
\alpha_0^2 = \left( \mu_0 H_0^2 / 4\pi \rho \right), \alpha_0
\]

is the Alfven velocity. Now for the displacement potentials \( \varphi \) and \( \psi \) for which

\[
\overrightarrow{u} = \varphi_{,1} + \psi_{,2} \quad \quad \overrightarrow{v} = \varphi_{,2} - \psi_{,1}
\] (2.7)

and equation (2.6) decomposes into the following system of equations

\[
\left(1 + \alpha_0^2 / c_1^2\right) \nabla^2 \varphi = c_1^2 - \ddot{\varphi}
\]

\[
\nabla^2 \psi = c_2^2 - \ddot{\psi}
\] (2.8)

where

\[
c_1^2 = (\lambda + 2\mu) / \rho \quad \text{and} \quad c_2^2 = \mu / \rho.
\]

The components of stresses in terms of \( \varphi \) and \( \psi \) are

\[
\tau_{22} = \lambda \nabla^2 \varphi + 2\mu \left( \varphi_{,22} - \psi_{,12} \right) \quad \tau_{11} = \lambda \nabla^2 \varphi + 2\mu \left( \varphi_{,11} + \psi_{,12} \right)
\]

\[
\tau_{12} = \mu [2\varphi_{,12} - \psi_{,11} + \psi_{,22}]
\] (2.9)

3. FORMULATION OF THE PROBLEM

If for convenience we replace the time like variable \( t \) by a space like variable \( \tau \), defined by \( \tau = c_1 t \)
then we may write (2.8) in the form
\[
\frac{\partial^2 \Phi}{\partial \tau^2} = \left(1 + \frac{\alpha_0^2}{c_1^2}\right) \nabla^2 \Phi ; \quad \beta^2 \frac{\partial^2 \Psi}{\partial \tau^2} = \nabla^2 \Psi
\]
where
\[
\beta^2 = \frac{c_1^2}{c_2^2} = \frac{(\lambda + 2\mu)}{\mu}
\]

We take the x-axis to be along the boundary and y-axis pointing into the medium. To find the solutions of the wave equations (3.1), we introduce the two-dimensional Fourier transform of the functions
\[
\Phi(x, y, \tau) \quad \text{and} \quad \Psi(x, y, \tau) \quad \text{as}
\]
\[
\Phi(\xi, y, \zeta) = \frac{1}{2\pi} \int \int \Phi(x, y, \tau) e^{i(\xi x + \zeta \tau)} \, dx \, d\tau ; \quad \Psi(\xi, y, \zeta) = \frac{1}{2\pi} \int \int \Psi(x, y, \tau) e^{i(\xi x + \zeta \tau)} \, dx \, d\tau
\]

**Boundary Conditions:** The boundary conditions for the variable pressure-pulse $P(x, \tau)$ moving over the boundary $y = 0$ of the conducting elastic medium are
\[
\tau_{22} = -P(x, \tau) \quad ; \quad \tau_{12} = 0 \quad \text{on} \quad y = 0
\]

Now if we multiply both sides of the equation (3.1) by $e^{i(\xi x + \zeta \tau)}$ and integrate over the entire x-plane, we find that the function plane $\Phi(\xi, y, \zeta)$ and $\Psi(\xi, y, \zeta)$ satisfy the following differential equations
\[
\frac{d^2 \Phi}{dy^2} = (\xi - k^2 \zeta^2) \Phi ; \quad \frac{d^2 \Psi}{dy^2} = (\xi - \beta^2 \zeta^2) \Psi
\]
where
\[
k^2 = \frac{c_1^2}{c_2^2 + \alpha_0^2}
\]
is the magnetic parameter of the problem. To satisfy the condition that the stresses tend to zero as $y \to \infty$, we take the solutions of (3.5) as
\[
\Phi(\xi, y, \zeta) = A \exp\left[-\left(\xi^2 - k^2 \zeta^2\right)\frac{1}{2} y\right] ; \quad \Psi(\xi, y, \zeta) = B \exp\left[-(\xi^2 - \beta^2 \zeta^2)\frac{1}{2} y\right]
\]
The corresponding wave function $\Phi(x, y, \tau)$ and $\Psi(x, y, \tau)$ are given by Fourier's inversion theorem for two-dimensional transforms. From these we obtain the following expressions...
\[
\frac{\tau_{22}}{2\mu} = -\frac{1}{2\pi} \int \int_{-\infty}^{\infty} \left[ \frac{i\xi (\xi - \beta^2 \zeta^2)}{2} \psi - \left( \xi - \frac{1}{2} \beta^2 \zeta^2 k^2 \right) |\phi| e^{-(\xi^2 + \zeta^2)} \right] d\xi d\zeta
\]

\[
\frac{\tau_{11}}{2\mu} = \frac{1}{2\pi} \int \int_{-\infty}^{\infty} \left[ i\xi (\xi - \beta^2 \zeta^2) \frac{1}{2} \psi - \left( \xi + \frac{\lambda}{2\mu} k^2 \zeta^2 \right) |\phi| e^{-(\xi^2 + \zeta^2)} \right] d\xi d\zeta
\]

\[
\frac{\tau_{12}}{2\mu} = \frac{1}{2\pi} \int \int_{-\infty}^{\infty} \left[ i\xi (\xi - k^2 \zeta^2) \frac{1}{2} \psi + \left( \xi - \frac{1}{2} \beta^2 \zeta^2 \right) \psi \right] e^{-(\xi^2 + \zeta^2)} d\xi d\zeta
\]

(3.7)

In a similar way we can find the displacements from (27)

\[
u = -\frac{1}{2\pi} \int \int_{-\infty}^{\infty} \left[ \frac{i\xi \phi + (\xi^2 - \beta^2 \zeta^2) \frac{1}{2} \psi}{2} e^{-(\xi^2 + \zeta^2)} \right] d\xi d\zeta
\]

\[
v = -\frac{1}{2\pi} \int \int_{-\infty}^{\infty} \left[ (\xi^2 - k^2 \zeta^2) \frac{1}{2} \psi - i\xi \psi \right] e^{-(\xi^2 + \zeta^2)} d\xi d\zeta
\]

(3.8)

4. SOLUTION

(I) When a variable pressure is applied on the boundary

Using boundary conditions, given in (34) we obtain

\[
(\xi^2 - \beta^2 \zeta^2 k^2 / 2) A - i\xi \left( \xi^2 - \beta^2 \zeta^2 \right) \frac{1}{2} B = -\bar{P}(\xi, \zeta) / 2\mu
\]

\[
i\xi \left( \xi^2 - k^2 \zeta^2 \right) \frac{1}{2} A + \left( \xi^2 - \beta^2 \zeta^2 / 2 \right) B = 0
\]

(4.1)

Where \( \bar{P}(\xi, \zeta) \) is the two dimensional Fourier transforms of \( P(x, \tau) \). From (4.1) we obtain

\[
A = \frac{1}{2\mu} \left( \xi^2 - \beta^2 \zeta^2 / 2 \right) \bar{P}(\xi, \zeta) ; \quad B = -\frac{i\xi}{2\mu} \left( \xi^2 - k^2 \zeta^2 \right) \bar{P}(\xi, \zeta)
\]

(4.2)

where

\[
f(\xi^2, \zeta^2) = -\left( \xi^2 - \beta^2 \zeta^2 k^2 / 2 \right) \left( \xi^2 - \beta^2 \zeta^2 / 2 \right)
\]

\[
g(\xi^2, \zeta^2) = \xi^2 \left( \xi^2 - k^2 \zeta^2 \right) \frac{1}{2} \left( \xi^2 - \beta^2 \zeta^2 \right) \frac{1}{2}
\]

(4.3)

Hence

\[
\tau_{22} = -\frac{1}{2\pi} \int \int_{-\infty}^{\infty} \frac{\bar{P}}{f + g} \left[ f e^{-(\xi^2 + \zeta^2)^2} + g e^{-(\xi^2 - \beta^2 \zeta^2)^2} \right] e^{-(\xi^2 + \zeta^2)} d\xi d\zeta
\]
The components of displacement vector are

\[ u = \frac{1}{4\pi\mu} \int \int f + g \frac{g}{\xi^2} e^{-\left(\xi^2 - \beta^2\xi^2\right)^{1/2}} \left[ \left( \xi^2 - \frac{1}{2} \beta^2 \xi^2 \right) e^{-\left(\xi^2 - \beta^2\xi^2\right)^{1/2} y} - e^{-\left(\xi^2 - \beta^2\xi^2\right)^{1/2} y} \right] e^{-i(\xi x + \xi^2)} d\xi d\eta \tag{4.5} \]

\[ v = -\frac{1}{4\pi\mu} \int \int \frac{P\xi}{f + g} \frac{g}{\xi^2} e^{-\left(\xi^2 - \beta^2\xi^2\right)^{1/2}} \left[ \left( \xi^2 - \frac{1}{2} \beta^2 \xi^2 \right) e^{-\left(\xi^2 - \beta^2\xi^2\right)^{1/2} y} - \xi^2 e^{-\left(\xi^2 - \beta^2\xi^2\right)^{1/2} y} \right] e^{-i(\xi x + \xi^2)} d\xi d\eta \tag{4.6} \]

(II) When a pulse of pressure is moving uniformly along the boundary

The stress set up in the interior of the semi-infinite elastic medium when a pulse of pressure of shape \( P = \chi(x) \) moves with uniform velocity \( v \) along the boundary \( y = 0 \), we then have

\[ p(x, \tau) = \chi(x - vt) = \chi(x - \beta_1 \tau), \quad \beta_1 = v / c_1 \]

and then

\[ \overline{P}(\xi, \eta) = \frac{1}{2\pi} \int \int \chi(x - \beta_1 \tau) e^{i(\xi x + \eta \xi)} dx d\tau \tag{4.7} \]

which by a trivial change of variable gives

\[ \overline{P}(\xi, \eta) = \frac{1}{2\pi} \int \int \chi(u) e^{i\xi u} du \int e^{i(\tau + \beta_1 \xi)} d\tau = 2\chi(\xi) \delta(\xi + \beta_1 \xi) \tag{4.8} \]

where \( 2\chi(\xi) = \int \chi(u) e^{i\xi u} du \)

Now if we make use of the fact that for any function \( \chi(\xi^2, \zeta^2) \)

\[ \int_{-\infty}^{\infty} \chi(\xi^2, \zeta^2) \delta(\xi + \beta_1 \xi) e^{-i\xi \eta} d\zeta = \chi(\xi^2, \beta_1 \xi^2) e^{-i\xi \eta} \tag{4.9} \]

we find for the component of stress and displacement as
\[ \tau_{22} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \chi(\xi) e^{i \xi (x - vt)} \left[ \frac{\theta}{\theta + \phi} e^{-(i - k^2 \beta_1^2) \frac{1}{2} y \xi} + \frac{\phi}{\theta + \phi} e^{-(i - \beta_2^2) \frac{1}{2} y \xi} \right] d\xi \]

\[ \tau_{11} = \frac{1}{\pi} \int_{-\infty}^{\infty} \chi(\xi) e^{i \xi (x - vt)} \left[ \frac{\phi}{\theta + \phi} e^{-(i - \beta_2^2) \frac{1}{2} y \xi} - \left(1 + \frac{\lambda}{2 \mu} \beta_1^2 \right) \left(1 - \frac{\beta_2^2}{2}ight) e^{-(i - k^2 \beta_1^2) \frac{1}{2} y \xi} \right] d\xi \]

\[ \tau_{12} = \frac{1}{\pi} \int_{-\infty}^{\infty} \chi(\xi) e^{i \xi (x - vt)} \left[ \frac{\phi}{\theta + \phi} e^{-(i - \beta_2^2) \frac{1}{2} y \xi} - \left(1 + \frac{\lambda}{2 \mu} \beta_1^2 \right) \left(1 - \frac{\beta_2^2}{2}ight) e^{-(i - k^2 \beta_1^2) \frac{1}{2} y \xi} \right] d\xi \]

(4.10)

\[ u = \frac{1}{2 \pi \mu} \int_{-\infty}^{\infty} \chi(\xi) e^{i \xi (x - vt)} \left[ \frac{\phi}{\theta + \phi} e^{-(i - \beta_2^2) \frac{1}{2} y \xi} - \left(1 + \frac{\lambda}{2 \mu} \beta_1^2 \right) \left(1 - \frac{\beta_2^2}{2}ight) e^{-(i - k^2 \beta_1^2) \frac{1}{2} y \xi} \right] d\xi \]

(4.11)

where

\[ \theta(\beta_2) = \left(1 - \frac{1}{2} \beta_2^2 k^2 \right)^{\frac{1}{2}} \left(1 - \frac{\beta_2^2}{2} \right) \]

\[ \phi(\beta_1, \beta_2) = \left(1 - \beta_2^2 \right)^{\frac{1}{2}} \left(1 - \frac{\beta_2^2}{2} \right) \]

(III) When a point force moves with uniform velocity over the boundary:

Here we shall consider the distribution of stress produced by the application to the boundary of a point force of magnitude \( P \) where point of application moves with uniform velocity \( v \). The form of the pressure pulse in this case is

\[ \chi(x) = P \delta(x) \quad \text{and} \quad \chi(\xi) = P / 2 \]

The resulting integrals in this case are

\[ \tau_{22} = -\frac{P}{2 \pi} \int_{-\infty}^{\infty} e^{i \xi (x - vt)} \left[ \frac{\theta}{\theta + \phi} e^{-(i - k^2 \beta_1^2) \frac{1}{2} y \xi} + \frac{\phi}{\theta + \phi} e^{-(i - \beta_2^2) \frac{1}{2} y \xi} \right] d\xi \]

\[ \tau_{11} = \frac{P}{2 \pi} \int_{-\infty}^{\infty} e^{i \xi (x - vt)} \left[ \frac{\phi}{\theta + \phi} e^{-(i - \beta_2^2) \frac{1}{2} y \xi} - \left(1 + \frac{\lambda}{2 \mu} \beta_1^2 \right) \left(1 - \frac{\beta_2^2}{2}ight) e^{-(i - k^2 \beta_1^2) \frac{1}{2} y \xi} \right] d\xi \]

(4.12)

\[ \tau_{12} = \frac{P}{2 \pi} \int_{-\infty}^{\infty} e^{i \xi (x - vt)} \left[ \frac{\phi}{\theta + \phi} e^{-(i - \beta_2^2) \frac{1}{2} y \xi} - \left(1 + \frac{\lambda}{2 \mu} \beta_1^2 \right) \left(1 - \frac{\beta_2^2}{2}ight) e^{-(i - k^2 \beta_1^2) \frac{1}{2} y \xi} \right] d\xi \]
u = \frac{P}{4\pi\mu} \int_{-\infty}^{\infty} \left[ -\frac{\phi}{(\theta + \varphi)i\xi} e^{-(\beta_2^2 \beta_1^2 - i/2)} e^{\left(\frac{1}{\beta_2^2 \beta_1^2 - 1}\right) \xi} \right] e^{ix(x-\nu t)} \, d\xi.

v = \frac{P}{4\pi\mu} \int_{-\infty}^{\infty} \left[ \left(1 - \frac{\beta_2^2}{\beta_1^2}ight) e^{\left(\frac{1}{\beta_2^2 \beta_1^2 - 1}\right) \xi} - e^{\left(\frac{1}{\beta_2^2 \beta_1^2 - 1}\right) \xi} \right] e^{ix(x-\nu t)} \, d\xi \quad (4.13)

5. DISCUSSION

If the magnetic field is weak or absent, then \( \alpha_0 \to 0 \) i.e \( k \to 1 \) and the expressions for stresses and displacements of the above three cases tally with the result of Sneddon[131] in classical elastokinetics. Moreover, if (i) \( k^2 \beta_1^2 > 1 \) and \( \beta_2^2 > 1 \) then taking \( x' = x - vt \) the expressions for \( u \) and \( v \) in (4.13) may be written as

\[
\begin{align*}
u &= \frac{P}{2\mu} \int \left[ \frac{\phi}{\theta + \varphi} H\left( x' - (\beta_2^2 - 1) \frac{1}{2} y \right) + \frac{2 - \beta_2^2}{2(\theta + \varphi)} H\left( x' - (k^2 \beta_1^2 - 1) \frac{1}{2} y \right) \right] \, d\xi, \\
v &= \frac{P}{2\mu} \int \left[ \frac{1}{\theta + \varphi} \left( \frac{2 - \beta_2^2}{2} \right) H\left( x' - (k^2 \beta_1^2 - 1) \frac{1}{2} y \right) + \left(1 - \frac{\beta_2^2}{\beta_1^2} \right) H\left( x' - (\beta_2^2 - 1) \frac{1}{2} y \right) \right] \, d\xi \quad (5.1)
\end{align*}
\]

If again \( k \to 1 \), the expression (5.1) reduces to the following form

\[
\begin{align*}
u &= \frac{P}{\mu\Delta} \left[ \left( 2 - \beta_2^2 \right) H\left( x' - \beta_1 y' \right) + 2\beta_1 \beta_2 H\left( x' - \beta_2 y' \right) \right] \\
v &= \frac{P}{\mu\Delta} \left[ -\beta_1 \left( 2 - \beta_2^2 \right) H\left( x' - \beta_1 y' \right) + 2\beta_1 H\left( x' - \beta_2 y' \right) \right] \quad (5.2)
\end{align*}
\]

where the notations have been changed as

\[
y = y', \quad \Delta = \left( 2 - \beta_2^2 \right)^2 + 4\beta_1 \beta_2, \quad \beta_1 = \left( \beta_1^2 - 1 \right)^{\frac{1}{2}}, \quad \beta_2 = \left( \beta_2^2 - 1 \right)^{\frac{1}{2}} \quad (5.3)
\]

The expressions for \( u \) and \( v \) in (5.2) tally with the result as studied by Fung[58] for supersonic case. The result also holds good for the expression of stresses. Considering (ii) \( k^2 \beta_1^2 < 1 \), \( \beta_2^2 < 1 \) and (iii) \( k^2 \beta_1^2 > 1 \) and \( \beta_2^2 < 1 \) we can ultimately reach to the solutions of Fung[58] for the subsonic and transonic cases also. Moreover the medium is undisturbed in front of Mach waves

\[
x' - (\beta_2^2 - 1) \frac{1}{2} y' = 0 \quad \text{and} \quad x' - (k^2 \beta_1^2 - 1) \frac{1}{2} y' = 0 \quad (5.4)
\]

while the medium is undisturbed in front of the following Mach waves in classical case

\[
x' - (\beta_2^2 - 1) \frac{1}{2} y' = 0 \quad \text{and} \quad x' - (\beta_2^2 - 1) \frac{1}{2} y' = 0 \quad (5.5)
\]

So in present case angle between the lines is decreased in comparison with the classical case, since

\[
1/\sqrt{k^2 \beta_1^2 - 1} > 1/\sqrt{\beta_1^2 - 1} \quad \text{for} \quad k^2 < 1
\]

\[
(5.6)
\]
REISSNER-SAGOCI PROBLEM IN MAGNETO-ELASTICITY

1. INTRODUCTION
For the last four decades a new domain has been developed in which investigations concern the interactions between strain and electromagnetic field. This new discipline is called magneto-elasticity. In classical theory of elasticity, Reissner and Sagoci in their investigations have considered [107,110] the torsional oscillations produced in a semi-infinite homogeneous isotropic medium by a periodic shear stress applied in an axially symmetric manner to a circular area of the plane surface of the boundary of the medium. The distribution of stresses in the interior of a semi-infinite elastic medium is determined when a load is applied to the surface by means of a rigid disc and it is assumed that the part of the boundary which lies beyond the edge of the disc is free from stress. The solution of the mixed boundary value problem is studied by Reissner and Sagoci through the introduction of a certain system of oblate spheroidal co-ordinates. Here the same problem has been studied considering the angle of rotation $\varphi$ [131] under the influence of magnetic field by the use of Hankel transformation to the solution of a pair of dual integrals. The authors have considered the small perturbations characterized by the displacement vector $\vec{u} \; [0, u_\theta, (r, z, t), 0]$ and also assume the perturbations to be independent of the angle $\theta$ as studied in the book of Parton and Perlin [98].

2. BASIC EQUATIONS AND RELATIONS
Maxwell's equations for an electromagnetic field may be written in the following form

$$\text{curl} \; \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}, \quad \text{div} \; \vec{B} = 0, \quad \text{curl} \; \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\text{div} \; \vec{D} = \rho_e, \quad \vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu_e \vec{H} \; (2.1)$$

We supplement these equations by Hooke's law

In presence of electromagnetic field, the equations of motion in absence of body forces

$$\frac{\partial \tau_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \; (2.2)$$

assume the form

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Finally current in a moving conductor is defined by the generalized Ohm's law [98]

\[ \bar{j} = \sigma \left[ \bar{E} + \left( \frac{\partial \bar{u}}{\partial t} \times \bar{B} \right) \right] + \rho_e \frac{\partial \bar{u}}{\partial t} \]  

(2.4)

where \( \bar{H} \) is the magnetic field vector, \( \bar{E} \) is the electric field vector, \( \bar{j} \) is the current density vector, \( \rho_e \) is the electric charge density, \( \bar{B} \) is the magnetic induction vector, \( \bar{D} \) is the electric induction vector, \( \mu_0 \) is the magnetic permeability, \( \varepsilon \) is the electric permittivity, \( \rho \) is the density of the medium and \( \sigma \) is the electrical conductivity.

3 GENERAL THEORY AND BOUNDARY CONDITIONS

Let us consider a semi-infinite solid \( z \geq 0 \) subjected to a homogeneous axial magnetic field \( \bar{H} = (0,0,H_0) \). The medium has infinite conductivity and permeability of the vacuum is \( \chi_0 = 4\pi \times 10^{-7} \text{ H/m(N/A}^2) \) [98]. We introduce a cylindrical system of co-ordinates in which the z-axis is directed along the axis of symmetry of the medium. A circular area \( r = r_0 \) of the surface is forced to rotate through an angle \( \Phi \) about an axis which is normal to the undeformed surface of the medium. It is assumed that the region of the surface lying outside the circle \( r \geq r_0 \) is free from stress. It has been shown by Reissner [106] that in this case only the circumferential component of the displacement vector is different from zero and that all the stress component vanishes except \( \tau_{z\theta} \) and \( \tau_{r\theta} \) which are given by the relations [131]

\[ \tau_{z\theta} = \mu \frac{\partial u_\theta}{\partial z} \quad ; \quad \tau_{r\theta} = \mu \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \]  

(3.1)

The boundary conditions of the problem are [131]

\[ u_\theta = f(r,t) \quad \text{for} \quad z = z_0 \quad r \leq r_0 \]  

(3.2)

\[ \tau_{z\theta} = 0 \quad \text{for} \quad z = z_0 , \quad r \geq r_0 \]  

(3.3)

In the case considered by Reissner and Sagoci [107] the surface displacement \( f(r,t) \) is of the form

\[ f(r,t) = r \Phi(t) \]  

(3.4)

and the relation between angle \( \Phi \) and the applied torque \( T \) may be derived from the equation
4 FORMULATION OF THE PROBLEM

Let us now consider the perturbation of the Maxwell’s electro-magnetic field in the form $\overline{H} = H_0 + \overline{h}$ and $\overline{E} = E_0 + \overline{e}$, where $\overline{h}$ and $\overline{e}$ are perturbations in the magnetic and electric field respectively. We then linearize the basic equations describing the motion in presence of elastic, electric and magnetic fields. In the M. K. S system the linearized system of equations take the form [138]

\[ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{\chi_0 H_0}{\mu} \frac{\partial h_\theta}{\partial z} = \frac{1}{c_T^2} \frac{\partial^2 u_\theta}{\partial t^2} \]

\[ e_r + \chi_0 H_0 \frac{\partial u_\theta}{\partial t} = 0 \]

\[ \frac{\partial e_r}{\partial z} + \chi_0 \frac{\partial h_\theta}{\partial t} = 0 \]

Here $\overline{e} (e_r, 0, 0)$ and $\overline{h} (0, h_\theta, 0)$ are perturbations arising in the electric and magnetic fields respectively and $c_T$ is the velocity of the transverse waves. To solve the partial differential equations (4.1), (4.2) and (4.3) we introduce the Hankel transform as

\[ U = \int_0^\infty r u_\theta J_1 (\xi r) \, dr \]

\[ V = \int_0^\infty r h_\theta J_1 (\xi r) \, dr \]

\[ W = \int_0^\infty r e_r J_1 (\xi r) \, dr \]

Multiplying both sides of the equations (4.1), (4.2) and (4.3) by $r J_1 (\xi r)$ and integrating with respect to $r$ from 0 to $\infty$, we obtain

\[ \left( \frac{1}{c_T^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + \xi^2 \right) U = \frac{\chi_0 H_0}{\mu} \frac{\partial V}{\partial z} \]

\[ W + \chi_0 H_0 \frac{\partial U}{\partial t} = 0 \]

\[ \frac{\partial W}{\partial z} + \chi_0 \frac{\partial V}{\partial t} = 0 \]
Eliminating $U, V$ and $W$ from (4.5) we obtain

$$
\left[ \left( \frac{1}{c_T^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + \xi^2 \right) - \frac{\chi_0 H_0^2}{\mu} \frac{\partial^2}{\partial z^2} \right] (U, V, W) = 0
$$

(4.6)

For the determination of $u_0, e_r$ and $h_0$ in terms of the solutions $U, V$ and $W$ we use the following

$$
u_0 = \int_0^{\xi} U(z, t, \xi) J_1(\xi r) d\xi
$$

$$
h_0 = \int_0^{\xi} V(z, t, \xi) J_1(\xi r) d\xi
$$

$$
e_r = \int_0^{\xi} W(z, t, \xi) J_1(\xi r) d\xi
$$

(4.7)

The non-vanishing component of stresses are

$$
\tau_{r0} = \left( \frac{\mu}{2} \right) \xi^2 U(z, t, \xi) \left[ J_0(\xi r) - J_2(\xi r) \right] d\xi
$$

(4.8)

$$
\tau_{r0} = \frac{\mu}{2} \xi^2 \frac{\partial U(z, t, \xi)}{\partial z} J_1(\xi r) d\xi
$$

(4.9)

The arbitrary functions introduced in the solution of equation (4.6) are determined by the boundary conditions. The solution must be such that the displacement and both component of stresses tend to zero as $z \to \infty$. When $z = 0$, the conditions (3.2) and (3.3) must be satisfied.

Using (3.2), (3.3), (4.7) and (4.9) we obtain the dual integral equations as

$$
\int_0^{\xi} U(0, t, \xi) J_1(\xi r) d\xi = f(r, t) \quad r \leq r_0
$$

(4.10)

$$
\int_0^{\xi} \frac{\partial U(0, t, \xi)}{\partial z} J_1(\xi r) d\xi = 0 \quad r \geq r_0
$$

(4.11)

5.SOLUTION IN THE STATIC CASE

As studied by Sneddon[131], we take partial differentiation with respect to time $t$ as zero in the equation (4.6) and in addition we are interested in the interaction between elastic and magnetic fields so that (4.6) reduces to the following form

$$
\left[ \left(1 + \frac{\chi_0 H_0^2}{\mu} \right) \frac{d^2}{dz^2} - \xi^2 \right] (U, V) = 0
$$

(5.1)
where the relationship between $U$ and $V$ is given by the first equation of (4.5).

Remembering the finiteness condition at infinity, the solution may be written as

$$U = A(m' \xi) e^{-m' \xi} ; \quad V = C(m' \xi) e^{-m' \xi}$$  \hspace{1cm} (5.2)

where $m' = \sqrt{1 + h_0} ; \quad h_0 = \chi_0 H_0^2 / \mu$

The relation between $A(m' \xi)$ and $C(m' \xi)$ in static case can be obtained from the first equation of (4.5), considering partial differentiation with respect to time $t$ as zero we obtain

$$C(m' \xi) = -\frac{\xi \mu (1 - m'^2)}{\chi_0 H_0 m'} A(m' \xi) = \frac{\xi H_0}{m'} A(m' \xi) \hspace{1cm} (5.3)$$

Substituting the value of $U$ from (5.2) into the equations (4.10) and (4.11) the following dual integral equations are obtained

$$\int_0^\infty \eta^{-3} F(\eta) J_1\left(\frac{\rho_0}{m'} \eta\right) d\eta = \varphi \left(\frac{\rho_0}{m'}\right) \hspace{1cm} (5.4)$$

$$\int_0^\infty F(\eta) J_1\left(\frac{\rho_0}{m'} \eta\right) d\eta = 0 \hspace{1cm} (5.5)$$

where

$$r = \rho_0 r_0 , \quad m' \xi = \eta / r_0 , \quad f(r) = \frac{r_0}{m'^2} \varphi \left(\frac{\rho_0}{m'}\right), \quad \eta^2 A(\eta / r_0) = r_0^2 F(\eta) \hspace{1cm} (5.6)$$

We obtain the solution putting $\alpha = -1$ and $p = 1$ in the result of Bushbridge [20] as

$$F(\eta) = \left(2 / \pi \right)^{1/2} \eta^{1/2} J_1(\eta) \left[\frac{1}{2} (1 - y^2) \right]^{1/2} y^2 g(y) dy + \int_0^1 u^2 (1 - u^2)^{1/2} du \int_0^1 g(yu) (\eta y)^{3/2} J_{3/2}(\eta y) dy$$

considering $\varphi \left(\frac{\rho_0}{m'}\right) = \varphi \left(\frac{\rho_0}{m'}\right)$, so that

$$F(\eta) = \frac{4 \varphi}{\pi} \left[\frac{\sin \eta}{\eta} - \cos \eta \right] \hspace{1cm} (5.7)$$

giving finally

$$u_0 = \frac{4 \varphi \rho_0}{\pi m'^2} \int_0^{\infty} \frac{\varphi \rho_0}{\eta^2} e^{-\eta J_1\left(\frac{\rho_0}{m'} \eta\right)} d\eta \hspace{1cm} (5.8)$$

Where $\zeta = z / r_0$ and $u_0$ and the component of stress may be written as
\[ u_0 = \frac{4\bar{\varphi} r_0}{\pi m'^2} \left[ S_0^1 - C_1^1 \right] \; , \; \tau_{x_0} = -\frac{4\bar{\varphi} \mu}{\pi} \left[ S_1^1 - C_2^1 \right] \] (5.9)

where

\[ S_n^m \left( \frac{\rho_0}{m'}, \zeta \right) = \int_0^\infty p^{n-2} \sin(p) J_m \left( \frac{p \rho_0}{m'} \right) e^{-\zeta} dp \]

\[ C_n^m \left( \frac{\rho_0}{m'}, \zeta \right) = \int_0^\infty p^{n-2} \cos(p) J_m \left( \frac{p \rho_0}{m'} \right) e^{-\zeta} dp \] (5.10)

When \( z = 0 \) it can be easily shown that

\[ S_0^1 = \frac{m'}{2 \rho_0} \left\{ \left( \frac{\rho_0^2}{m'^2} - 1 \right)^{1/2} + \frac{\rho_0^2}{m'^2} \left[ \frac{n}{2} - \tan^{-1} \left( \frac{\rho_0}{m'} - \frac{1}{2} \right) \right] \right\} \; , \; \downarrow \]

\[ C_1^1 \left( \frac{\rho_0^2 - m'^2}{1} \right)^{1/2} \rho_0 \uparrow \frac{\rho_0}{m'} \geq 1 \] (5.11)

\[ S_1^1 = \frac{m'}{\rho_0} \left[ 1 - \left( \frac{1}{1 - \frac{\rho_0^2}{m'^2}} \right)^{1/2} \right] \; \downarrow \]

\[ C_2^1 = \frac{m'}{\rho_0} \left[ 1 - \left( \frac{1}{1 - \frac{\rho_0^2}{m'^2}} \right)^{-1/2} \right] \uparrow \frac{\rho_0}{m'} \leq 1 \] (5.12)

so

\[ u_0 = \varphi r_0 \frac{\rho_0}{m'^3} \left\{ 1 - 2 \tan^{-1} \left( \frac{\rho_0}{m'} - \frac{1}{2} \right) \right\} - \frac{2}{n m'^2} \left( 1 - \frac{m'^2}{\rho_0^2} \right)^{1/2} \; , \; \frac{\rho_0}{m'} \geq 1 \] (5.13)

\[ \tau_{x_0} = -\frac{4\bar{\varphi} \mu}{\pi} \left[ \frac{1}{\left( \frac{\rho_0^2}{m'^2} - 1 \right)^{1/2}} \right] \; , \; \frac{\rho_0}{m'} \leq 1 \] (5.14)

Using (4.7) and (5.2), \( h_0 \) can be easily calculated.
6. DISCUSSION

Like all other problems in magneto-elasticity, it has been assumed here that the heat exchange between the parts of the solid is slow. It is obvious from the above expressions that the circumferential component of displacement has been affected by the magnetic field. When the magnetic field is weak or absent:

\[ h_0 \to 0, \quad m' \to 1 \quad \text{and then} \]

\[
\begin{align*}
  u_\theta &= \varphi r_0 \left\{ \rho_0 \left[ 1 - 2 \tan^{-1} \left( \rho_0^2 - 1 \right)^{1/2} \right] - \frac{2}{\pi} \left( 1 - \frac{1}{\rho_0^2} \right)^{1/2} \right\}, \quad \rho_0 \geq 1 \\
  \tau_{x\theta} &= -\frac{4\varphi \mu}{\pi} \left[ \left( \frac{1}{\rho_0^2} - 1 \right)^{1/2} \right], \quad \rho_0 \leq 1
\end{align*}
\]

which are in agreement with the corresponding result in classical problem as discussed by Reissner and Sagoci[107] and presented in the standard book of Sneddon[131].
1 INTRODUCTION

We observe a rapid development of the coupled theory of elastic bodies. As a typical example we may mention magneto-elasticity, in which the investigations are concerned with the interactions between the strain and electromagnetic field. Nowacki [91], Kaliski [66,67], Dunkin and Eringen [45] and Suhubi [138], recently, Nath and Sengupta have studied some two dimensional problems in magneto-elasticity [83]. They also investigated Reissner-Sagoci problem in magneto-elasticity [84]. Sengupta and his research associates [3,82] have studied the problems of the coupled theory of elastic bodies, in which one of the interacting field is magnetic field. The present paper deals with the investigations on the distribution of stresses and displacements in a semi-infinite homogeneous, isotropic, conducting elastic medium immersed under the influence of strong primary magnetic field \( \mathbf{H} = (0,0,H_0) \), in connection with the moving loads over the boundary of the medium. Three cases, namely, (i) supersonic, (ii) subsonic and (iii) transonic have been considered in this paper. In the classical theory of elasticity, this type of moving load
problem was first proposed by Lamb [69] and investigated by Cole and Huth [17]. Sneddon [13] gave the solution for the subsonic case and also extended the problem to include a moving load tangential to free surface. In designing the highways or airport runways or where the earth crust suffers a pressure waves, the study of the titled problem may be significant. Moreover as the earth is placed in its own magnetic field, the study of the present problem is perhaps justified.

2. BASIC EQUATIONS AND RELATIONS

The system of equations in electro-dynamics may be written as [46, 61]

\[
\text{curl} \vec{E} = \frac{\mu_0}{c} \frac{\partial \vec{h}}{\partial t}, \quad \text{curl} \vec{h} = \frac{4\pi}{c} \vec{j}, \quad \text{div} \vec{h} = 0, \quad \text{div} \vec{E} = 0
\]

\[
\vec{j} = \lambda_0 \left[ \vec{E} + \frac{\mu_0}{c} \left( \frac{\partial \vec{u}}{\partial t} \times \vec{H} \right) \right], \quad \vec{D} = \varepsilon_0 \left[ \vec{E} + \frac{1}{c} \frac{\varepsilon_0 - 1}{\varepsilon_0} \left( \frac{\partial \vec{u}}{\partial t} \times \vec{H} \right) \right]
\]

in which \( \vec{h} \) and \( \vec{E} \) as perturbations of the magnetic and electric fields respectively, \( \vec{j} \) denotes the current density vector, \( c \) is the speed of light, \( \mu_0 \) and \( \varepsilon_0 \) are the magnetic and electric permeability respectively, \( \vec{D} \) is the electric induction vector, \( \vec{u} \) is the displacement vector and \( \lambda_0 \) is the electric conductivity.

The basic equation of motion in magneto-elasticity in absence of body forces may be written as [61]

\[
\sigma_{ji,j} + T_{ji,j} = \rho \ddot{u}_i
\]

where \( T_{ij} \) denotes the Maxwell electro-magnetic stress tensor \( T_{ij} \) are the Lorentz forces. Comma denotes the partial differentiation with respect to the space coordinates and dot denotes the time derivative.

The Maxwell Tensor is related to the vector \( \vec{h} \) in the following manner

\[
T_{ij} = \left( \frac{\mu_0}{4\pi} \right) \left[ \delta_{ij} h_j + H_j h_i - \delta_{ij} H_k h_k \right] \quad [i, j = 1, 2, 3]
\]

Eliminating the vector \( \vec{E} \) and \( \vec{j} \) and the stresses and strains and expressing the vector \( T_{ij} \) by the component of the vector \( \vec{h} \) we arrive at the systems of equations [61]
\[ \nabla^2 \vec{h} - \beta \hat{\vec{h}} = -\beta \text{curl} (\vec{u} \times \vec{H}), \quad \beta = 4\pi \lambda_0 \frac{\mu_0}{c^2} \tag{2.4} \]

\[ \mu \nabla^2 \vec{u} + (\lambda + \mu) \text{grad div} \vec{u} + \left( \frac{\mu_0}{4\pi} \right) (\text{curl} \vec{h} \times \vec{H}) = \rho \ddot{\vec{u}} \]

where \( \lambda, \mu \) are Lamé's constants.

For the elastic solid ideally conducting electricity \( \exists \) [\( \beta = \infty, \vec{h} = \text{curl}(\vec{u} \times \vec{H}) \)] we get

\[ \mu \nabla^2 \vec{u} + (\lambda + \mu) \text{grad div} \vec{u} + \left( \frac{\mu_0}{4\pi} \right) [\text{curl} \text{curl}(\vec{u} \times \vec{H}) \times \vec{H}] = \rho \ddot{\vec{u}} \tag{2.5} \]

If the magnetic field \( \vec{H} \) is of the form \( \vec{H} = (0,0,H_0) \) then equation (2.5) reduces to

\[ \mu \nabla^2 \vec{u} + (\lambda + \mu + \alpha_0^2) \text{grad div} \vec{u} = \rho \ddot{\vec{u}} \tag{2.6} \]

where \( \alpha_0 = \frac{\mu_0 H_0^2}{4\pi \rho} \), \( \alpha_0 \) is the Alfvén velocity.

Equation (2.6) decomposes into the following system of equations

\[ \left( 1 + \frac{\alpha_0^2}{c_1^2} \right) \nabla^2 \phi = c_1^{-2} \ddot{\phi}, \quad \nabla^2 \psi = c_2^{-2} \ddot{\psi} \tag{2.7} \]

where \( \phi(x,y,t) \) and \( \psi(x,y,t) \) are displacement potentials, given by

\[ u = \phi_x - \psi_y \quad ; \quad v = \phi_y + \psi_x \tag{2.8} \]

and \( c_1 = \sqrt{\frac{(\lambda + 2\mu)}{\rho}} \) and \( c_2 = \sqrt{\frac{\mu}{\rho}} \) denotes the dilatation wave speed and shear wave speed respectively. The component of stresses in terms of the displacements \( (u, v) \) are

\[ \sigma_{yy} = (\lambda + 2\mu) \frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x}, \quad \sigma_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{2.9} \]

3. FORMULATION OF THE PROBLEM AND BOUNDARY CONDITIONS

Let us consider a homogeneous elastic semi-space \( y \geq 0 \) immersed under the influence of constant strong primary magnetic field \( \vec{H} = (0,0,H_0) \) selecting the origin on the free plane boundary of the semi-space and \( y \)-axis pointing into the medium we
suppose that on plane boundary of the medium, there acts a loading \( P(x + Ut) \) which moving with a constant speed \( U \) in the negative \( x \)-axis direction for a long time so that a steady state prevails in the neighbourhood of the loading as seen by an observer moving with the load.

If we consider the Galilean transformation in [58]

\[
x' = x + Ut, \quad y' = y, \quad t' = t
\]

(3.1)
equation (2.7) reduces to the following form

\[
\left(1 + \frac{\alpha_0^2}{c_1^2}\right) \nabla'^2 \phi = \frac{U^2}{c_1^2} \frac{\partial^2 \phi}{\partial x'^2}, \quad \nabla'^2 \equiv \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}
\]

(3.2)

On introducing Mach numbers [58]

\[
M_1 = \frac{U}{c_1}, \quad M_2 = \frac{U}{c_2}
\]

(3.3)

and the parameters [14]

\[
\beta_1 = \sqrt{M_1^2 - 1}, \quad \beta_2 = \sqrt{M_2^2 - 1} ; \quad M_1, M_2 > 1
\]

(3.4)

and

\[
\overline{\beta}_1 = \sqrt{1 - M_1^2}, \quad \overline{\beta}_2 = \sqrt{1 - M_2^2} ; \quad M_1, M_2 < 1
\]

(3.5)

We obtain the following partial differential equations from (3.2)

\[
\left\{ \begin{array}{c}
\left( \frac{\beta_1^2}{\beta_1^2 + \frac{\alpha_0^2}{c_1^2}} \right) \frac{\partial^2 \phi}{\partial x'^2} - \left( 1 + \frac{\alpha_0^2}{c_1^2} \right) \frac{\partial^2 \phi}{\partial y'^2} = 0 \\
\beta_1^2 \frac{\partial^2 \psi}{\partial x'^2} - \frac{\partial^2 \psi}{\partial y'^2} = 0 \\
\left( \frac{\beta_1^2}{\beta_1^2 + \frac{\alpha_0^2}{c_1^2}} \right) \frac{\partial^2 \phi}{\partial x'^2} + \left( 1 + \frac{\alpha_0^2}{c_1^2} \right) \frac{\partial^2 \phi}{\partial y'^2} = 0 \\
\overline{\beta}_1^2 \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} = 0
\end{array} \right\}
\]

(3.6)

(3.7)
Expression of the stress components in (2.9) may be reduced in terms of the displacement potentials and Mach numbers as

\[
\sigma_{yy} = \mu \left( \frac{M_2^2 - 2}{\partial x'^2} \frac{\partial^2 \phi}{\partial x'^2} + 2 \frac{\partial^2 \psi}{\partial x' \partial y'} \right)
\]

\[
\sigma_{xy} = \mu \left( 2 \frac{\partial^2 \phi}{\partial x' \partial y'} - \left( M_2^2 - 2 \right) \frac{\partial^2 \psi}{\partial x'^2} \right)
\]

(3.8)

For a concentrated line load, the boundary conditions in the moving co-ordinates may be written as [58]

\[
\sigma_{yy} = -P \delta(x')
\]

\[
\sigma_{xy} = 0
\]

at \( y' = 0 \)

(3.9)

where \( \delta(x') \) denotes the Dirac delta function. Since \( x \) and \( t \) enter the boundary conditions only in the combination \( (x + Ut) \) i.e. boundary conditions are independent of \( t' \).

The boundary conditions in (3.9) become at \( y' = 0 \), on integration

\[
\left\{ \begin{array}{l}
(M_2^2 - 2) \frac{\partial \phi}{\partial x'} + 2 \frac{\partial \psi}{\partial y'} = -\frac{P}{\mu} H(x') \\
2 \frac{\partial \phi}{\partial y'} - (M_2^2 - 2) \frac{\partial \psi}{\partial x'} = 0
\end{array} \right.
\]

(3.10)

\[
H(x') = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{\frac{ix}{\epsilon + r\lambda}} d\lambda
\]

where \( H(x') \) is the Heaviside unit function [57,58].

The solution of our problem is given by the function \( \phi \) and \( \psi \) which satisfy the differential equations (3.6) and (3.7) obeying the boundary condition on the free surface (3.10) with the appropriate radiation and finiteness conditions at infinity.

4. SOLUTION OF THE PROBLEM

The nature of the solution depends on the Mach numbers \( M_1, M_2 \). Hence the three subcases can be distinguished (a) \( M_1 > 1, M_2 > 1 \) (supersonic) (ii) \( M_1 < 1, M_2 < 1 \) (subsonic) (c) \( M_1 < 1, M_2 > 1 \) (transonic). Since \( C_1 > C_2 \) so that \( M_2 > M_1 \), the above three cases exhaust all possibilities.
(a) Supersonic case \((M_2 > M_1 > 1)\)

We consider the differential equations (3.6) and seek solutions in the form

\[
\phi(x', y'), \psi(x', y') = \left\{ \hat{\phi}(y'), \hat{\psi}(y') \right\} e^{i\lambda x'}
\]

From (2.8) and (3.6) we obtain

\[
\phi(x', y') = Ae^{i\lambda x'} e^{-i\beta_1 y'}
\]
\[
\psi(x', y') = Be^{i\lambda x'} e^{-i\beta_2 y'}
\]

where \(\beta_1^* = \left( \beta_1^2 - \frac{\alpha_0^2}{c_1^2} \right) \left( 1 + \frac{\alpha_0^2}{c_1^2} \right) \approx \beta_1^2 - M_1^2 \frac{\alpha_0^2}{c_1^2} \) (4.3)

[neglecting higher powers of \(\frac{\alpha_0}{c_1}\)]

Substituting (4.2) in (3.10) we obtain two simultaneous equations

\[
(M_2^2 - 2)i\lambda A(\lambda) - 2B(\lambda)i\lambda \beta_2 = P(\lambda)
\]
\[
-2i\lambda \beta_1 A(\lambda) - (M_2^2 - 2)i\lambda B(\lambda) = 0
\]

where \(P(\lambda) = -\frac{\mu}{\mu \lambda} \) is the Fourier transform of \(-\frac{P}{\mu} H(x')\). Now the expressions for \(A(\lambda)\) and \(B(\lambda)\) are

\[
A(\lambda) = (M_2^2 - 2) \frac{1}{\Delta} \frac{P(\lambda)}{i\lambda} = \frac{P(M_2^2 - 2)}{\mu \Delta \lambda^2} \quad \Delta = (2 - M_2^2)^2 + 4\beta_1^* \beta_2
\]

\[
B(\lambda) = -\frac{2P\beta_1^*}{\mu \Delta \lambda^2}
\]

Hence from (2.8), (3.8) and (4.2) we obtain the expression for stresses and displacements as \([47, 154]\)

\[
\sigma_{yy} = \frac{P}{\Delta} \left[ (2 - M_2^2)(2\beta_1^* + \frac{\lambda}{\mu} M_1^2) \delta(x' - \beta_1 y') - 4\beta_1^* \beta_2 \delta(x' - \beta_2 y') \right]
\]
\[
\sigma_{xy} = -\frac{P}{\Delta} 2\beta_1^* (2 - M_2^2) \left[ \delta(x' - \beta_1 y') - \delta(x' - \beta_2 y') \right]
\]
\[
u = \frac{P}{\mu \Delta} \left[ -\beta_1^* (2 - M_2^2) H(x' - \beta_1 y') + 2\beta_1^* \beta_2 H(x' - \beta_2 y') \right]
\]

\[
\sigma_{xx} = \frac{P}{\mu \Delta} \left[ (2 - M_2^2) H(x' - \beta_1 y') + 2\beta_1^* \beta_2 H(x' - \beta_2 y') \right]
\]
These equations show that the disturbances are marked by two Mach waves

\[ x' - \beta_1^* y' = 0 \quad ; \quad x' - \beta_2 y' = 0 \]  \hspace{2cm} (4.7)

The medium is undisturbed in front of these Mach waves.

(b) Subsonic cases \((M_2 > 1 > M_1)\)

Seeking solution as \(m = \pm 1\) we obtain the formal solution of the differential equation (3.7) as

\[
\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(x)e^{i\lambda x'}e^{-\beta_1^*\lambda |y'|}d\lambda
\]

\[
\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(\lambda)e^{i\lambda x'}e^{-\beta_2\lambda |y'|}d\lambda
\]

where \(\beta_1^{*2} = \left(\frac{\beta_1^2 + \alpha_0^2}{\varepsilon_1^2}\right)\sqrt{\left(1 + \frac{\alpha_0^2}{\varepsilon_1^2}\right) \equiv \beta_1^2 + M_1^2 \frac{\alpha_0^2}{\varepsilon_1^2}}\) \hspace{2cm} (4.8)

[neglecting higher powers of \(\left(\frac{\alpha_0}{\varepsilon_1}\right)^2\)]

Now substituting the above values of \(\phi\) and \(\psi\) in the boundary condition (3.10) we obtain as before

\[
i\lambda(M_2^2 - 2)A(\lambda) - 2\beta_2 |\lambda| B(\lambda) = P(\lambda)
\]

\[
-2\beta_1^* |\lambda| A(\lambda) - (M_2^2 - 2)i\lambda B(\lambda) = 0
\]

where \(P(\lambda)\) is Fourier transform of \(-\frac{p}{\mu}H(x')\)

From (4.10) we can write the expressions for \(A(\lambda)\) and \(B(\lambda)\) as

\[
A(\lambda) = \frac{P(M_2^2 - 2)}{\mu \Delta \lambda^2} \quad \bar{\lambda} = (2 - M_2^2)^2 - 4\beta_1^* \beta_2
\]

\[
B(\lambda) = \frac{-2\beta_1^* p}{\mu \Delta |\lambda| i\lambda}
\]

Now the components of displacement and stresses may be written by (2.8), (3.8) and (4.8) \([17, 15]\)
\[
\begin{align*}
    u &= \frac{K_1 P}{\mu} \left( 1 - \frac{\theta_1}{\pi} \right) - \frac{K_2 P}{\mu} \left( 1 - \frac{\theta_2}{\pi} \right) \\
    v &= \frac{P}{\mu \pi} (K_2 \log r_2 - \beta_1^* K_1 \log r_1) \\
    \sigma_{yy} &= -\frac{1}{\pi} \left[ K_1 P (2 - M_2^2) \frac{\sin \theta_1}{r_1} - 2 K_2 \beta_2 P \frac{\sin \theta_2}{r_2} \right] \\
    \sigma_{xy} &= -\frac{1}{\pi} 2 K_1 \beta_1^* P \left[ \frac{\cos \theta_1}{r_1} - \frac{\cos \theta_2}{r_2} \right]
\end{align*}
\]

where
\[
K_1 = \frac{(2 - M_2^2)}{\Delta} \quad \quad K_2 = \frac{2 \beta_1^*}{\Delta}
\]

\[
x' + i \beta_1^* y' = r_1 e^{i \theta_1}, \quad x' + i \beta_2 y' = r_2 e^{i \theta_2}, \quad 0 \leq \theta_1, \theta_2 \leq \pi
\]

(c) Transonic case \( M_1 < 1, M_2 > 1 \)

In this case the load is moving at a speed slower than the longitudinal wave speed but faster than the shear wave speed. Seeking solution as in (4.1) for the differential equations (3.7)_1 and (3.6)_2 we may write the formal solution as

\[
\begin{align*}
    \phi &= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\lambda) e^{i \lambda x'} e^{-i \beta_1^* |\lambda| y'} d\lambda \\
    \psi &= \frac{1}{2\pi} \int_{-\infty}^{\infty} B(\lambda) e^{-i \lambda \beta_2 y'} e^{i \lambda x'} d\lambda
\end{align*}
\]

where
\[
\beta_1^* \equiv \beta_1^2 + M_1^2 \frac{\alpha_0^2}{c_1^2}
\]

Using the boundary conditions (3.10) we obtain as before.

\[
\begin{align*}
    i \lambda (M_2^2 - 2) A(\lambda) - 2 i \lambda \beta_2 B(\lambda) &= P(\lambda) \\
    -2 \beta_1^* |\lambda| A(\lambda) - (M_2^2 - 2)i \lambda B(\lambda) &= 0
\end{align*}
\]

The expressions for \( A(\lambda) \) and \( B(\lambda) \) may be written as

\[
\begin{align*}
    A(\lambda) &= \frac{P}{\mu} \frac{(M_2^2 - 2)}{(M_2^2 - 2)^2 \lambda^2 - 4 \beta_1^* \beta_2 i \lambda |\lambda|} \\
    B(\lambda) &= -\frac{P |\lambda|}{\mu i \lambda} \frac{2 \beta_1^*}{(M_2^2 - 2)^2 \lambda^2 - 4 \beta_1^* \beta_2 i \lambda |\lambda|}
\end{align*}
\]
So the displacement components may be written as \[47, l32\] before

\[
\begin{align*}
\mathbf{u} &= \frac{P}{\mu} \left\{ K_4 \left( \pi - 2 \tan^{-1} \beta_1^* \frac{y'}{x'} \right) + \frac{2 \beta_1^* \beta_2}{(2 - M_2^2)} \left[ \log |x' - \beta_2 y'| \right] \\
&\quad - K_3 \left[ \log (x'^2 + \beta_1^* y'^2) + \frac{2 \beta_1^* \beta_2}{(2 - M_2^2)} \right] \\
&\quad - \frac{1}{2 \pi} \left[ \log (x'^2 + \beta_2 y'^2) + \frac{2 \beta_1^* \beta_2}{(2 - M_2^2)} \right] \right\} \\
\mathbf{v} &= \frac{P}{\mu} \left\{ K_4 \left[ -\beta_2 \log (x'^2 + \beta_1^* y'^2) + \frac{2 \beta_1^*}{(2 - M_2^2)} \log |x' - \beta_2 y'| \right] \\
&\quad + K_3 \left[ \pi - 2 \tan^{-1} \frac{\beta_1^* y'}{x'} - \frac{2 \beta_1^*}{2 - M_2^2} \right] \right\}
\end{align*}
\tag{4.16}
\]

where \( K_3 = \frac{-4 \beta_1^* \beta_2 (2 - M_2^2)}{\Delta_3} \), \( K_4 = \frac{(2 - M_2^2)^3}{\Delta_3} \), \( \Delta_3 = (2 - M_2^2)^4 + 16 \beta_1^* \beta_2^2 \)

Similarly the component of stresses may be calculated as before.

5. DISCUSSION

If the magnetic field is weak or absent, then \( \alpha_0 \to 0 \) and the result that obtained are in good agreement with the corresponding result as studied by Fung [53]. In supersonic case the expressions in (4.7) show that the disturbances are marked by two mach waves

\[x' - \beta_1^* y' = 0 \quad \text{and} \quad x' - \beta_2 y' = 0\]

(5.1)

In the classical case the Mach waves are given by \( x' - \beta_1 y' \) and \( x' - \beta_2 y' = 0 \)

Now since

\[
\beta_1^* \approx \beta_1^2 - M_1^2 \frac{\alpha_0^2}{c_1^2} \quad \text{and} \quad \frac{1}{\beta_1} > \frac{1}{\beta_1^2 - M_1^2 \frac{\alpha_0^2}{c_1^2}} \]

(5.2)

the angle between the lines is decreased in comparison with the classical case [53].

In the subsonic case there is a singularity under the point of application of the load, where the vertical displacement is logarithmically singular, which is clear from the solutions given by (4.12).
The denominator of $K_1$ and $K_2$ given in the expression (4.12) vanishes when

$$\bar{\Lambda} = (2 - M_2^2)^2 - 4\beta_1 \beta_2 = 0 \quad (5.3)$$

If the definitions in (3.3), (3.5) and (4.9) are substituted in the equation $\bar{\Lambda} = 0$ we obtain

$$\left(1 + \frac{U_0^2}{c_1^2}\right)^{\frac{1}{2}} \left(2 - \frac{U^2}{c_2^2}\right)^2 - 4 \left(1 - \frac{U^2}{c_1^2}\right)^\frac{1}{2} \left(1 - \frac{U^2}{c_2^2}\right)^2 = 0 \quad (5.4)$$

As $\alpha_0 \to 0$, we easily obtain

$$\left(2 - \frac{U^2}{c_2^2}\right)^2 - 4 \left(1 - \frac{U^2}{c_1^2}\right)^\frac{1}{2} \left(1 - \frac{U^2}{c_2^2}\right)^2 = 0 \quad (5.5)$$

Which is the characteristic equation of Rayleigh surface waves [103]. Hence the equation (5.4) may be regarded as the Rayleigh type of waves under the influence of magnetic field satisfying the specified condition, which has mentioned before. Hence if the load moves steadily at the Rayleigh type of wave speed, the responses will be infinitely large.

In the transonic case it is evident that the solution becomes singular both under the load and along the characteristics $x' - \beta_2 y' = 0$. 