CHAPTER – 2

PROPAGATION OF WAVES IN PRESENCE OF A COMPRESSIONAL WAVE SOURCE
PAPER-4

INFLUENCE OF GRAVITY ON PROPAGATION OF WAVES IN A MEDIA IN PRESENCE OF A COMPRESSIONAL SOURCE

1. INTRODUCTION

Stoneley[136] studied the effect of the ocean on the transmission of Rayleigh waves considering the bottom of the ocean as a solid half-space. In classical theory of elasticity, Jeffreys[63], Muskat[80] and Sommerfeld [133] have discussed the wave propagation for the case, where distance of a point source from the plane interface is finite. In particular, Press and Ewing [100] studied the propagation of waves when the point source being present in the liquid layer. Their results are directly related to an important practical problem, that of the ‘refraction arrival’ from a source to a receiver in seismology of near earthquakes and in seismic refraction investigations. In the classical problem of elastic waves and vibrations, the gravity effect is generally neglected. The effect of gravity was first studied by Bromwich[16] in the problem of propagation of waves in solid in particular on an elastic globe. Subsequently, the investigations of the effect of gravity was considered by Love[73] who exhibited that the velocity of Rayleigh waves is increased to a significant extent by the gravitational field when wave lengths are large.

More recently, Biot [13] developed a theory of initial stress and use it to investigate the influence of gravity on Rayleigh waves, assuming the force of gravity to create a type of initial stress of hydrostatic nature and the medium to be incompressible. The initial stress is produced in the body by a slow process of creep where the shear stress tend to become small or vanish after a long interval of time. De and Sengupta studied the effect of gravity on surface waves [36], on the

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propagation of waves in an elastic layer [37] and on plane Lamb’s problem [38 ] respectively.

Das and Sengupta[33] investigated the effect of gravity on visco-elastic surface waves in solid elastic medium. Assuming that the boundaries are all parallel planes, in the axis symmetric problem of propagation of waves under the influence of gravity in a media composed of a liquid layer and an underlying solid half space when the compressional wave source being present in the upper liquid part, the following two cases have been considered (i)the waves are smaller than ordinary earthquake Rayleigh waves (ii)the waves are considerably longer than ordinary earthquake Rayleigh waves

2. BASIC EQUATIONS AND RELATIONS

We shall use here the subscript (1) for liquid and subscript (2) for the solid part respectively. In the Cartesian co-ordinate system the two dimensional equations of motion in an elastic solid medium in absence of body forces are

\[
\rho_2 \frac{\partial^2 u_2}{\partial t^2} = (\lambda_2 + \mu_2) \frac{\partial \Theta}{\partial x} + \mu_2 \nabla^2 u_2
\]

\[
\rho_2 \frac{\partial^2 w_2}{\partial t^2} = (\lambda_2 + \mu_2) \frac{\partial \Theta}{\partial z} + \mu_2 \nabla^2 w_2
\] (1)
The two dimensional equations of motion in the cylindrical co-ordinate system \((r, \theta, z)\) may be written as:

\[
\begin{align*}
\left(\lambda_2 + 2\mu_2\right)
&= \frac{\partial^2 q_2}{\partial r^2} + \frac{1}{r} \frac{\partial q_2}{\partial r} - \frac{q_2}{r^2} + \frac{\partial^2 w_2}{\partial z^2} + \mu_2 \left(\frac{\partial^2 q_2}{\partial z^2} - \frac{\partial^2 w_2}{\partial z \partial r}\right) = \rho_2 \frac{\partial^2 q_2}{\partial t^2} \\
\left(\lambda_2 + 2\mu_2\right)
&= \frac{\partial^2 q_2}{\partial z \partial r} + \frac{1}{r} \frac{\partial q_2}{\partial r} + \frac{\partial^2 w_2}{\partial z^2} - \mu_2 \left(\frac{\partial q_2}{\partial z} - \frac{\partial w_2}{\partial r}\right) - \mu_2 \left(\frac{\partial^2 q_2}{\partial z^2} - \frac{\partial^2 w_2}{\partial r^2}\right) = \rho_2 \frac{\partial^2 w_2}{\partial z^2}
\end{align*}
\]

where \(\Theta\) is the cubical dilatation, \(q_2\) and \(w_2\) are the displacements in the \(r\) and \(z\) directions and the angle \(\theta\) does not appear because of the axial symmetry. Now we define \(q_2\) and \(w_2\) in terms of the potential function \(\phi_2\) and the function \(W_2\) as:

\[
q_2 = \frac{\partial \phi_2}{\partial r} - \frac{\partial W_2}{\partial z} ; \quad w_2 = \frac{\partial \phi_2}{\partial z} + \frac{\partial (r W_2)}{r \partial r}
\]

Substituting the values of \(q_2\) and \(w_2\) from (3) in (2) and using the relation \(W_2 = -\frac{\partial \psi_2}{\partial r}\) we obtain

\[
\nabla^2 \phi_2 = \frac{1}{\alpha_2^2} \frac{\partial^2 \phi_2}{\partial t^2} ; \quad \nabla^2 \psi_2 = \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2} ; \quad \nabla^2 \equiv \frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}
\]

and

\[
\alpha_2 = \left(\frac{\lambda_2 + 2\mu_2}{\rho_2}\right)^{\frac{1}{2}} , \quad \beta_2 = \left(\frac{\mu_2}{\rho_2}\right)^{\frac{1}{2}}
\]

Now for the liquid part, the equations of motion, under the influence of gravity, may be written in terms of the potential function \(\phi_1\) as:

\[
\frac{\partial^2 \phi_1}{\partial t^2} = \alpha_2^2 \nabla^2 \phi_1 + g \frac{\partial \phi_1}{\partial z}
\]
In the above, $\alpha_1$, $\alpha_2$ are the velocity of compressional waves in liquid and solid respectively and $\beta_2$ is the velocity of distortional waves in the solid. The displacements may be expressed in terms of the potentials $\phi_1$, $\phi_2$ and $\psi_2$ as

$$\begin{align*}
q_1 &= \frac{\partial \phi_1}{\partial t} \\
q_2 &= \frac{\partial \phi_2}{\partial t} + \frac{\partial^2 \psi_2}{\partial t^2} \frac{\partial \phi_2}{\partial t} \\
w_1 &= \frac{\partial \phi_1}{\partial z} \\
w_2 &= \frac{\partial \phi_2}{\partial z} + \frac{\partial^2 \psi_2}{\partial z^2} - \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2}
\end{align*}$$

(7)

3. GENERAL THEORY AND BOUNDARY CONDITIONS

We take $z = 0$ as the free surface of the liquid, $z = H$ as the surface of separation of the two media and $z$ axis vertically downwards. The source lies at the point $(0, 0, h)$ in the liquid part.

**Boundary condition**: The boundary conditions are

$$\begin{align*}
\phi_1 &= 0 \quad \text{at} \quad z = 0 \\
\omega_1 &= \omega_2 \quad \text{at} \quad z = H \\
(p_{zt})_2 &= 0 \quad \text{at} \quad z = H \\
(p_{zz})_1 &= (p_{zz})_2 \quad \text{at} \quad z = H
\end{align*}$$

(8)\(9)\(10)\(11)

where

$$\begin{align*}
(p_{zt})_2 &= \mu_2 \left( \frac{\partial q_2}{\partial z} + \frac{\partial w_2}{\partial t} \right) \\
(p_{zz})_1 &= \lambda_1 \nabla^2 \phi_1 \\
(p_{zz})_2 &= \lambda_2 \nabla^2 \phi_2 + 2\mu_2 \frac{\partial w_2}{\partial z}
\end{align*}$$

(12)

4. SOLUTION OF THE PROBLEM

(4.1) Case (i) The waves are smaller than ordinary earthquake Rayleigh waves: In this case we consider the equation (6) for the liquid part. But the gravity terms in the equations for the solid part of the system are omitted by Scholte [14]; and so we use the equation (4) for the solid part. Now we
follow Pekeris in dividing the liquid layer into two parts by the plane $z = h$ so that the potential
is represented by two different expression in the form given by Ewing et al.

$\phi_1$ is represented by two different expression in the form given by Ewing et al.

$$\phi_1' = \int_0^\infty A(K)J_0(Kr)\sin\lambda_1z \exp \{ \frac{iv_1z}{2\alpha_1^2} \} dK \quad 0 \leq z \leq h \quad (13)$$

$$\phi_1'' = \int_0^\infty [B(K)\sin\lambda_1z + C(K)\cos\lambda_1z] \exp \{ \frac{iv_2z}{2\alpha_2^2} \} J_0(Kr) dK \quad h \leq z \leq H \quad (14)$$

where

$$\lambda_1 = \left( \frac{v_1^2 - g^2}{4\alpha_1^4} \right), \quad \lambda_2 = K_2^2 - K_1^2, \quad K_\alpha = \frac{\omega}{\alpha_1} \quad (15)$$

Here the condition (8) at the free surface $z = 0$ is satisfied by the assumed form of (13). Also $\phi_1'$ and

$\phi_1''$ satisfy the following conditions

$$\phi_1' = \phi_1'' \quad \text{at} \quad z = h \quad (16)$$

and

$$\frac{\partial \phi_1'}{\partial z} - \frac{\partial \phi_1''}{\partial z} = 2\exp(i\omega t) \int_0^\infty J_0(Kr) dK = 2\exp(i\omega t) \int_0^\infty J_0(Kr) dK \quad \text{at} \quad z = h \quad (17)$$

For the solid part

$$\phi_2 = \int_0^\infty Q_2(K) \exp[-iv_2z] J_0(Kr) dK, \quad \psi_2 = \int_0^\infty S_2(K) \exp[-iv_2z] J_0(Kr) dK \quad (18)$$

where

$$v_2^2 = K_2^2 - K_1^2, \quad \lambda_2 = \frac{\omega}{\alpha_2}, \quad K_\beta = \frac{\omega}{\beta_2} \quad (19)$$

Using (13) and (14) in (16) and (17) and then substituting the values of $\phi_1'$, $\phi_2$, $\psi_2$ in the boundary

conditions (9), (10) and (11) we obtain

$$\exp\left[ -\frac{gH}{2\alpha_1^2} \right] \left[ \frac{g}{2\alpha_1^2} \sin\lambda_1H \right] B(K) + iv_2Q_2(K) \exp[-iv_2H] - K^2S_2(K) \exp[-v_2H]$$

$$= \{ 2 \sin\lambda_1H \left[ \frac{g}{2\alpha_1^2} \cos\lambda_1H \right] + C_0 \}$$

(20)
\[2i\overline{v}_2 Q_2(K) \exp[-i\overline{v}_2 H] + (\overline{v}_2^2 - K^2) S_2^{(K)} \exp[-i\overline{v}_2 H] = 0\] (21)

\[\rho_1 \exp\left\{\frac{g\hbar}{2\alpha_1}\right\} \{\omega^2 \sin\lambda_1 H + g[\overline{\lambda}_1 \cos\lambda_1 H - \left(\frac{g}{2\alpha_1}\right) \sin\lambda_1 H]}B(K) + (2\mu_2 K^2 - \rho_2 \omega^2)Q_2 \exp[-i\overline{v}_2 H] - 2\mu_2 K^2 i\overline{v}_2 S_2 \exp[-i\overline{v}_2 H] = -\frac{2\rho_1}{\overline{\lambda}_1 C^0} \exp\left[\frac{g(h - H)}{2\alpha_1}\right]\{\omega^2 \cos\lambda_1 H - g[\overline{\lambda}_1 \sin\lambda_1 H + \left(\frac{g}{2\alpha_1}\right) \cos\lambda_1 H]\} \sin\lambda_1 h\] (22)

where
\[C^0 = \left[1 - \left(\frac{g}{\sqrt{\overline{\lambda}_1 \alpha_1}}\right) \sin\lambda_1 h \cos\lambda_1 h\right]\] (23)

Determining the functions \(B, Q_2\) and \(S_2\) from (20) to (22) and then evaluating \(A\) from (16) and (17) we have in this case

\[\phi_t' = 2\exp\{\int_0^\infty \frac{2\alpha_1}{\overline{\lambda}_1 \Delta(K) C^0} \left\{\rho_2^2 \beta_2^2 [4K^2 \overline{v}_2 \overline{v}_2' + (2K^2 - K_{\beta_2}^2)^2] [\overline{\lambda}_1 \cos\lambda_1 (H - h) - \frac{g}{2\alpha_1^2} \sin\lambda_1 (H - h)] + \frac{\rho_1 \omega^2 i\overline{v}_2}{\beta_2^2} \{\omega^2 \sin\lambda_1 (H - h) + \overline{\lambda}_1 \cos\lambda_1 (H - h)\} - \frac{\rho_1 \omega^2 i\overline{v}_2}{\beta_2^2} \{\omega^2 \sin\lambda_1 (H - h) + \overline{\lambda}_1 \cos\lambda_1 (H - h)\}\} dK, \quad 0 \leq z \leq h\] (24)

\[\phi_t'' = 2\exp\{\int_0^\infty \frac{2\alpha_1}{\overline{\lambda}_1 \Delta(K) C^0} \left\{\rho_2^2 \beta_2^2 [4K^2 \overline{v}_2 \overline{v}_2' + (2K^2 - K_{\beta_2}^2)^2] [\overline{\lambda}_1 \cos\lambda_1 (H - z) - \frac{g}{2\alpha_1^2} \sin\lambda_1 (H - z)] + \frac{\rho_1 \omega^2 i\overline{v}_2}{\beta_2^2} \{\omega^2 \sin\lambda_1 (H - z) + \overline{\lambda}_1 \cos\lambda_1 (H - z)\} - \frac{\rho_1 \omega^2 i\overline{v}_2}{\beta_2^2} \{\omega^2 \sin\lambda_1 (H - z) + \overline{\lambda}_1 \cos\lambda_1 (H - z)\}\} dK, \quad h \leq z \leq H\] (25)
\[ \phi_2 = -2 \exp(i\omega t) \int_0^\infty J_0(K \lambda_1 h) \exp\left[ \frac{g(h-H)}{2\alpha_1^2} \right] \rho_1 \omega^2 (2K^2 - K_\beta^2) \exp[-i\nu_2(z-H)]dK \] (26)

\[ \psi_2 = 4e(i\omega t) \frac{J_0(K \lambda_1 h) \exp\left[ \frac{g(h-H)}{2\alpha_1^2} \right] i\nu_2 \rho_1 \omega^2}{C^0 \Delta} \exp[-i\nu_2'(z-H)]dK \] (27)

where

\[ \Delta = [\lambda_1 \cos \lambda_1 H - \left( \frac{g}{2\alpha_1^2} \right) \sin \lambda_1 H] [4K^2 \nu_2 \nu_2' + (2K^2 - K_\beta^2)^2 \rho_2 \beta^2 + \nu_2'^2 \rho_2 \beta^2 + \nu_2^2 \rho_2 \beta^2] + [\omega^2 \sin \lambda_1 H + g(\lambda_1 \cos \lambda_1 H - \left( \frac{g}{2\alpha_1^2} \right) \sin \lambda_1 H)] \rho_1 i\nu_2 \omega^2 \] (28)

The integrals can be represented by the sum of branch line integrals and residues. The residues correspond to the pole \( K = K_\alpha \) given by the roots of the equation

\[ \Delta(K) = 0 \] (29)

(4.2) Case (ii) The waves are longer than ordinary earthquake Rayleigh waves: In this case we consider the equation (6) for the liquid part as before. The equation of motion for the solid part now also affected by the gravitational field. If \( g \) be the acceleration due to gravity then component of body forces are \( X = 0, Z = g \). We shall assume that the initial stress due to gravity is hydrostatic in nature.

The state of initial stress are

\[ s_{11} = s_{33} = s \quad s_{13} = 0 \] (30)

where \( s \) is a function of depth. The equilibrium conditions of the initial stress field are

\[ \frac{\partial s}{\partial x_1} = 0 \quad \frac{\partial s}{\partial x_3} + \rho_2 g = 0 \] (31)

The dynamical equations of the two dimensional problem under the initial stress field are
where

\[ s_{jk} = 2 \mu e_{jk} + \lambda e \delta_{jk} \]

(33)

\[ e_{jk} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_K} + \frac{\partial u_K}{\partial x_j} \right) \quad e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \]

Equation (32) in cylindrical co-ordinates and in terms of potentials \( \phi_2 \) and \( \psi_2 \) may be written as

\[
\left( V^2 - \frac{1}{\alpha_2^2} \frac{\partial^2}{\partial t^2} \right) \phi_2 + \left( \frac{g}{\alpha_2^2} \left( \frac{\partial^2 \psi_2}{\partial r^2} - \frac{1}{r} \frac{\partial \psi_2}{\partial r} \right) = 0 \right)
\]

(34)

\[
\left( V^2 - \frac{1}{\beta_2^2} \frac{\partial^2}{\partial t^2} \right) \psi_2 - \left( \frac{g}{\beta_2^2} \right) \phi_2 = 0
\]

(35)

Now we seek solution for \( \phi_2 \) and \( \psi_2 \) in the form given by (18) and we obtain from (34) and (35) as

\[
\phi_2 = \int_0^\infty \left[ a_2 (K) \exp[-i \lambda_2^2 z] + Q_2^* (K) \exp[-i \lambda_2^2 z] \right] J_0 (Kr) dK \]

(36)

\[
\psi_2 = \int_0^\infty \left[ a_2^* Q_2^* (K) \exp[-i \lambda_2^2 z] + a_2 Q_2 (K) \exp[-i \lambda_2^2 z] \right] J_0 (Kr) dK \]

(37)

where

\[
\lambda_2^2, \lambda_2^2 = K^2 - \frac{1}{2} \left[ K^2_{\alpha_2} + K^2_{\beta_2} \pm \left( K^2_{\alpha_2} - K^2_{\beta_2} \right)^2 + \frac{4g^2 K^2}{\alpha_2^2 \beta_2^2} \right]^{1/2}
\]

(38)

\[
a_2' = \frac{\alpha_2^2}{-gK^2} \left( -K^2 + \lambda_2^2 + K^2_{\alpha_2} \right), \quad a_2^* = \frac{\alpha_2^2}{-gK^2} \left( -K^2 + \lambda_2^2 + K^2_{\alpha_2} \right)
\]
Substituting the values of $\phi_1^\prime$, $\phi_2$ and $\psi_2$ from (25), (36) and (37) in the boundary condition we obtain

as before

$$
\exp\left[-\frac{gH}{2\alpha_1^2}\right] \left[ \bar{\lambda}_1 \cos \bar{\lambda}_1 H - \left( \frac{g}{2\alpha_1^2} \right) \sin \bar{\lambda}_1 H \right] B(K) + b_2 Q_2' \exp[-i\lambda_2^2 H] + b_2' Q_2'' \exp[-i\lambda_2^2 H] = 0
$$

and

$$
\exp\left[-\frac{gH}{2\alpha_1^2}\right] \left[ \bar{\lambda}_1 \cos \bar{\lambda}_1 H - \left( \frac{g}{2\alpha_1^2} \right) \sin \bar{\lambda}_1 H \right] \exp[g(h-H) / 2\alpha_1^2] = 0
$$

where

$$
b_2' = i\lambda_2 + a_2' \left( \lambda_2^2 - K\beta_2^2 \right), \quad b_2'' = i\lambda_2 + a_2' \left( \lambda_2^2 - K\beta_2^2 \right)
$$

$$
c_2' = 2i\lambda_2 + a_2' \left( 2\lambda_2^2 - K\beta_2^2 \right), \quad c_2'' = 2i\lambda_2 + a_2' \left( 2\lambda_2^2 - K\beta_2^2 \right)
$$

$$
d_2' = -\lambda_2 K^2 - \rho_2 a_2 \lambda_2^2 + 2\mu_2 i\alpha_2' \lambda_2' \left[ \lambda_2^2 - K\beta_2^2 \right]
$$

$$
d_2'' = -\lambda_2 K^2 - \rho_2 a_2 \lambda_2^2 + 2\mu_2 i\alpha_2' \lambda_2' \left[ \lambda_2^2 - K\beta_2^2 \right]
$$

Solving (39) to (41) we obtain the values of $B$, $Q_2'$ and $Q_2''$. Using these values we obtain from (13), (14), (36) and (37) the following expressions

$$
\phi_1' = 2 \exp[\text{i} \omega t] \frac{\exp[g(h-H) / 2\alpha_1^2] K \sin \bar{\lambda}_1 Z}{\bar{\lambda}_1 \Delta' C^0} \left[ \bar{\lambda}_1 \cos \bar{\lambda}_1 (H-h) - \left( \frac{g}{2\alpha_1^2} \right) \sin \bar{\lambda}_1 (H-h) \right]
$$
\[ \begin{align*}
\sum \left( c_2' d_2'' - d_2' c_2'' \right) + \rho_1 \left( \omega_0^2 \sin \lambda_1 (H - h) + g \left[ \lambda_1 \cos \lambda_1 (H - h) - \left( \frac{g}{2 \alpha_1} \right) \sin \lambda_1 (H - h) \right] \right) \\
\end{align*} \]

\[ \begin{align*}
\left( b_2' c_2'' - c_2' b_2'' \right) \exp \left[ \frac{-gz}{2 \alpha_1} \right] J_0 (Kr) dK \\
\end{align*} \]

\[ \phi_1' = 2 \exp [i \omega t] \int_0^\infty \frac{\exp \left[ \frac{g(h - H)}{2 \alpha_1} \right] K \sin \lambda_1 h}{\lambda_1 \Delta' C^0} \left[ \lambda_1 \cos \lambda_1 (H - z) - \left( \frac{g}{2 \alpha_1} \right) \sin \lambda_1 (H - z) \right] \\
\end{align*} \]

\[ \begin{align*}
\left( b_2' c_2'' - c_2' b_2'' \right) \exp \left[ \frac{-gz}{2 \alpha_1} \right] J_0 (Kr) dK \\
\end{align*} \]

\[ \phi_2 = -2 \exp [i \omega t] \int_0^\infty \frac{J_0 (Kr) K \sin \lambda_1 h}{\Delta' C^0} \exp \left[ \frac{g(h - H)}{2 \alpha_1} \right] \rho_1 \omega_0^2 \left[ c_2 \exp [-i \lambda_2' z] - c_2' \exp [-i \lambda_2' z] \right] dK \\
\end{align*} \]

\[ \begin{align*}
\psi_2 = -2 \exp [i \omega t] \int_0^\infty \frac{J_0 (Kr) K \sin \lambda_1 h}{\Delta' C^0} \exp \left[ \frac{g(h - H)}{2 \alpha_1} \right] \rho_1 \omega_0^2 \left[ c_2 a_2'' \exp [-i \lambda_2' z] - c_2' a_2' \exp [-i \lambda_2' z] \right] dK \\
\end{align*} \]

\[ \begin{align*}
\Delta' = [\lambda_1 \cos \lambda_1 H - \left( \frac{g}{2 \alpha_1} \right) \sin \lambda_1 H] [c_2' d_2' - d_2' c_2''] \\
\end{align*} \]

Again the integrals (43) to (46) can be expressed by the sum of branch line integrals and residues. The residues corresponds to the pole \( K = K_n \) given by the roots of the equation \( \Delta'(K) = 0 \).

Now the normal mode of solution for the expressions given by (24) to (27) may be written in the final form after expanding \( C_0^{-1} \) and retaining only the first two terms.
\[
\phi_2 = \frac{2}{\pi} \left( \frac{2\pi}{r} \right)^2 \sum_{n=1}^{\infty} \frac{1}{(K_n)^{1/2}} \exp[i(\omega t - K_n r - \pi/4)] \Theta_2(K_n) \sin\left\{ K_n h \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \exp\left[ -K_n (z - H) \left( 1 - \frac{c^2}{\beta_2^2} \right)^{1/2} \right]
\]

(49)

\[
\psi_2 = \frac{2}{\pi} \left( \frac{2\pi}{r} \right)^2 \sum_{n=1}^{\infty} \frac{1}{(K_n)^{1/2}} \exp[i(\omega t - K_n r - \pi/4)] \Xi_2(K_n) \sin\left\{ K_n h \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \exp\left[ -K_n (z - H) \left( 1 - \frac{c^2}{\beta_2^2} \right)^{1/2} \right]
\]

(50)

where

\[
\Theta_2(K_n) = -\rho_1 \rho_2 \frac{c^2}{\beta_2^2} \cdot \frac{\exp[g(h-H)]\left(2 - \frac{c^2}{\beta_2^2}\right)}{2\alpha_1^2} \frac{K_n h}{\left( \frac{c_1^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2}} \cdot \frac{1}{M} \cdot \frac{g}{\alpha_1^2} \cdot \frac{1}{\left( \frac{c_1^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2}} \right. \\
\times \sin\left\{ K_n h \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \cos\left\{ K_n h \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} 
\]

(51)

\[
\Xi_2(K_n) = -2H \rho_1 \rho_2 \frac{c^2}{\beta_2^2} \cdot \frac{\exp[g(h-H)]\left(1 - \frac{c^2}{\alpha_2^2}\right)^{1/2}}{2\alpha_1^2} \frac{1}{M} \cdot \frac{g}{\alpha_1^2} \cdot \frac{1}{\left( \frac{c_1^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2}} \\
\times \sin\left\{ K_n h \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \cos\left\{ K_n h \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} 
\]

(52)
\[
\sin\left(\frac{K_n h}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2}\right) \frac{1}{2} \right] \cos\left(\frac{K_n h}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2}\right) \frac{1}{2} \right] (52)
\]

\[
\alpha_{\text{new}} \frac{e^i \phi_i' = e^i \phi_i''}
\]

and \( \phi_i' = \phi_i'' = \frac{2}{H} \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} \sum \frac{1}{n} \left(\frac{K_n}{r}\right)^{\frac{1}{2}} \exp\left[i(\omega t - K_n r - \frac{\pi}{4})\right] \theta_1(K_n) \times
\]

\[
\sin\left(\frac{K_n h}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2}\right) \frac{1}{2} \right] \sin\left(\frac{K_n h}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2}\right) \frac{1}{2} \right] (53)
\]

where

\[
\theta_1(K_n) = \frac{\rho_1}{\rho_2} \frac{c^4}{\beta_2^4} \frac{K_n H}{\left(\frac{1 - c^2}{\alpha_2^2}\right)^{\frac{1}{2}}} \left[ 1 + \frac{\frac{g}{\alpha^2}}{\left(\frac{1 - c^2}{\alpha_2^2} - \frac{g^2}{4\alpha_1^4 K_n^2}\right)^{\frac{1}{2}}} \right] \times
\]

\[
\sin\left(\frac{K_n h}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2}\right) \frac{1}{2} \right] + \cos\left(\frac{K_n h}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2}\right) \frac{1}{2} \right] (54)
\]

\[
M = \frac{\rho_1}{\rho_2} \frac{c^4}{\beta_2^4} \sin(K_n H \lambda_1^0) \left[ 1 - \frac{1 - c^2}{\alpha_2^2} \right]^{\frac{1}{2}} \cos(K_n H \lambda_1^0) + \frac{g}{\alpha^2} \left(1 - \frac{1 - c^2}{\alpha_2^2}\right) \frac{1}{2} \cos(K_n H \lambda_1^0) - K_n H \sin(K_n H \lambda_1^0) - \frac{\lambda_1^0 K_n^2}{\lambda_1^0} \frac{1}{2} \right] - 4 \left[ \left(1 - \frac{1 - c^2}{\alpha_2^2}\right) \frac{1}{2} \left(1 - \frac{1 - c^2}{\beta_2^2}\right) \frac{1}{2} + \left(1 - \frac{1 - c^2}{\alpha_2^2}\right) \frac{1}{2} \right]
\]
The period equation (29) may now be written as

\[
\sin \lambda_1 H + \frac{g \lambda_1}{\omega^2} \cos \lambda_1 H - \frac{g^2}{2 \alpha_1^2} \sin \lambda_1 H
\]

\[
\cos \lambda_1 H - \frac{g}{2 \alpha_1^2 \lambda_1} \sin \lambda_1 H
\]

\[
= \rho_2 \beta_2^4 \rho_1 \frac{c^4}{\alpha_1^2 - 1 - \frac{g^2}{4 \alpha_1^4 K_n^2}} \frac{1}{\alpha_1^2 - 1 - \frac{g^2}{4 \alpha_1^4 K_n^2}}
\]

\[
\times \left[ 4 \left( 1 - \frac{c^2}{\alpha_1^2} \right) \frac{1}{2} \left( 1 - \frac{c^2}{\beta_2^2} \right) \frac{1}{2} - \left( 2 - \frac{c^2}{\beta_2^2} \right)^2 \right]
\]

in all the above cases as \( g \to 0 \) the results obtained are in agreement with corresponding result of the classical problem. Similar solution may be written for the expressions \( \phi_2, \psi_2, \phi_1' \) and \( \phi_1'' \) in (43) to (46).

### 5. NUMERICAL CALCULATION

Equation (29) namely \( \Delta(K) = 0 \) where \( \Delta(K) \) is defined in (28) may be written using (15) and (19) and approximating \( \cos x \equiv 1 \) and \( \sin x \equiv x \) (i.e. considering the arguments of sine and cosine are small) as
\[ \frac{\rho_2}{\rho_1} \left[ 1 - \frac{gH}{2\alpha_1^2} \right] \left[ \left( 2 - \frac{c^2}{\beta_2^2} \right)^2 - 4 \left( 1 - \frac{c^2}{\alpha_2^2} \right)^2 \left( 1 - \frac{c^2}{\beta_2^2} \right)^2 \right] = \left( 1 - \frac{c^2}{\alpha_2^2} \right)^{\frac{1}{2}} \frac{c^2}{\beta_2^2} \left[ \frac{c^2}{\beta_2^2} KH + \frac{g}{\beta_2^2 K} \left( 1 - \frac{gH}{2\alpha_1^2} \right) \right] \]

in which \( c = \frac{\omega}{K} \) is the phase velocity. Now for sedimentary ocean bottom

\[ \rho_2 = 2\rho_1, \quad \alpha_2 = \sqrt{3} \beta_2, \quad \beta_2 = 1.5 \alpha_1 \]

and since

\[ \frac{gH}{2\alpha_1^2} = \frac{gH}{2\beta_1^2} \cdot \frac{\beta_2^2}{\alpha_1^2} = 1.125 \quad G \cdot KH \]

where \( G = \frac{g}{\beta_2^2 K} \) is the gravity parameter

Using (59) and (60), the equation (58) may be written as

\[ 2[1-1.125GKH] \left[ \left( 2 - \frac{c^2}{\beta_2^2} \right)^2 - 4 \left( 1 - \frac{c^2}{\beta_2^2} \cdot \frac{1}{3} \right)^2 \left( 1 - \frac{c^2}{\beta_2^2} \right)^2 \right] \]

\[ = \left( 1 - \frac{c^2}{\beta_2^2} \cdot \frac{1}{3} \right)^{\frac{1}{2}} \frac{c^2}{\beta_2^2} \left[ \frac{c^2}{\beta_2^2} KH + G(1-1.125GKH) \right] \]

Equation (61) is now dimensionless in \( \frac{c^2}{\beta_2^2} \) for a particular value of gravity parameter \( G \) and \( KH \) and

\[ \text{from which} \]

\[ \text{from which a real root of} \quad \frac{c^2}{\beta_2^2} \]

may be evaluated for different values of \( G \) and \( KH \). Similar calculations

may be drawn in case of granitic ocean bottom and basaltic ocean bottom.
6. DISCUSSION

In both cases there is dispersion of waves due to gravity. If \( g \to 0 \) in case (1) we obtain \( \lambda_1 \to v_1 \) and the result thus obtained are in good agreement with the classical problem as studied by Ewing et al. Also if \( \rho_1 \to 0 \) and \( g \to 0 \) in \( \Delta(K) = 0 \) we easily obtain the equation of Rayleigh waves in classical elasto-kinetics as

\[
\left( 2 - \frac{c^2}{\beta_2^2} \right)^2 = 4 \left( \frac{c^2}{\alpha_2^2} - 1 \right)^2 \left( \frac{c^2}{\beta_2^2} - 1 \right)^2
\]

(62)

Now if \( \rho_1 \to 0 \) in \( \Delta'(K) = 0 \) as given in (48) we easily obtain the Rayleigh surface waves under the influence of gravity as

\[
c'_2 d''_2 - d'_2 c''_2 = 0
\]

(63)

where \( c'_2, d'_2, c'_2 \) and \( d'_2 \) have been defined in (42). Also the changes in the above potentials produced by a varying depth \( z \) are represented by the factor \( \sin \left( K_n z \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^2 \right) \) in (53).
and the factor

\[ \sin \left\{ K_n h \left( \frac{c^2}{\alpha^2} - 1 - \frac{g^2}{4\alpha^2 K_n^2} \right)^{1/2} \right\} \]

depends on the depth of the source. From equation (29) and (61) and the Table 1 we may conclude that the phase velocity \( c \) increases with the increasing value of the gravity parameter \( G \) for a particular value of \( KH \). It is to be noted that this investigation may rise into importance when the degree of accuracy is demanded into considerable amount.
INFLUENCE OF GRAVITY ON THE PROPAGATION OF WAVES IN A MEDIA IN PRESENCE OF A COMPRESSIONAL WAVE SOURCE

1. INTRODUCTION

Using the theory of plane waves, Stoneley [136] studied the effect of the ocean on transmission of Rayleigh waves and calculated phase and group velocities for a water layer assumed to be three km. thick over a solid substratum. He confined his attention to the longer period Rayleigh waves and concluded that the effect of the water layer was unimportant. In a note added to that paper, Jeffreys [63] proved from Stoneley’s equation that there exists a minimum of group velocity at some period shorter than those investigated by Stoneley. Scholte [111] while attempting to explain microseism generation by transfer of energy from gravity surface waves to elastic waves in the bottom, considered the combined effect of gravity and compressibility in a layer of water in contact with an elastic solid bottom. Muskat [80] and Sommerfeld [133] have discussed the wave propagation for the case, where distance of a point source from the plane interface is finite. In particular, Press and Ewing [100] studied the propagation of waves when the point source being present in the liquid layer. Their results are directly related to an important practical problem, that of the ‘refraction arrival’ from a source to a receiver in seismology of near earthquakes and in seismic refraction investigations. In the classical problem of elastic waves and vibrations, the gravity effect is generally neglected. The effect of gravity was first studied by Bromwich [16] in the problem of propagation of waves in solid in particular on an elastic globe. Subsequently, the investigations of the effect of gravity was considered by Love [73], who exhibited that the velocity of Rayleigh waves is increased to a significant extent by the gravitational field when wave lengths

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are large. More recently, Biot [13] developed a theory of initial stress and use it to investigate the influence of gravity on Rayleigh waves, assuming the force of gravity to create a type of initial stress of hydrostatic nature and the medium to be incompressible. The initial stress is produced in the body by a slow process of creep where the shear stress tend to become small or vanish after a long interval of time. Assuming that the boundaries are all parallel planes in the axis symmetric problem of propagation of waves under the influence of gravity in a media composed of a liquid layer and an underlying solid half space when the compressional wave source being present in the lower solid substratum and the waves are smaller than ordinary earthquake Rayleigh waves.

2. BASIC EQUATIONS AND RELATIONS

We shall use here the subscript (1) for liquid and subscript (2) for the solid part respectively. In the Cartesian co-ordinate system the two dimensional equations of motion in an elastic solid medium in absence of body forces are

\[
\rho_2 \frac{\partial^2 u_2}{\partial t^2} = (\lambda_2 + \mu_2)\frac{\partial \Theta}{\partial x} + \mu_2 \nabla^2 u_2
\]

\[
\rho_2 \frac{\partial^2 w_2}{\partial t^2} = (\lambda_2 + \mu_2)\frac{\partial \Theta}{\partial z} + \mu_2 \nabla^2 w_2
\]

(2.1)

The above equations may be written in polar co-ordinate system as [55]

\[
(\lambda_2 + 2\mu_2) \left( \frac{\partial^2 q_2}{\partial r^2} + \frac{1}{r} \frac{\partial q_2}{\partial r} - \frac{q_2}{r^2} + \frac{\partial^2 w_2}{\partial \theta \partial r} \right) + \mu_2 \left( \frac{\partial^2 q_2}{\partial z^2} - \frac{\partial^2 w_2}{\partial \theta \partial r} \right) = \rho_2 \frac{\partial^2 q_2}{\partial t^2}
\]

(2.2)

\[
(\lambda_2 + 2\mu_2) \left( \frac{\partial^2 q_2}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial q_2}{\partial \theta} + \frac{\partial^2 w_2}{\partial \theta^2} \right) - \mu_2 \left( \frac{\partial q_2}{\partial z} - \frac{\partial w_2}{\partial \theta} \right) - \mu_2 \left( \frac{\partial^2 q_2}{\partial \theta \partial z} - \frac{\partial^2 w_2}{\partial \theta \partial z} \right) = \rho_2 \frac{\partial^2 w_2}{\partial t^2}
\]

Where \( \lambda, \mu \) are Lame elastic constants, \( u \) and \( w \) are the displacements in the direction of \( x \) and \( z \) axis, \( \Theta \) is cubical dilatation, \( q \) and \( w \) are the displacements in the direction of \( r \) and \( z \) respectively.
Now we define two functions $\phi_2$ and $W_2$ as

$$q = \frac{\partial \phi_2}{\partial r} - \frac{\partial W_2}{\partial z} ; \quad w = \frac{\partial \phi_2}{\partial z} + \frac{1}{r} \frac{\partial ((rw_2))}{\partial r} ; \quad W_2 = -\frac{\partial \psi_2}{\partial r}$$

(2.3)

Using (2.3) in equation (2.2) we obtain

$$\nabla^2 \phi_2 = \frac{1}{\alpha_2^2} \frac{\partial^2 \phi_2}{\partial t^2} ; \quad \nabla^2 \psi_2 = \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2}$$

(2.4)

where

$$\alpha_2^2 = \frac{\lambda_2 + 2\mu_2}{\rho_2} ; \quad \beta_2^2 = \frac{\mu_2}{\rho_2} ; \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

(2.5)

Here $\alpha_2$ is the velocity of compressional waves, $\beta_2$ is the velocity of distortional waves and $\rho_2$ is the density of the solid medium.

3. GENERAL THEORY AND BOUNDARY CONDITIONS

We consider a gravitating and compressible liquid layer of width $H$ bounded by the parallel planes $z = 0$ and $z = H$ lying over a solid semi-space. The surface of separation being $z = H$ and the $z$-axis is taken vertically downwards. We shall use here the subscript (1) for liquid and subscript (2) for the solid part respectively. The displacements may be expressed in terms of the potentials $\phi_1$, $\phi_2$ and $\psi_2$ [55] as

$$q_1 = \frac{\partial \phi_1}{\partial r} ; \quad w_1 = \frac{\partial \phi_1}{\partial z} \quad (3.1)$$

$$q_2 = \frac{\partial \phi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial z \partial r} ; \quad w_2 = \frac{\partial \phi_2}{\partial z} + \frac{\partial^2 \psi_2}{\partial z^2} - \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2} = \frac{\partial \phi_2}{\partial z} + \frac{\partial^2 \psi_2}{\partial z^2} + K_{\beta_2} \psi_2 \quad (3.2)$$

where

$$K_{\beta_2} = \omega / \beta_2$$

(3.3)

A time factor $\exp (i\omega t)$ is understood with the expressions for the potentials.

**Boundary conditions**: The boundary conditions are

$$\phi_1 = 0 \quad \text{at} \quad z = 0$$

$$w_1 = w_2 ; \quad (p_{xz})_2 = 0 ; \quad (p_{zz})_1 = (p_{zz})_2 \quad \text{at} \quad z = H$$

(3.4) (3.5)

where the component of stresses may be written
\[(p_{zz})_1 = \lambda_1 \nabla^2 \phi_1 \quad ; \quad (p_{zz})_2 = \lambda_2 \nabla^2 \phi_2 + 2\mu_2 \frac{\partial w_2}{\partial z}\]
\[(p_{xz})_2 = \mu_2 \left( \frac{\partial q_2}{\partial z} + \frac{\partial w_2}{\partial \tau} \right) \quad \tag{3.6}\]

**SOLUTION OF THE PROBLEM**

Here we consider the case when the waves are smaller than ordinary earthquake Rayleigh waves.

Now under the influence of gravity the velocity potential \(\phi_1\) for liquid satisfy the equation\[71\]
\[
\frac{\partial^2 \phi_1}{\partial t^2} = \alpha_1^2 \nabla^2 \phi_1 + g \frac{\partial \phi_1}{\partial z} \quad \tag{4.1}\]
\n(4.1) may be written in terms of displacement potential \(\phi_1\) as

\[
\frac{\partial^2 \phi_1}{\partial t^2} = \alpha_1^2 \nabla^2 \phi_1 + g \frac{\partial \phi_1}{\partial z} \quad \tag{4.2}\]

where \(\alpha_1\) is the velocity of the compressional waves in the liquid.

The gravity terms in the equations for the solid part of the system are omitted by Scholte \[111\], so for the solid part we use the equations (2.4).

In order to satisfy the boundary condition (3.4) \(\phi_1\) may be written as \[55\]
\[
\phi_1 = \int_0^\infty J_0(Kr)e^{-\frac{g_1^2}{2\alpha_1^2}} \sin \bar{\eta}_1 zdK \quad (0 \leq z \leq H) \quad \tag{4.3}\]

where \(\bar{\eta}_1 = \left( v_1^2 - \frac{g_1^2}{4\alpha_1^4} \right) \); \(v_1^2 = K_{a_1}^2 - K^2 \); \(K_{a_1} = \frac{\omega}{\alpha_1}\) \[4.4\]

As a source in a solid half-space can produce both compressional and distortional waves and that an expression for spherical waves emitted by the point source \((0, H + d = h)\) can be written in the form \[55\]
\[
\varphi_0 = \int_0^\infty J_0(Kr)e^{-i\bar{\nu}_2 |z-h|^\frac{\omega}{\nu_2}} dK \quad \tag{4.5}\]

where \(\bar{\nu}_2 = K_{a_2}^2 - K^2 \); \(K_{a_2} = \omega / \alpha_2\) \[4.6\]

The expressions for \(\varphi_2\) and \(\psi_2\) may be written in the following form \[55\]
\[ \varphi_2 = \varphi_a + \int_0^\infty Q_2(K) J_0(K_r) e^{-i\tilde{\nu}_2 z + \text{int}} \, dK \] (4.7)

\[ \psi_2 = \int_0^\infty S_2(K) J_0(Kr) e^{-i\tilde{\nu}_2 z + \text{int}} \, dK \] (4.8)

where \[ \tilde{\nu}_2 = K^2 - K^2 \quad ; \quad K_{\tilde{\nu}_2} = \omega/\beta_2 \] (4.9)

If \( Z < H + d \) we have

\[ \varphi_2 = \int_0^\infty \frac{K}{\tilde{\nu}_2} e^{-i\tilde{\nu}_2 (z-H-d) + \text{int}} J_0(Kr) \, dK + \int_0^\infty Q_2 J_0(Kr) e^{-i\tilde{\nu}_2 z + \text{int}} \, dK \] (4.10)

\[ \psi_2 = \int_0^\infty S_2 J_0(Kr) e^{-i\tilde{\nu}_2 z + \text{int}} \, dK \] (4.11)

Using the above expression for \( \varphi_1, \varphi_2 \) and \( \psi_2 \) in the boundary condition given in (3.5) we obtain the following

\[ e^{2\alpha_1^2} \left[ \tilde{\eta}_1 \cos \tilde{\eta}_1 H - \left( g/2\alpha_1^2 \right) \sin \tilde{\eta}_1 H \right] \Delta + i\tilde{\nu}_2 Q_2 e^{-i\nu_2 H} - K^2 S_2 e^{-i\nu_2 H} = K e^{-i\nu_2 d} \] (4.12)

\[ 2i\tilde{\nu}_2 Q_2 e^{-i\nu_2 H} + (\tilde{\nu}_2^2 - K^2) S_2 e^{-i\nu_2 H} = 2K e^{-i\nu_2 d} \] (4.13)

\[ \rho_1 e^{2\alpha_1^2} \left[ \omega^2 \sin \tilde{\eta}_1 H + g \left( \tilde{\eta}_1 \cos \tilde{\eta}_1 H - \left( gH/2\alpha_1^2 \right) \sin \tilde{\eta}_1 H \right) \right] \left[ \left( 2\mu_2 K^2 - \rho_2 \omega^2 \right) \tilde{\nu}_2 e^{-i\nu_2 H} + 2 \mu_2 K^2 i\tilde{\nu}_2 S_2 e^{-i\nu_2 H} \right]

\[ = -\left( 2\mu_2 K^2 - \rho_2 \omega^2 \right) \tilde{\nu}_2 \left( 2K^2 - K_{\tilde{\nu}_2}^2 \right) \] (4.14)

The values of \( \Lambda, Q_2 \) and \( S_2 \) in terms of \( K \) and other parameters can be found if the determinant \( \Delta \) of the above equations is not equal to zero.

Now using \( \mu_2 = \rho_2 \beta_2^2 \) we can write the expression for \( \Delta \) as

\[ \Delta(K) = e^{-gH} \left[ \tilde{\eta}_1 \cos \tilde{\eta}_1 H - \frac{g}{2\alpha_1^2} \sin \tilde{\eta}_1 H \right] \rho_2 \beta_2^2 \left[ 4K^2 \tilde{\nu}_2 \tilde{\nu}_2' + \left( 2K^2 - K_{\tilde{\nu}_2}^2 \right)^2 \right] \]

\[ \rho_1 \left[ \omega^2 \sin \tilde{\eta}_1 H + g \left( \tilde{\eta}_1 \cos \tilde{\eta}_1 H - \frac{g}{2\alpha_1^2} \sin \tilde{\eta}_1 H \right) \right] \tilde{\nu}_2 \omega^2 \left/ \beta_2^2 \right. \] (4.15)

Now \( A, Q_2 \) and \( S_2 \) may be written in the following form

\[ A = 2\rho_2 \omega^2 \left( 2K^2 - K_{\tilde{\nu}_2}^2 \right)^2 \left[ K/\Delta \right] e^{-i\nu_2 d} \] (4.16)
\[ Q_2 = \frac{Ke^{\frac{-gh^2}{2\alpha_1^2}}}{i\nu_2\Delta} e^{i\nu_2(z-H-d)} \left[ \rho_1 \left\{ \omega^2 \sin \eta_1 H + g \left( \frac{\eta_1 \cos \eta_1 H}{2\alpha_1^2} \sin \eta_1 H \right) \right\} \right. \]
\[ \left. + \frac{i\nu_2}{\beta_2^2} \left( \eta_1 \cos \eta_1 H - \frac{g}{2\alpha_1^2} \sin \eta_1 H \right) \eta_2 \beta_2 \left\{ 4K^2 \nu_2^2 + \left( 2K^2 - K_\beta^2 \right)^2 \right\} \right] \] (4.17)

\[ S_2 = \frac{-4K(2\mu_2 K^2 - \rho_2 \omega^2)}{\Delta} e^{\frac{-gh^2}{2\alpha_1^2}} \left( \eta_1 \cos \eta_1 H - \frac{g}{2\alpha_1^2} \sin \eta_1 H \right) e^{-i\nu_2(z-H-d)} \] (4.18)

Now substituting the above expressions for \( A \), \( Q_2 \) and \( S_2 \) in (4.3), (4.7) and (4.8) to obtain \( \varphi_1 \), \( \varphi_2 \) and \( \psi_2 \) as

\[ \varphi_1 = -2J \int_0^\infty \frac{\rho_2 \omega^2}{\Delta} e^{\frac{-gh^2}{2\alpha_1^2}} J_0(Kr) \sin \eta_1 z dK \] (4.19)

\[ \varphi_2 = \int_0^\infty \frac{K}{i\nu_2} e^{-i\nu_2(z-H-d)} J_0(Kr)dK + \int_0^\infty \frac{K}{i\nu_2} e^{-i\nu_2(z-H+d)} J_0(Kr)dK - \frac{2Ke^{\frac{-gh^2}{2\alpha_1^2}}}{i\nu_2\Delta} \]
\[ \int_0^\infty \left[ \eta_1 \cos \eta_1 H - \frac{g}{2\alpha_1^2} \sin \eta_1 H \right] \left( 2K^2 - K_\beta^2 \right)^2 \rho_2 \beta_2 e^{-i\nu_2 z} J_0(Kr)dK \] (4.20)

\[ \psi_2 = -4J \int_0^\infty \frac{K}{\Delta} 2\mu_2 K^2 - \rho_2 \omega^2 e^{\frac{-gh^2}{2\alpha_1^2}} \left[ \eta_1 \cos \eta_1 H - \frac{g}{2\alpha_1^2} \sin \eta_1 H \right] e^{-i\nu_2(z-H)} J_0(Kr)dK \] (4.21)

Time factor \( \exp(iot) \) is understood with every term in the above expressions.

The first two terms in (4.20) may be combined as follows.

\[ 2J \int_0^\infty \frac{K}{i\nu_2} \cos \nu_2(z - H) e^{-i\nu_2 z} J_0(Kr)dK \]
\[ \text{for } H \leq z \leq H + d \] (4.22)

or,
\[ 2J \int_0^\infty \frac{K}{i\nu_2} e^{-i\nu_2(z-H)} \cos \nu_2 dJ_0(Kr)dK \]
\[ \text{for } H + d \leq z < \infty \] (4.23)

From (3.1) and (3.2) we can evaluate the displacements \( q \) and \( w \). Thus for the solid bottom \( z=H \)

we make use of \( \varphi_2 \) given by (4.22) and the third integral in (4.20) and \( \psi_2 \) given in (4.21).
So we obtain the following displacements

\[ q_H = -2K_2 \int_0^\infty \frac{K^2}{V(K)} \left[ \frac{\rho_1}{\rho_2} \frac{K^2}{\bar{\eta}_l} \sin \frac{\bar{\eta}_lH}{\cos \bar{\eta}_lH - \frac{g}{2\alpha_1^2 \bar{\eta}_l^2}} + \frac{g}{w^2} \right] - 2iu_2 P^2 \cos \gamma_H g^2 w^2 \left[ e^{\gamma_1d}J_1(Kr) \right] \]  

(4.24)

\[ w_H = -2K_2 \int_0^\infty \frac{2K^2 - K_2^2}{V(K)} e^{-\gamma_2d} J_0 (Kr) K dK \]  

(4.25)

where

\[ V(K) = 4K^2 \bar{\omega}_2 \bar{\omega} + 2 \left( 2K^2 - K_2^2 \right)^2 + \frac{\rho_1}{\rho_2} \omega^4 \sin \frac{\bar{\eta}_lH}{\bar{\eta}_l \cos \bar{\eta}_lH - \frac{g}{2\alpha_1^2 \sin \bar{\eta}_lH}} + w^2 g \frac{iu_2}{\beta_2^4} \]  

(4.26)

and

\[ \Delta(K) = V(K) \rho_2 \beta_2^2 e^{\frac{gH}{2\alpha_1^2}} \left( \bar{\eta}_l \cos \bar{\eta}_lH - \frac{g}{2\alpha_1^2 \sin \bar{\eta}_lH} \right) \]  

(4.27)

The above integrals can be represented by the sum of branch line integrals and residues. The residues correspond to the pole \( K = K_0 \) given by the roots of the equation

\[ V(K) = 0 \]  

(4.28)

Now the amplitudes of waves determined by branch line integrals diminish as \( r^2 \). As we are interested in an approximation which hold for large values of \( r \), the terms corresponding to branch points are left out of consideration and only the residues are computed. The asymptotic values of the displacements are obtained as
\[ q_{11} = \frac{2}{H^2} \sqrt{\frac{2\pi}{r} \sum_n \frac{1}{\sqrt{K_n}}} P(K_n) e^{-i\omega_{2n}d} e^{i(\omega t - K_n r + \frac{\pi}{4})} \]  
(4.29)

\[ W_H = \frac{2}{H^2} \sqrt{\frac{2\pi}{r} \sum_n \frac{1}{\sqrt{K_n}}} Q(K_n) e^{-i\omega_{2n}d} e^{i(\omega t - K_n r + \frac{\pi}{4})} \]  
(4.30)

where the phase velocity \( c_n \) and \( \omega_{2n} \) for each mode are given by

\[ c_n = \frac{\omega}{k_n}, \quad \omega_{2n} = K_{a_2}^2 - K_n^2 \]  
(4.31)

and

\[
P(K_n) = \frac{K_n^2 H^2 C_n^2}{R(K_n) \beta_2^2} \left[ \frac{\rho_1 C_n^2}{\rho_2 \beta_2^2} \sqrt{1 - \frac{c_n^2}{\alpha_2^2}} W(K_n) - 2 \sqrt{1 - \frac{c_n^2}{\alpha_2^2}} \sqrt{1 - \frac{c_n^2}{\beta_2^2}} \right]
\]

\[ Q(K_n) = \frac{K_n^2 C_n^2}{R(K_n) \beta_2^2} \left(2 - \frac{C_n^2}{\beta_2^2}\right) \sqrt{1 - \frac{C_n^2}{\alpha_2^2}} \]

where

\[
R(K_n) = \frac{\rho_1 C_n^4}{\rho_2 \beta_2^4} \left[ \frac{1 - \frac{C_n^2}{\alpha_2^2}}{\frac{C_n^2}{\alpha_1^2} - 1 - \frac{g^2}{4 \alpha_1^2 K_n^2}} \sqrt{\frac{C_n^2}{\alpha_1^2} - 1 - \frac{g^2}{4 \alpha_1^2 K_n^2}} - \frac{K_n H(1 - \frac{C_n^2}{\alpha_2^2})}{S(K_n)} \right]
\]

\[-4 \left[ 3 - 2 \frac{C_n^2}{\alpha_2^2} \right] \sqrt{1 - \frac{C_n^2}{\beta_2^2} + (1 - \frac{C_n^2}{\alpha_2^2}) \sqrt{1 - \frac{C_n^2}{\beta_2^2} - 2 \left(2 - \frac{C_n^2}{\alpha_2^2}\right) \sqrt{1 - \frac{C_n^2}{\alpha_2^2}}} \right] \]
\[
W(K_n) = \frac{\sin \left( K_n H \sqrt{\frac{c_i^2}{a_i^2} - 1 - \frac{g^2}{4a_i^2K_n^2}} \right)}{\cos \left( K_n H \sqrt{\frac{c_i^2}{a_i^2} - 1 - \frac{g^2}{4a_i^2K_n^2}} \right)} - \frac{g}{\omega^2}
\]

and
\[
S(K_n) = \left\{ W(K_n) \right\}^2 + 1
\]

The period equation (4.28) can be written in dimensionless form

\[
W(K_n) = \frac{\rho_2}{\rho_1} \frac{\beta_2^4}{C_n^4} \sqrt{\frac{c_i^2}{a_i^2} - 1 - \frac{g^2}{4a_i^2K_n^2}} \left[ \sqrt{\frac{1 - C_n^2}{\alpha_2^2}} \sqrt{\frac{1 - C_n^2}{\beta_2^2}} - \left( \frac{2 - C_n^2}{\beta_2^2} \right)^2 \right] (4.32)
\]

It defines as usual a relationship between the period \( T = 2\pi / C_n K_n = 2\pi / \omega \) and the phase velocity with the elastic constants of the system as parameter.

5. NUMERICAL CALCULATION

The period equation (4.28) may be written in the following dimensionless simpler form

approximating \( \sin \tilde{\eta}_i H = \tilde{\eta}_i H \) and \( \cos \tilde{\eta}_i H = 1 \)

\[
\left( 2 - \frac{C_n^2}{\beta_2^2} \right) - 4 \left[ \frac{1 - C_n^2}{\alpha_2^2} \sqrt{\frac{1 - C_n^2}{\beta_2^2}} \right] \frac{\rho_2}{\rho_1} = \frac{C_n^2}{\beta_2^2} \left[ \frac{C_n^2}{\beta_2^2} (K_n H)/(1 - \frac{gH}{2a_i^2}) + G \right] \sqrt{\frac{1 - C_n^2}{\alpha_2^2}} (5.1)
\]

Where \( c = \omega / K \) is the phase velocity and \( G = g / K \beta_2^2 \), is the gravity parameter.

Now for granitic ocean bottom [55] in which

\[
\frac{\rho_2}{\rho_1} = 2.5 \quad \frac{\alpha_2}{\beta_2} = \sqrt{3} \quad \frac{\beta_2}{\alpha_1} = 2
\]

We can evaluate \( c / \beta_2 \) from (5.1) for a particular value of the gravity parameter and \( KH \) as given in
the following table.

<table>
<thead>
<tr>
<th>G</th>
<th>KH</th>
<th>$c^2/\beta^2$</th>
<th>$c/\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>10</td>
<td>0.5483978</td>
<td>0.740539</td>
</tr>
<tr>
<td>0.3</td>
<td>10</td>
<td>0.6763711</td>
<td>0.822418</td>
</tr>
<tr>
<td>0.4</td>
<td>10</td>
<td>0.7474303</td>
<td>0.864541</td>
</tr>
</tbody>
</table>

The phase velocity may also be calculated for the sedimentary ocean bottom and basaltic ocean bottom in a similar manner.

6. DISCUSSION

Now neglecting the gravitational field i.e. making $g \to 0$ ($G \to 0$) the period equation (4.28) reduces to the following form

$$\tan \left( KH \sqrt{\frac{C^2}{\alpha_1^2} - 1} \right) = \rho_2 \beta_2^4 \frac{C^2}{\alpha_1^2} \left[ \frac{4}{\sqrt{1 - \frac{C^2}{\alpha_1^2}}} \sqrt{1 - \frac{C^2}{\beta_2^2}} - \left( 2 - \frac{C^2}{\beta_2^2} \right)^2 \right]$$

(6.1)

Which is in complete agreement with the corresponding result as studied by Ewing et al [55]. Now using the same results and approximations as done in section 5 for granaitic ocean bottom, we obtain $c/\beta_2=0.512717173$. The dispersion of wave occurs and the phase velocity increases in presence of gravitation field $g$, which is clear from the form of the period equation (4.28). It is also clear from numerical calculation that the ratio $c/\beta_2$ increases with the increasing value of the gravity parameter $G$. The second term in (4.20) may be interpreted as a spherical wave emitted by the image of the source in the interface. It’s simple form is $\exp(-iK_{ai}R)$ where $R^2 = r^2 + (z-H+d)^2$. Now if $g \to 0$, $\eta_i \to \tilde{v}_i$, the above problem reduces to the classical one as studied by Ewing, Jardetzky and Press[55]. In the above limit the asymptotic values of displacements as given in (4.29) and (4.30) are in complete agreement with the corresponding results of Ewing et al [55].