The present paper is concerned with the investigation of the steady state response to applied loads moving with constant speed for infinity long time, over the free surface of a homogeneous elastic layer under which there lies an infinite semi-space in contact. Here only supersonic-supersonic case has been studied. If the thickness of the layer tends to zero then the results are in good agreement with the corresponding classical problem.

Key Words : Steady-State Response; Galilean Transformation; Supersonic; Mach-Numbers; Heaviside Unit Function.

INTRODUCTION

In classical theory of elasticity Lamb¹, Cole and Huth² and Sneddon³ have studied the problem of steady-state response to moving loads in an elastic solid medium. This type of investigation is found also in the foundation problems of soil mechanics. In designing highways or airport runways, the earthmass may be composed of two different mediums, one in the form of a layer of finite length and another as an underlying infinite semi-space, where the study of the titled problem is significant. Moreover, this problem may attract the attention because of its engineering significance in the question of ground motion under pressure waves⁴. It is to be noted that Sengupta and his research workers have published a good number of papers in this field also (e.g., 1973, p. 2241-48, 1980, p. 183-200 etc)⁵&⁶.

2. GENERAL THEORY AND BOUNDARY CONDITIONS

Let us consider a homogeneous elastic solid layer \( M' \) of finite thickness \( h \) lies over a homogeneous isotropic elastic semi-space \( M(y \geq 0) \). Let us introduce a set of orthogonal cartesian axes \( 0-xyz \), the origin \( 0 \) being any point on the surface of separation \( y = 0 \) and \( y \)-axis is directed vertically downwards. Let us also consider a pressure pulse \( P(x + Ut) \) which is moving with a constant speed in the negative \( x \)-axis direction for a long time, over the free surface of \( M' \), so that a steady state prevails in the neighbourhood of the loading as seen by an observer moving with the load. In a state of plane strain the elastic displacements \( u \) and \( v \) are derivable from the displacement potentials \( \phi(x, y, t) \) and \( \psi(x, y, t) \) as⁴.
$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial y}; \quad v = \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}.$$  \hspace{1cm} (2.1)

Henceforth, we shall use the suffix (1) and (2) for the medium $M'$ and $M$ respectively.

This displacement potentials $\varphi_1, \psi_1, \varphi_2$ and $\psi_2$ satisfy the following wave equations in the medium $M'$ and $M$ respectively:

$$\nabla^2 \varphi_i = \alpha_i^{-2} \frac{\partial^2 \varphi_i}{\partial x^2}, \quad \nabla^2 \psi_i = \beta_i^{-2} \frac{\partial^2 \psi_i}{\partial x^2} \quad [i = 1, 2], \hspace{1cm} (2.2)$$

where $\alpha_i = \left\{ \frac{\lambda_i + 2\mu_i}{\rho_i} \right\}^{1/2}$ denote the dilatation wave speed and $\beta_i = \left( \frac{\mu_i}{\rho_i} \right)^{1/2}$ denote the shear wave speed in the solid media. If we introduce a Galilean transformation:

$$x' = x + Ut, \quad y' = y, \quad t' = t \hspace{1cm} (2.3)$$

and since the response in the two media is in a steady-state $\varphi$ and $\psi$ are also independent of $t'$ as seen by an observer moving with the load i.e., $x$ and $t$ enter $\varphi$ and $\psi$ only in the combination $x + Ut$ under these assumptions eq. (2.2) can be simplified into the following form:

$$\nabla^2 \varphi_i = \frac{U^2}{\alpha_i^2} \frac{\partial^2 \varphi_i}{\partial x^2}, \quad \nabla^2 \psi_i = \frac{U^2}{\beta_i^2} \frac{\partial^2 \psi_i}{\partial x^2} \quad [i = 1, 2]. \hspace{1cm} (2.4)$$

If we introduce the Mach numbers:

$$M_1 = U/\alpha_1, \quad M'_1 = U/\beta_1, \quad M_2 = U/\alpha_2, \quad M'_2 = U/\beta_2 \hspace{1cm} (2.5)$$

and the parameters:

$$\bar{\beta}_1 = \sqrt{M'_1 - 1}, \quad \bar{\alpha}_1 = \sqrt{M_1^2 - 1}, \quad \bar{\beta}_2 = \sqrt{M'_2 - 1}, \quad \bar{\alpha}_2 = \sqrt{M_2^2 - 1} \hspace{1cm} (2.6)$$

We obtain the following partial differential equations:

$$-\frac{\beta_1^2 \beta_2^2}{\bar{\alpha}_1^2} \frac{\partial^2 \varphi_1}{\partial x^2} - \frac{\partial^2 \varphi_1}{\partial y^2} = 0; \quad -\frac{\beta_1^2 \beta_2^2}{\bar{\alpha}_1^2} \frac{\partial^2 \psi_1}{\partial x^2} - \frac{\partial^2 \psi_1}{\partial y^2} = 0 \hspace{1cm} (2.7)$$

$$-\frac{\beta_1^2 \beta_2^2}{\bar{\alpha}_1^2} \frac{\partial^2 \varphi_2}{\partial x^2} - \frac{\partial^2 \varphi_2}{\partial y^2} = 0; \quad -\frac{\beta_1^2 \beta_2^2}{\bar{\alpha}_1^2} \frac{\partial^2 \psi_2}{\partial x^2} - \frac{\partial^2 \psi_2}{\partial y^2} = 0 \hspace{1cm} (2.8)$$

The nature of the response of the layered media to the travelling pressure pulse will differ according to the $M_i$'s are greater than unity or not.

The following nine cases may be considered super-super, super-sub, super-tran, sub-super, sub-sub, sub-tran, Tran-super, Tran-sub, Tran-Tran Here only the supersonic-supersonic
(\(M'_1 > M_1 > 1, M'_2 > M_2 > 1\)) case has been studied in this paper.

**Boundary Conditions:** In moving co-ordinates the boundary conditions are\(^{4&7}\)

\[
\begin{align*}
\sigma_{yy} & = -P(x') \\
\sigma_{yx} & = 0 \quad \text{at } y' = -h
\end{align*}
\]

\[
\begin{align*}
u_1 &= u_2; \quad \nu_1 = v_2; \quad (\sigma_{yx})_1 = (\sigma_{yx})_2; \quad (\sigma_{yy})_1 = (\sigma_{yy})_2 \quad \text{at } y' = 0,
\end{align*}
\]

\((2.10)\)

where

\[
\sigma_{yy} = \mu_1 \left[ (M'_i - 2) \frac{\partial^2 \varphi_i}{\partial \xi^2} + 2 \frac{\partial^2 \psi_i}{\partial \xi \partial \eta'} \right] + 2 \frac{\partial^2 \psi_i}{\partial \eta'^2}
\]

\((i = 1, 2)\)

\((2.11)\)

which have been written in terms of Mach numbers.

### 3. Solution of the Problem

If we seek solution of eqs. (2.7) and (2.8) in the form

\[
\begin{align*}
\varphi_1 (x', y'), \psi_1 (x', y'), \varphi_2 (x', y'), \psi_2 (x', y') &= \{ \overline{\varphi}_1 (y'), \overline{\psi}_1 (y'), \overline{\varphi}_2 (y'), \overline{\psi}_2 (y') \} e^{i\lambda y'}
\end{align*}
\]

\((3.1)\)

then

\[
\begin{align*}
\overline{\varphi}_1 &= A_1(\lambda) e^{i\lambda_2 y'} + A_1'(\lambda) e^{-i\lambda_2 y'} \quad \overline{\psi}_1 = B_1(\lambda) e^{i\lambda_2 y'} + B_1'(\lambda) e^{-i\lambda_2 y'} \quad (3.2)
\end{align*}
\]

\[
\begin{align*}
\overline{\varphi}_2 &= A_2(\lambda) e^{-i\lambda_2 y'} \quad \overline{\psi}_2 = B_2(\lambda) e^{-i\lambda_2 y'} \quad (3.3)
\end{align*}
\]

In the medium \(M\) only backward running waves are admitted to satisfy the radiation condition at infinity and for the medium \(M'\) the presence of the reflected waves are also obvious. Since \(\lambda\) is arbitrary, we may let the constants \(A_1, A_1', B_1, B_1', A_2\) and \(B_2\) and depend on \(\lambda\) and integrate over \(\lambda\). Hence, we assume a general solution in the form
\[
\begin{align*}
\varphi(x', y') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ A_1(\lambda) e^{i\lambda\beta_1} y' + A'_1(\lambda) e^{-i\lambda\beta_1} y' \right] e^{i\lambda x'} d\lambda \\
\psi(x', y') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ B_1(\lambda) e^{i\lambda\beta_1} y' + B'_1(\lambda) e^{-i\lambda\beta_1} y' \right] e^{i\lambda x'} d\lambda \\
\varphi_2(x', y') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} A_2(\lambda) e^{i\lambda(x' - \beta_2 y')} d\lambda \\
\psi_2(x', y') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} B_2(\lambda) e^{i\lambda(x' - \beta_2 y')} d\lambda.
\end{align*}
\]

and

From the first condition of (2.9) we obtain on integrating with respect to \(x'\) and dividing by \(\mu_2\) and then using (3.4)

\[
\frac{\mu_1}{\mu_2} \left[ (M_1^2 - 2) \frac{\partial \varphi_1}{\partial x'} + 2 \frac{\partial \psi_1}{\partial y'} \right] = -\frac{1}{\mu_2} \int_0^{x'} P(x') dx' \frac{\mu_1}{\mu_2} \left[ (M_1^2 - 2) (A_1 e^{-i\lambda\beta_1} + A'_1 e^{i\lambda\beta_1} h) \\
+ 2\beta_1 (B_1 e^{-i\lambda\beta_1} - B'_1 e^{i\lambda\beta_1} h) \right] = P(\lambda) \frac{\mu_1}{i\lambda},
\]

where \(P(\lambda)\) is the Fourier transform of \(-\frac{1}{\mu_2} \int_0^{x'} p(x') dx'\). Similarly from other boundary conditions and using (2.1), (2.11) and (3.4) we obtain the following

\[
2\beta_1 [A_1 e^{-i\lambda\beta_1} h - A'_1 e^{i\lambda\beta_1} h] - (M_1^2 - 2) [B_1 e^{-i\lambda\beta_1} h + B'_1 e^{i\lambda\beta_1} h] = 0,
\]

\[
(A_1 + A'_1) = \beta'_1 (B_1 - B'_1) + A_2 + \beta_2 B_2,
\]

\[
\beta_1 (A_1 - A'_1) + (B_1 + B'_1) = -\beta_2 A_2 + B_2,
\]

\[
\mu_1 [2 \beta_1 (A_1 - A'_1) - (M_1^2 - 2) (B_1 + B'_1)] = \mu_2 [-2 \beta_2 A_2 - (M_2^2 - 2) B_2]
\]

and

\[
\mu_1 [(M_1^2 - 2) (A_1 + A'_1) + 2 \beta'_1 (B_1 - B'_1)] = \mu_2 [(M_2^2 - 2) A_2 - 2 \beta'_1 B_2].
\]

From (3.7) and (3.10) and then from (3.8) and (3.9) we obtain respectively

\[
B_1 - B'_1 = \frac{\mu_2/\mu_1}{M_1^2 \beta_1} (M_2^2 - 2) (M_1^2 - 2) A_2 - \frac{\beta'_1 [M_1^2 - 2] + 2 \mu_2/\mu_1}{M_1^2 \beta_1} B_2.
\]
and

\[ B_1 + B_1' = \frac{[(\mu_2/\mu_1) - 1] 2 \beta_2}{M_1^2} A_2 + \frac{[2 + (M_2^2 - 2) \mu_2/\mu_1]}{M_1^2} B_2. \] ... (3.12)

Now from (3.8) and (3.12) and then from (3.7) and (3.11) we obtain respectively

\[ A_1 - A_1' = -\left[ \frac{[ (\mu_2/\mu_1) - 1] 2 \beta_2 + \beta_2 M_1^2}{\beta_1 M_1^2} A_2 + \frac{[(M_1^2 - 2) - (M_2^2 - 2) \mu_2/\mu_1]}{\beta_1 M_1^2} B_2 \right]. \] .... (3.13).

and

\[ A_1 + A_1' = \frac{[2 + \mu_2/\mu_1 (M_2^2 - 2)]}{M_1^2} A_2 + \frac{[2 - 2 \mu_2/\mu_1]}{M_1^2} B_2. \] ... (3.14)

From (3.11) and (3.12) and then from (3.13) and (3.14) we obtain

\[ B_1 = c_1 A_2 + d_1 B_2 \quad \text{and} \quad B_1' = c_2 A_2 + d_2 B_2. \] ... (3.15)

and

\[ A_1 = a_1 A_2 + b_1 B_2 \quad \text{and} \quad A_1' = a_2 A_2 + b_2 B_2. \]

where

\[ c_1 = \frac{1}{2} \left[ \frac{(\mu_2/\mu_1 - 1) 2 \beta_2}{\beta_1} \pm \left\{ \frac{\mu_2/\mu_1 (M_2^2 - 2) - (M_1^2 - 2)}{\beta_1^2} \right\} \right]/M_1^2 \beta_1, \]

\[ d_1 = \frac{1}{2} \left[ \frac{2 (M_2^2 - 2) \mu_2/\mu_1}{\beta_1} \pm \left\{ (M_1^2 - 2) + 2 \mu_2/\mu_1 \right\} \right]/M_1^2 \beta_1, \]

\[ a_1 = \frac{1}{2} \left[ \frac{2 \beta_1 + \mu_2/\mu_1 (M_2^2 - 2) \beta_1}{\beta_1} \pm \left\{ (\mu_2/\mu_1 - 1) \beta_2 + \beta_2 M_2^2 \right\} \right]/M_1^2 \beta_1, \]

and

\[ b_1 = \frac{1}{2} \left[ \frac{2 \beta_1 \beta_2 (1 - \mu_2/\mu_1)}{\beta_1} \pm \left\{ (M_1^2 - 2) - (M_2^2 - 2) \mu_2/\mu_1 \right\} \right]/M_1^2 \beta_1. \] .... (3.16)

Substituting the values of \( B_1, B_1', A_1 \) and \( A_1' \) from (3.15) into (3.5) and (3.6) we obtain

\[ \{(M_1^2 - 2)_+ a + 2 \beta_1 c\} A_2 + \{(M_1^2 - 2)_+ b + 2 \beta_1 d\} B_2 = \frac{P(\lambda)}{i\lambda} \frac{\mu_2}{\mu_1} \] ... (3.17)

and

\[ \{2 \beta_1 a - (M_1^2 - 2)_+ c\} A_2 + \{2 \beta_1 b - (M_1^2 - 2)_+ d\} B_2 = 0, \] ... (3.18)

where
\begin{align}
+^a &= a_1 e^{-i \lambda \beta_1} h + a_2 e^{i \lambda \beta_1} h, \\
-^a &= b_1 e^{-i \lambda \beta_1} h + b_2 e^{i \lambda \beta_1} h,
\end{align}

\begin{align}
+^b &= c_1 e^{-i \lambda \beta_1} h + c_2 e^{i \lambda \beta_1} h, \\
-^b &= d_1 e^{-i \lambda \beta_1} h + d_2 e^{i \lambda \beta_1} h.
\end{align}

From (3.17) and (3.18) we obtain

\begin{align}
A_2 &= \frac{\mu_2}{\mu_1} \left[ 2 \beta_1 - b - (M_1^2 - 2) + d \right] / \Delta^* \\
S_2 &= \frac{\mu_2}{\mu_1} \left[ 2 \beta_1 - a - (M_1^2 - 2) + c \right] / \Delta^*,
\end{align}

and

\begin{align}
B_2 &= \frac{\mu_2}{\mu_1} \left[ 2 \beta_1 - a - (M_1^2 - 2) + c \right] / \Delta^*,
\end{align}

where

\begin{align}
\Delta^* &= \left\{ \left( M_1^2 - 2 \right)^+ a + 2 \beta_1' c \right\} \left\{ 2 \beta_1 - b - (M_1^2 - 2) + d \right\} \\
&\quad - \left\{ \left( M_1^2 - 2 \right)^+ b + 2 \beta_1' d \right\} \left\{ 2 \beta_1 - a - (M_1^2 - 2) + c \right\}
\end{align}

using (3.21) and (3.22) \( A_1, A_1', B_1 \) and \( B_1' \) can be easily calculated through which the expressions for potentials are

\begin{align}
\varphi_2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\lambda)}{i\lambda} \frac{1}{\Delta^*} \frac{\mu_2}{\mu_1} \left[ 2 \beta_1 - b - (M_1^2 - 2)^+ d \right] e^{i\lambda(x' - \beta_2 y')} d\lambda, \\
\psi_2 &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\lambda)}{i\lambda} \frac{1}{\Delta^*} \frac{\mu_2}{\mu_1} \left[ 2 \beta_1 - a - (M_1^2 - 2)^+ c \right] e^{i\lambda(x' - \beta_2 y')} d\lambda,
\end{align}

\begin{align}
\varphi_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\lambda)}{i\lambda} \frac{1}{\Delta^*} \frac{\mu_2}{\mu_1} \left[ A_1^0 e^{i\lambda \beta_1} y' + B_1^0 e^{-i\lambda \beta_1} y' \right] e^{i\lambda x'} d\lambda, \\
\psi_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\lambda)}{i\lambda} \frac{1}{\Delta^*} \frac{\mu_2}{\mu_1} \left[ C_1^0 e^{i\lambda \beta_1} y' + D_1 e^{-i\lambda \beta_1} y' \right] e^{i\lambda x'} d\lambda,
\end{align}

where

\begin{align}
A_1^0 &= \left\{ 2 \beta_1 (a_1 - b_1 a_1) + (M_1^2 - 2) (b_1 c - a_1 d) \right\}, \\
B_1^0 &= \left\{ 2 \beta_1 (a_2 - b_2 a_2) + (M_1^2 - 2) (b_2 c - a_2 d) \right\}.
\end{align}
\[ C_1^0 = \{2\, \beta_1 (c_1 - b - ad_1) + (M_1^2 - 2) (d_1 + c_1 + d) \} \]

and

\[ D_1^0 = \{2\, \beta_1 (c_2 - b - ad_2) + (M_2^2 - 2) (d_2 + c_2 + d) \}. \]

So

\[
u_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\lambda)}{\Delta^*} \frac{\mu_2}{\mu_1} \left\{ (2\, \beta_2 a - (M_1^2 - 2) d_2) e^{i\lambda(x - \beta_2 y')} - (2\, \beta_1 - (M_1^2 - 2) b) e^{i\lambda(x' - \beta_1 y')} \right\} d\lambda, \quad \ldots \tag{3.28}
\]

\[
u_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\lambda)}{\Delta^*} \frac{\mu_2}{\mu_1} \left[ (A_1 e^{i\lambda \beta_1 y'} + B_1 e^{-i\lambda \beta_1 y'}) - \beta_1' \left( C_1 e^{i\lambda \beta_1 y'} - D_1 e^{-i\lambda \beta_1 y'} \right) \right] e^{i\lambda x} d\lambda, \quad \ldots \tag{3.30}
\]

and

\[
u_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{p(\lambda)}{\Delta^*} \frac{\mu_2}{\mu_1} \left[ \beta_1 (A_1 e^{i\lambda \beta_1 y'} + B_1 e^{-i\lambda \beta_1 y'}) + \beta_1 (C_1 e^{i\lambda \beta_1 y'} + D_1 e^{-i\lambda \beta_1 y'}) \right] e^{i\lambda x} d\lambda, \quad \ldots \tag{3.31}
\]

4. PARTICULAR CASE

When \( p(x') \) is a concentrated line load: When the load \( p(x') \) is a concentrated line load then  
\( p(x') = p\delta(x') \) then neglecting \( h^2 \) and higher power of \( h \), \( V' \) can be written as the following:

\[
A^* = A + i\lambda h B, \quad \ldots \tag{4.1}
\]

where

\[
A = \{(M_1^2 - 2) (a_1 + a_2) + 2 \beta_1 (c_1 - c_2) \} \{2\beta_1 (b_1 - b_2) - (M_1^2 - 2) (d_1 + d_2) \}
\]

\[
- \{(M_1^2 - 2) (b_1 + b_2) + 2 \beta_1 (d_1 - d_2) \} \{2\beta_1 (a_1 - a_2) - (M_1^2 - 2) (c_1 + c_2) \}.
\]

After a little simplification we obtain the above value of \( A \) as
and where

\[ A = [(2 - M_1^2)^2 + 4\beta_2 \beta_1^2] \left( \frac{\mu_2}{\mu_1} \right)^2 \quad \ldots \quad (4.2) \]

\[ B = [(M_1^2 - 2) (a_1 + a_2) + 2 \beta_1' (c_1 - c_2)] [(M_1^2 - 2) \beta_1' (d_1 - d_2) - 2\beta_1^2 (b_1 + b_2)] \]

\[ - [2\beta_1 (b_1 - b_2) - (M_1^2 - 2) (d_1 + d_2)] [(M_1^2 - 2) \beta_1 (a_1 - a_2) + 2\beta_1^2 (c_1 + c_2)] \]

\[ + [2\beta_1^2 (a_1 + a_2) - (M_1^2 - 2) \beta_1 (c_1 - c_2)] [(M_1^2 - 2) (b_1 + b_2) + 2\beta_1' (d_1 - d_2)] \]

\[ + [2\beta_1 (a_1 - a_2) - (M_1^2 - 2) (c_1 + c_2)] [(M_1^2 - 2) \beta_1 (b_1 - b_2) + 2\beta_1^2 (d_1 + d_2)]. \quad \ldots \quad (4.3) \]

With this approximation we can write

\[ \frac{1}{\Delta^*} = \frac{1}{A} [1 - i\hbar B/A]. \quad \ldots \quad (4.4) \]

Hence:

\[ u_2 = \frac{P}{A\mu_1} \left( 2\beta_1 [b_1 H(x' - \beta_2 y' - \beta_1 h) - b_2 H(x' - \beta_2 y' + \beta_1 h)] - (M_1^2 - 2) \right) \]

\[ [d_1 H(x' - \beta_2 y' - \beta_1 h) + d_2 H(x' - \beta_2 y' + \beta_1 h)] + 2\beta_1 \beta_2 [a_1 H(x' - \beta_2 y' - \beta_1 h) - a_2 H(x' - \beta_2 y' + \beta_1 h)] \]

\[ - a_2 H(x' - \beta_2 y' + \beta_1 h)] - \beta_1' (M_1^2 - 2) [c_1 H(x' - \beta_2 y' - \beta_1 h) + c_2 H(x' - \beta_2 y' + \beta_1 h)] \]

\[ + \frac{PB_1}{A\mu_1} \left( 2\beta_1 [b_1 \delta(x' - \beta_2 y' - \beta_1 h) - b_2 \delta(x' - \beta_2 y' + \beta_1 h)] \right) \]

\[ - (M_1^2 - 2) [d_1 \delta(x' - \beta_2 y' - \beta_1 h) \]

\[ + d_2 \delta(x' - \beta_2 y' + \beta_1 h)] + 2\beta_1 \beta_2 [a_1 \delta(x' - \beta_2 y' - \beta_1 h) - a_2 \delta(x' - \beta_2 y' + \beta_1 h)] \]

\[ - \beta_1' (M_1^2 - 2) [c_1 \delta(x' - \beta_2 y' - \beta_1 h) + c_2 \delta(x' - \beta_2 y' + \beta_1 h)]; \quad \ldots \quad (4.5) \]

and

\[ v_2 = \frac{P}{A\mu_1} \beta_1 (M_1^2 - 2) [d_1 H(x' - \beta_2 y' - \beta_1 h) + d_2 H(x' - \beta_2 y' + \beta_1 h)] \]

\[ - 2\beta_1 \beta_2 [b_1 H(x' - \beta_2 y' - \beta_1 h) - b_2 H(x' - \beta_2 y' + \beta_1 h)] - 2\beta_1 [a_1 H(x' - \beta_2 y' - \beta_1 h) - a_2 H(x' - \beta_2 y' + \beta_1 h) - b_1 H(x' - \beta_2 y' - \beta_1 h) + b_2 H(x' - \beta_2 y' + \beta_1 h)] \]

- 2\beta_1 \beta_2 [b_1 H(x' - \beta_2 y' - \beta_1 h) - b_2 H(x' - \beta_2 y' + \beta_1 h)] - 2\beta_1 [a_1 H(x' - \beta_2 y' - \beta_1 h)
\[ -a_2 H(x' - \beta_2 y' + \beta_1 h)] + (M_1^2 - 2) [c_1 H(x' - \beta_2 y' - \beta_1 h) + c_2 H(x' - \beta_2 y' + \beta_1 h)]] \]

\[ + \frac{PB}{A^2 \mu_1} \cdot h \left( \beta_2 (M_1^2 - 2) \left[ d_1 \delta(x' - \beta_2 y' - \beta_1 h) + d_2 \delta(x' - \beta_2 y' + \beta_1 h) \right] - 2 \beta_1 \beta_2 \right) \]

\[ + [b_1 \delta(x' - \beta_2 y' - \beta_1 h) - b_2 \delta(x' - \beta_2 y' + \beta_1 h)] - 2 \beta_1 [a_1 \delta(x' - \beta_2 - \beta_1 h)] \]

\[ - a_2 \delta(x' - \beta_2 y' + \beta_1 h)] + (M_1^2 - 2) [c_1 \delta(x' - \beta_2 y' - \beta_1 h) + c_2 \delta(x' - \beta_2 y' + \beta_1 h)]] \]

\[ \rightarrow \cdots (4.6) \]

Similarly, \( u_1 \) and \( v_1 \) can be calculated. In the above, the following formulas have been used for the concentrated line load from Sneddon (1973, p. 41, 4878) and Fung (1968, p. 267)^4

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\lambda) e^{i\lambda x} d\lambda = -1/\mu_2 \int_{0}^{\infty} p(x') dx' \]

\[ = -\frac{p}{\mu_2} H(x' - \beta_2 y' - \beta_1 h) \]

\[ i \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda p(\lambda) e^{i\lambda \xi} d\lambda = \frac{d}{d\xi} \int_{-\infty}^{\infty} p(\lambda) e^{i\lambda \xi} d\lambda = -\frac{p}{\mu_2} \frac{d}{d\xi} H(\xi) \]

\[ = -\frac{p}{\mu_2} \delta(\xi) [\xi = x' - \beta_2 y' - \beta_1 h]. \]

**DISCUSSION**

It is evident from (3.28) to (3.31) that the displacements in both the media are affected by the thickness of the layer. Now as \( h \to 0 \) in (3.28) and (3.29)

\[ \Delta^* \to \Delta = \left( \frac{\mu_2}{\mu_1} \right)^2 [(2 - M_2^2)^2 + 4\beta_2 \beta_y'] ; \quad \beta_y' [2\beta_1 - a - (M_1^2 - 2),_+] \to 2\beta_2 \beta_y' \frac{\mu_2}{\mu_1} \]

\[ [2\beta_1 - b - (M_1^2 - 2),_+] \to (2 - M_2^2) \frac{\mu_2}{\mu_1} ; \quad \beta_y [2\beta_1 - a - (M_1^2 - 2),_+] \to 2\beta_2 \beta_y' \frac{\mu_2}{\mu_1} \]

So when \( p(\lambda) \) is a concentrated line load then (3.28) and (3.29) reduce to
\[
\begin{align*}
  u_2 &= \frac{P}{\mu_2 \Delta} \left[ (2 - M_2^2) H(x' - \beta_2 y') + 2\beta_2 \overrightarrow{\beta_2} H(x' - \overrightarrow{\beta_2} y') \right] \quad \cdots \ (5.2) \\
  \text{and} \quad \nu_2 &= \frac{P}{\mu_2 \Delta} \left[ -\overrightarrow{\beta_2} (2 - M_2^2) H(x' - \beta_2 y') + 2\beta_2 \overrightarrow{\beta_2} H(x' - \overrightarrow{\beta_2} y') \right] \quad \cdots \ (5.3)
\end{align*}
\]

\[ \Delta = (2 - M_2^2)^2 + 4\beta_2 \overrightarrow{\beta_2}, \]

which is the same result as we make \( h \to 0 \) in (4.5) and (4.6) which is in complete agreement with the result as studied by Fung (p. 263)\(^4\).

The existence of Dirac delta function in the expression of \( u_2 \) and \( \nu_2 \) in (4.5) and (4.6) indicates the force concentration due to the movement of the concentrated line load over the free surface of the elastic layer of small thickness on the top of an infinite elastic semi-space. This stress concentration occurs in the neighbourhood of the line along which the load moves.

It is clear from (4.5) and (4.6) for the displacement that the disturbances are marked by the mach waves such that

\[
\begin{align*}
  x' - \overrightarrow{\beta_2} y' - \beta_1 h &= 0, \\
  x' - \overrightarrow{\beta_2} y' - \overrightarrow{\beta_1} h &= 0, \\
  x' - \overrightarrow{\beta_2} y' - \overrightarrow{\beta_1} h &= 0, \\
  x' - \overrightarrow{\beta_2} y' - \overrightarrow{\beta_1} h &= 0, \\
  x' - \overrightarrow{\beta_2} y' + \beta_1 h &= 0, \\
  x' - \overrightarrow{\beta_2} y' + \overrightarrow{\beta_1} h &= 0, \\
  x' - \overrightarrow{\beta_2} y' + \beta_1 h &= 0, \\
  x' - \overrightarrow{\beta_2} y' + \overrightarrow{\beta_1} h &= 0,
\end{align*}
\]

and

\[
  x' - \overrightarrow{\beta_2} y' + \overrightarrow{\beta_1} h = 0,
\]

which are emanated from

\[
\begin{align*}
  \left( 0, -\frac{\beta_1}{\beta_2} \right); \quad \left( 0, -\frac{\beta_1}{\beta_2} \right); \quad \left( 0, -\frac{\beta_1}{\beta_2} \right); \quad \left( 0, -\beta_1 \right)
\end{align*}
\]

and their images

\[
\begin{align*}
  \left( 0, \frac{\beta_1}{\beta_2} \right); \quad \left( 0, \frac{\beta_1}{\beta_2} \right); \quad \left( 0, \frac{\beta_1}{\beta_2} \right); \quad \left( 0, \frac{\beta_1}{\beta_2} \right)
\end{align*}
\]

Instead of two much waves \( x' - \overrightarrow{\beta_2} y' = 0, x' - \overrightarrow{\beta_2} y' = 0 \) emanated from \((0, - h)\) in case of semi-infinite elastic medium as studied by Fung\(^4\). From (5.2) and (5.3), we also observe that there
are two mach waves as $h \to 0$. Also we can reach directly to the expressions given by (4.5) (4.6) from the boundary conditions (29) and (2.10) if we write $(\sigma_{yy})_1 = -p\delta(x')$ instead $(\sigma_{yy})_1 = -p(x')$ and other boundary conditions remain unchanged.

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Some two-dimensional problems in magneto-elasticity

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Abstract

The distribution of stresses and displacements in a semi-infinite elastic medium immersed under the influence of strong primary magnetic field $\vec{H}(0, 0, H_0)$ has been investigated for the following three cases—(i) variable pressure is applied to the boundary of the medium, (ii) a pulse of pressure moving uniformly along the boundary of the medium, and (iii) a point force moves with uniform velocity over the boundary of the medium. If the magnetic field is weak or absent then the results obtained for the above cases are in good agreement with the corresponding classical problem.

(Keywords: magneto-elasticity/Alfven velocity/pressure pulse/supersonic)

Introduction

We observe a rapid development of the coupled theory of elastic bodies. By this name we understand an interrelation of two or more branches of phenomenological physics, so far being developed separately. As a typical example we may mention magneto-elasticity. In this newly developed branch, problems of waves and vibrations in a homogeneous conducting elastic medium which is assumed to be situated under the influence of a constant primary magnetic field are receiving greater attention by many investigators. Here the authors have studied the problem of moving load over the boundary of a semi-infinite elastic medium under the influence of initial constant magnetic field $\vec{H} = (0, 0, H_0)$.

Basic Equations and Relations

The system of equations in electro-dynamics can be written as:

$$\text{curl} \vec{E} = \frac{\mu_0}{c} \frac{\partial \vec{H}}{\partial t}, \quad \text{curl} \vec{H} = \frac{4\pi}{c} \vec{j}, \quad \text{div} \vec{E} = 0, \quad \text{div} \vec{H} = 0$$

$$\vec{j} = \lambda_0 \left[ \vec{E} + \frac{\mu_0}{c} \left( \frac{\partial \vec{u}}{\partial t} \times \vec{H} \right) \right], \quad \vec{D} = \varepsilon_0 \left[ \vec{E} + \frac{1}{c} \frac{\mu_0 \varepsilon_0}{\varepsilon_0} \left( \frac{\partial \vec{u}}{\partial t} \times \vec{H} \right) \right]$$


in which $\vec{h}$ and $\vec{E}$ are perturbations of the magnetic and electric fields respectively, $\vec{j}$ denotes the current density vector, $c$ is the speed of light, $\mu_0$ and $\varepsilon_0$ are the magnetic and electric permeability respectively, $\vec{D}$ is electric induction vector, $\vec{u}$ is the displacement vector and $\lambda_0$ is the electric conductivity.

The basic equations of motion in magneto-elasticity in absence of body forces can be written as:

$$\sigma_{\mu,j} + T_{\mu,j} = \rho \ u_i$$  (2)

where $T_{\mu,j}$ denotes the Maxwell electro-magnetic stress-tensor, $T_{\mu,j}$ are the Lorentz forces. The Maxwell tensor is related to the vector $\vec{h}$ in the following manner.

$$T_{\mu} = \frac{\mu_0}{4\pi} \left[ H_i h_j + H_j h_i - \delta_{ij} H_k h_k \right] \ [i,j=1,2,3]$$  (3)

Eliminating the vector $\vec{E}$ and $\vec{j}$ and the stresses and strains and expressing the vector $T_{\mu}$ by the component of the vector $\vec{u}$ we arrive at the systems of equations:

$$\nabla^2 \vec{h} - \beta \frac{\mu_0}{4\pi} \nabla \times (\vec{u} \times \vec{H}), \ \ \beta = 4\pi \lambda_0 \mu_0 / c^2$$

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla \nabla \nabla \vec{u} + \frac{\mu_0}{4\pi} \left[ \nabla \nabla \nabla (\nabla \times \vec{H}) \times \vec{H} \right] = \rho \ \nabla \vec{u}$$  (4)

For the elastic solid ideally conducting electricity:

$[\beta = \infty ; \ \vec{h} = \nabla \times (\vec{u} \times \vec{H})]$, we get

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla \nabla \nabla \vec{u} + \frac{\mu_0}{4\pi} \left[ \nabla \nabla \nabla (\nabla \times \vec{H}) \times \vec{H} \right] = \rho \ \nabla \vec{u}$$  (5)

If the magnetic field $\vec{H}$ is of the form $\vec{H} = (0, 0, H_0)$ then eqn. (5) reduces to

$$\mu \nabla^2 \vec{u} + (\lambda + \mu + \alpha_0^2 \rho) \nabla \nabla \nabla \vec{u} = \rho \ \nabla \vec{u}$$  (6)
where $c_0^2 = \frac{\mu_0 H_0^2}{4 \pi \rho}$, $c_0$ is the Alfvén velocity. Now for the displacement potentials $\phi$ and $\psi$ for which $u = \phi_1 + \psi_2$ and $v = \phi_2 - \psi_1$ (7)

eqn (6) decomposes into the following system of equations.

\[ (1 + \alpha_0^2 / c_1^2) \nabla^2 \phi = c_1^{-2} \dot{\phi} \]  
\[ \nabla^2 \psi = c_2^{-2} \ddot{\psi} \]  

where we have written $c_1^2 = (\lambda + 2\mu) / \rho$ and $c_2^2 = \mu / \rho$. The components of stresses in terms of the function $\varphi$ and $\psi$ are

\[ \tau_{22} = \lambda \nabla^2 \varphi + 2\mu (\varphi_{22} - \psi_{12}) \]  
\[ \tau_{11} = \lambda \nabla^2 \varphi + 2\mu (\varphi_{11} + \psi_{12}) \]  
\[ \tau_{12} = \mu [2 \varphi_{12} - \psi_{11} + \psi_{22}] \]  

Formulation of the Problem

If for convenience we replace the time variable $t$ by a space like variable $\tau$ defined by $\tau = c_1 t$ then we may write (8) in the form

\[ \frac{\partial^2 \varphi}{\partial \tau^2} = (1 + \alpha_0^2 / c_1^2) \nabla^2 \varphi \]  
\[ \beta^2 \frac{\partial^2 \psi}{\partial \tau^2} = \nabla^2 \psi \]  

where

\[ \beta^2 = c_1^2 / c_2^2 = (\lambda + 2\mu) / \mu \]  

We take the x-axis to be along the boundary and y-axis pointing into the medium. To find the solutions of the wave equations (10), we introduce the two dimensional Fourier transform of the functions
\[ \varphi(x, y, \tau) \quad \text{and} \quad \psi(x, y, \tau) \quad \text{as} \]

\[
\overline{\varphi}(\xi, y, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y, \tau) \, e^{i(\xi x + \zeta \tau)} \, dx \, d\tau
\]

\[
\overline{\psi}(\xi, y, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, y, \tau) \, e^{i(\xi x + \zeta \tau)} \, dx \, d\tau
\]

**Boundary conditions**: The boundary conditions for the variable pressure pulse \( p(x, \tau) \) moving over the boundary \( y = 0 \) of the conducting elastic medium are

\[
\tau_{22} = -p(x, \tau) \quad , \quad \tau_{12} = 0 \quad \text{on} \quad y = 0
\]

Now, if we multiply both sides of the eqn. (10) by \( e^{i(\xi x + \zeta \tau)} \) and integrate over the entire \( x \)-plane we find that the function \( \overline{\varphi}(\xi, y, \zeta) \) and \( \overline{\psi}(\xi, y, \zeta) \) satisfy the following differential equations

\[
\frac{d^2 \overline{\varphi}}{dy^2} = (\xi^2 - \kappa^2 \zeta^2) \overline{\varphi} \quad \quad \frac{d^2 \overline{\psi}}{dy^2} = (\zeta^2 - \beta^2 \zeta^2) \overline{\psi}
\]

where \( \kappa^2 = \frac{c_1^2}{c_1^2 + \alpha_0^2} \) is the magnetic parameter of the problem. To satisfy the condition that the stresses tend to zero as \( y \to \infty \) we take the solutions of (14) as

\[
\overline{\varphi}(\xi, y, \zeta) = A \exp \left[ - (\xi^2 - \kappa^2 \zeta^2)^{1/2} y \right] \quad , \quad \overline{\psi}(\xi, y, \zeta) = \beta \exp \left[ - (\zeta^2 - \beta^2 \zeta^2)^{1/2} y \right]
\]

The corresponding wave function \( \varphi(x, y, \tau) \) and \( \psi(x, y, \tau) \) are given by Fourier's inversion theorem for two dimensional transforms. From these we obtain the following expressions.

\[
\frac{\tau_{22}}{2\mu} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ i \xi (\xi^2 - \beta^2 \zeta^2)^{1/2} \overline{\psi} - (\zeta^2 - \frac{1}{2} \beta^2 \zeta^2 k^2) \overline{\varphi} \right] e^{-i(\xi x + \zeta \tau)} \, d\xi \, d\zeta
\]

\[
\frac{\tau_{11}}{2\mu} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ i \xi (\xi^2 - \beta^2 \zeta^2)^{1/2} \overline{\psi} - \left( \zeta^2 + \frac{\lambda}{2\mu} \zeta^2 k^2 \right) \overline{\varphi} \right] e^{-i(\xi x + \zeta \tau)} \, d\xi \, d\zeta
\]
In a similar way we can find the displacements from (7)

\[
\begin{align*}
\tau & = - \frac{1}{2 \mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ i \xi \phi + (\xi^2 - 2 \xi \phi \beta^2 \phi^3) \right] e^{-i(\xi x + \xi \phi)} d\xi d\phi, \\
\sigma & = - \frac{1}{2 \mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \left( \phi^2 - \phi^2 \phi^3 \right) \right] e^{-i(\xi x + \xi \phi)} d\xi d\phi.
\end{align*}
\]

**Solution**

(i) *When a variable pressure is applied on the boundary:*

Using the boundary condition, we obtain

\[
\begin{align*}
\left( \xi^2 - \frac{1}{2} \beta^2 \phi^3 k^2 \right) A - i \xi \left( \xi^2 - \beta^2 \phi^3 \phi^3 \right) B &= - \bar{p} (\xi, \phi) / 2\mu, \\
i \xi \left( \xi^2 - k^2 \phi^3 \phi^3 \right) A + \left( \xi^2 - \frac{1}{2} \beta^2 \phi^3 \phi^3 \right) B &= 0
\end{align*}
\]

where \( \bar{p} (\xi, \phi) \) is the two dimensional Fourier transforms of \( p (x, \phi) \). From (18) we obtain after simplification

\[
\begin{align*}
A &= \frac{1}{2 \mu} \left( \frac{\xi^2 - \frac{1}{2} \beta^2 \phi^3 \phi^3}{f + g} \right) \bar{p} (\xi, \phi), \\
B &= - \frac{i \xi \left( \xi^2 - \beta^2 \phi^3 \phi^3 \right)}{2\mu} \frac{\bar{p} (\xi, \phi)}{(f + g)}
\end{align*}
\]

where

\[
\begin{align*}
f (\xi^2, \phi^2) &= - \left( \xi^2 - \frac{1}{2} \beta^2 \phi^3 \phi^3 \right) \\
g (\xi^2, \phi^2) &= \xi^2 \left( \xi^2 - \beta^2 \phi^3 \phi^3 \right)
\end{align*}
\]

Hence,
The components of the displacement vector are

\[
\begin{align*}
\tau_{22} &= -\frac{1}{2\pi} \int \int_{-\infty}^{\infty} \frac{\bar{p}}{(f + g)} \left[ f e^{-\left(\xi^2 - k^2 \zeta^2\right) \gamma} + g e^{-\left(\xi^2 - \beta^2 \zeta^2\right) \gamma} \right] e^{-i(\xi x + \zeta \tau)} \, d\xi \, d\zeta \\
\tau_{11} &= \frac{1}{2\pi} \int \int_{-\infty}^{\infty} \frac{\bar{p}}{(f + g)} \left[ g e^{-\left(\xi^2 - \beta^2 \zeta^2\right) \gamma} - \left(\xi^2 + \frac{\lambda}{2\mu} k^2 \zeta^2\right) \left(\xi^2 - \frac{1}{2} \beta^2 \zeta^2\right) e^{-\left(\xi^2 - k^2 \zeta^2\right) \gamma} \right] e^{-i(\xi x + \zeta \tau)} \, d\xi \, d\zeta \\
\tau_{12} &= \frac{1}{2\pi} \int \int_{-\infty}^{\infty} \frac{i \xi \bar{p}}{(f + g)} \left(\xi^2 - k^2 \zeta^2\right) \gamma \left(\xi^2 - \frac{1}{2} \beta^2 \zeta^2\right) \left[ e^{-\left(\xi^2 - \beta^2 \zeta^2\right) \gamma} - e^{-\left(\xi^2 - k^2 \zeta^2\right) \gamma} \right] e^{-i(\xi x + \zeta \tau)} \, d\xi \, d\zeta 
\end{align*}
\]

(21)

The components of the displacement vector are

\[
\begin{align*}
u &= -\frac{1}{4\pi\mu} \int \int_{-\infty}^{\infty} \frac{\xi \bar{p}}{(f + g)} \left[ \frac{g}{\xi^2} e^{-\left(\xi^2 - k^2 \zeta^2\right) \gamma} - \left(\xi^2 - \frac{1}{2} \beta^2 \zeta^2\right) e^{-\left(\xi^2 - \beta^2 \zeta^2\right) \gamma} \right] e^{-i(\xi x + \zeta \tau)} \, d\xi \, d\zeta \\
v &= -\frac{1}{4\pi\mu} \int \int_{-\infty}^{\infty} \frac{(\xi^2 - k^2 \zeta^2) \gamma}{(f + g)} \left[ \left(\xi^2 - \frac{1}{2} \beta^2 \zeta^2\right) e^{-\left(\xi^2 - \beta^2 \zeta^2\right) \gamma} - \xi^2 e^{-\left(\xi^2 - k^2 \zeta^2\right) \gamma} \right] e^{-i(\xi x + \zeta \tau)} \, d\xi \, d\zeta 
\end{align*}
\]

(22)

(ii) When a pulse of pressure is moving uniformly along the boundary

The stress set up in the interior of the semi-infinite elastic medium when a pulse of pressure of shape \( p = \chi(x) \) moves with uniform velocity \( v \) along the boundary \( y = 0 \). We then have

\[
p(x, \tau) = \chi(x - vt) = \chi(x - \beta_1 \tau), \quad \beta_1 = v/c_1
\]

and then

\[
\bar{p}(\xi, \zeta) = \frac{1}{2\pi} \int \int_{-\infty}^{\infty} \chi(x - \beta_1 \tau) e^{i(\xi x + \zeta \tau)} \, dx \, dt
\]

(23)
which by a trivial change of variable gives

\[
\bar{p} (\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi (u) e^{i \xi u} du \int_{-\infty}^{\infty} e^{i \xi (\tau + \beta_1 \xi)} d\tau = 2 \chi (\xi) \delta (\xi + \beta_1 \xi) \tag{24}
\]

where

\[
2 \chi (\xi) = \int_{-\infty}^{\infty} \chi (u) e^{i \xi u} du \tag{25}
\]

Now if we make use of the fact that for any function \(\chi (\xi^2, \eta^2)\)

\[
\int_{-\infty}^{\infty} \chi (\xi^2, \eta^2) \delta (\xi + \beta_1 \xi) e^{-i \xi \zeta} d\zeta = \chi (\xi^2, \beta_1^2 \xi^2) e^{-i \xi \nu} \tag{26}
\]

we find for the component of stress as

\[
\tau_{22} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \chi (\xi) e^{i \xi (x - \eta)} \left[ \frac{\Phi}{\Theta + \Phi} e^{-i (1 - k^2 \beta_1^2) \nu} y_\xi + \frac{\Phi}{\Theta + \Phi} e^{-i (1 - \beta_3^2) \nu} y_\zeta \right] d\xi \tag{27}
\]

\[
\tau_{11} = \frac{1}{\pi} \int_{-\infty}^{\infty} \chi (\xi) e^{i \xi (x - \eta)} \left[ \frac{\Phi}{\Theta + \Phi} e^{-i (1 - \beta_3^2) \nu} y_\xi - \left(1 + \frac{\lambda}{2\mu} \beta_1^2 k^2\right) \left(1 - \frac{1}{2} \beta_2^2 \right) e^{-i (1 - \beta_3^2) \nu} y_\zeta \right] d\xi \tag{28}
\]

\[
\tau_{12} = \frac{1}{\pi} (1 - k^2 \beta_3^2) \nu \left(1 - \frac{1}{2} \beta_2^2 \right) \int_{-\infty}^{\infty} \chi (\xi) e^{i \xi (x - \eta)} \left[ e^{-i (1 - \beta_3^2) \nu} y_\xi - e^{-i (1 - \beta_3^2) \nu} y_\zeta \right] d\xi
\]

and

\[
u = \frac{1}{2\pi \mu} \int_{-\infty}^{\infty} e^{i \xi (x - \eta)} \chi (\xi) \left[ \frac{\Phi}{(\Theta + \Phi) \xi^2} e^{-i (1 - \beta_3^2) \nu} y_\xi + \left(1 - \frac{1}{2} \beta_2^2 \right) \frac{\Phi}{(\Theta + \Phi) \xi^2} e^{-i (1 - \beta_3^2) \nu} y_\zeta \right] d\xi \tag{29}
\]
\[ \nu = -\frac{1}{2\pi\mu} \int_{-\infty}^{\infty} \chi(\xi) \frac{(1 - k^2 \beta_1^2)^{1/2}}{(\theta + \varphi) \xi} \left[ \left(1 - \frac{1}{2} \beta_2^2 \right) e^{-(1 - k^2 \beta_1^2) \varphi \xi} - e^{-(1 - k^2 \beta_1^2) \theta \xi} \right] d\xi \]

where

\[ \theta (\beta_2) = -\left(1 - \frac{1}{2} \beta_2^2 k^2 \right) \left(1 - \frac{1}{2} \beta_2^2 \right) \]

\[ \varphi (\beta_1, \beta_2) = (1 - k^2 \beta_1^2)^{1/2} (1 - \beta_2^2)^{1/2}, \quad \beta_2 = \frac{\nu}{c_2} = \beta_1 \]

(iii) When a point force moves with uniform velocity over the boundary

Here we shall consider the distribution of stress produced by the application to the boundary of a point force of magnitude \( P \) where point of application moves with uniform velocity \( \nu \). The form of the pressure pulse in this case is

\[ \chi(x) = P \delta(x) \quad \text{and} \quad \chi(\xi) = P / 2 \]

The resulting integrals in this case are

\[ \tau_{22} = -\frac{P}{2\pi} \int_{-\infty}^{\infty} e^{\xi(x-v)} \left[ \frac{\theta}{\theta + \varphi} e^{-(1 - k^2 \beta_1^2) \varphi \xi} + \frac{\varphi}{\theta + \varphi} e^{-(1 - k^2 \beta_1^2) \theta \xi} \right] d\xi \]

\[ \tau_{11} = \frac{P}{2\pi} \int_{-\infty}^{\infty} e^{\xi(x-v)} \left[ e^{-(1 - k^2 \beta_1^2) \varphi \xi} - \frac{1 + \lambda}{2\mu} \frac{\beta_1^2 k^2}{\theta + \varphi} \left(1 - \frac{1}{2} \beta_2^2 \right) e^{-(1 - k^2 \beta_1^2) \theta \xi} \right] d\xi \]

\[ \tau_{12} = \frac{P}{2\pi} \int_{-\infty}^{\infty} e^{\xi(x-v)} \left(1 - \frac{1}{2} \beta_2^2 \right) \left(\frac{1 + \lambda}{2\mu} \frac{\beta_1^2 k^2}{\theta + \varphi} \right) \left[ e^{-(1 - k^2 \beta_1^2) \varphi \xi} - e^{-(1 - k^2 \beta_1^2) \theta \xi} \right] d\xi \quad (29) \]

\[ u = \frac{P}{4\pi\mu} \int_{-\infty}^{\infty} \left[ \frac{\varphi}{(\theta + \varphi)} \frac{1}{i \xi} e^{-(1 - k^2 \beta_1^2) \varphi \xi} - \frac{1 - \frac{1}{2} \beta_2^2}{(\theta + \varphi) i \xi} e^{-(1 - k^2 \beta_1^2) \theta \xi} \right] e^{\xi(x-v)} d\xi \quad (30) \]
\[ v = - \frac{P}{4\pi \mu} \int_{-\infty}^{\infty} \frac{(1 - k^2 \beta_1^2)^{1/2}}{(\theta + \varphi) \xi} \left[ \left( 1 - \frac{1}{2} \beta_2^2 \right) e^{-(1 - k^2 \beta_1^2)^{1/2} y_\xi} - e^{-(1 - k^2 \beta_1^2)^{1/2} y_\xi} \right] e^{\xi(x - \eta)} d\xi \]

**Discussion**

If the magnetic field is weak or absent then \( \alpha_0 \to 0 \) i.e. \( k \to 1 \) and the expressions for stresses and displacement of the above three cases tally with the result of Sneddon in classical elastokinetics. Moreover, if (i) \( k^2 \beta_1^2 > 1 \) and \( \beta_2^2 > 1 \) then taking \( x' = x - vt \) the expressions for \( u \) and \( v \) in (30) we obtain

\[
\begin{align*}
\nu &= \frac{P}{2 \mu} \left[ \frac{\varphi}{\theta + \varphi} H \left( \frac{1 - k^2 \beta_1^2}{1 - \frac{1}{2} \beta_2^2} \right) \right] x' - \frac{2 - \beta_2^2}{2(\theta + \varphi)} H \left[ x' - (k^2 \beta_1^2 - 1)^{1/2} y \right] \] 
\end{align*}
\]

If again \( k \to 1 \) then the expression (31) reduces to the following form

\[
\begin{align*}
u &= \frac{P}{2 \mu} \left[ \left( k^2 \beta_1^2 - 1 \right)^{1/2} \right] x' - \frac{2 - \beta_2^2}{2(\theta + \varphi)} H \left[ x' - (k^2 \beta_1^2 - 1)^{1/2} y \right] 
\end{align*}
\]

where the notations have been changed as

\[
y = y' , \Delta = (2 - \beta_2^2)^2 + 4 \beta_1 \beta_2 , \quad \beta_1 = (\beta_1^2 - 1)^{1/2} , \quad \beta_2 = (\beta_2^2 - 1)^{1/2}
\]

The expressions for \( u \) and \( v \) in (32) tally with the result as studied by Fung for supersonic case. The result also holds good for the expression of stresses. Considering (ii) \( k^2 \beta_1^2 < 1 \), \( \beta_2^2 < 1 \) and (iii) \( k^2 \beta_1^2 > 1 \), \( \beta_2^2 < 1 \) we can ultimately reach to the solutions of Fung for the subsonic and transonic cases also.

Moreover, it is to be noted that the medium is undisturbed in front of the mack waves

\[
x' - (\beta_2^2 - 1)^{1/2} y' = 0 \quad \text{and} \quad x' - (k^2 \beta_1^2 - 1)^{1/2} y' = 0
\]
whether the medium is undisturbed in front of the mack waves
\[ x' \left( \beta_2^2 - 1 \right)^{1/2} y' = 0 \quad \text{and} \quad x' \left( \beta_1^2 - 1 \right)^{1/2} y' = 0 \] (35)
in the classical case. So in the present case the angle between the lines is decreased in comparison with the classical case since
\[ \frac{1}{\sqrt{k^2 \beta_2^2 - 1}} > \frac{1}{\sqrt{\beta_1^2 - 1}} \quad (k^2 < 1) \] (36)

References
Influence of gravity on propagation of waves in a medium in presence of a compressional source

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Abstract. The present paper is concerned with two-dimensional wave propagation in a homogeneous isotropic elastic solid medium underlying a liquid layer, under the influence of a gravitational field, when a compressional wave source is present in the liquid. In the absence of gravity the result is in good agreement with the corresponding result of the classical problem.

Keywords. Compressional wave source; initial stress; refraction arrival; axial symmetry

1. Introduction

Stoneley (1926) studied the effect of the ocean on the transmission of Rayleigh waves considering the bottom of the ocean as a solid half-space. In classical theory of elasticity, Jeffreys (1926), Muskat (1933) and Sommerfeld (1949, p. 55) have discussed wave propagation for the case where the distance of a point source from the plane interface is finite. In particular, Press & Ewing (1950) studied the propagation of waves when the point source is present in the liquid layer. Their results are directly related to an important practical problem, that of the ‘refraction arrival’ from a source to a receiver in seismology of near-earthquakes and in seismic refraction investigations. In the classical problem of elastic waves and vibrations the gravity effect is generally neglected. The effect of gravity in the problem of propagation of waves in solids, in particular on an elastic globe, was first studied by Bromwich (1898). Subsequently, investigation of the effect of gravity was considered by Love (1911, pp 144–178) who showed that the velocity of Rayleigh waves is increased to a significant extent by the gravitational field when wavelengths are large. More recently, Biot (1965, pp 44–45, 273–281) developed a theory of initial stress and used it to investigate the influence of gravity on Rayleigh waves, assuming creation of a type of initial stress of hydrostatic nature by the force of gravity and the medium as incompressible. The initial stress is produced in the body by a slow process of creep, where the shear stresses tend to become small or vanish after a long interval of time. De & Sengupta (1973, 1974, 1976) studied the effect of gravity on surface waves, on the propagation of waves in an elastic layer and Lamb’s problem on a plane. Das & Sengupta (1992) investigated the effect of gravity on visco-elastic surface waves in solids. Assuming
that the boundaries are all parallel planes, the authors have studied the axisymmetric problem of propagation of waves under the influence of gravity in a medium which is composed of a liquid layer and of an underlying solid half-space. The authors have considered the following two cases by way of investigating the problem, (1) that the waves are smaller than ordinary earthquake Rayleigh waves, and (ii) the waves are considerably longer than ordinary earthquake Rayleigh waves.

2. Basic equations and relations

We shall use here the subscript '1' for liquid and subscript '2' for the solid part respectively. In the Cartesian co-ordinate system the two-dimensional equations of motion in an elastic solid medium in absence of body forces are

\[
\rho_2 \frac{\partial^2 u_2}{\partial t^2} = (\lambda_2 + \mu_2) \frac{\partial \Theta}{\partial x} + \mu_2 \nabla^2 u_2,
\]

\[
\rho_2 \frac{\partial^2 w_2}{\partial t^2} = (\lambda_2 + \mu_2) \frac{\partial \Theta}{\partial z} + \mu_2 \nabla^2 w_2
\]

The two-dimensional equations of motion in the cylindrical co-ordinate system \((r, \theta, z)\) may be written as (Ewing et al. 1957, p. 9)

\[
(\lambda_2 + 2\mu_2) \left( \frac{\partial^2 q_2}{\partial r^2} + \frac{1}{r} \frac{\partial q_2}{\partial r} - \frac{q_2}{r^2} + \frac{\partial^2 w_2}{\partial z^2} \right) + \mu_2 \left( \frac{\partial^2 q_2}{\partial z^2} - \frac{\partial^2 w_2}{\partial z \partial r} \right) = \rho_2 \frac{\partial^2 q_2}{\partial t^2},
\]

\[
(\lambda_2 + 2\mu_2) \left( \frac{\partial^2 q_2}{\partial z^2} + \frac{1}{r} \frac{\partial q_2}{\partial z} + \frac{\partial^2 w_2}{\partial r^2} \right) - \mu_2 \left( \frac{\partial q_2}{\partial z} - \frac{\partial w_2}{\partial r} \right) - \mu_2 \left( \frac{\partial^2 q_2}{\partial z \partial r} - \frac{\partial^2 w_2}{\partial t^2} \right) = \rho_2 \frac{\partial^2 w_2}{\partial z^2},
\]

where \(\Theta\) is the cubical dilatation, and \(q_2\) and \(w_2\) are the displacements in the \(r\) and \(z\) directions. The angle \(\theta\) does not appear because of the axial symmetry. Now we define \(q_2\) and \(w_2\) in terms of the potential function \(\phi_2\) and the function \(W_2\) as (Ewing et al. 1957, p. 9)

\[
q_2 = \frac{\partial \phi_2}{\partial r} - \frac{\partial \psi_2}{\partial z}; \quad w_2 = \frac{\partial \phi_2}{\partial z} + \frac{\partial (rW_2)}{\partial r}.
\]

Substituting the values of \(q_2\) and \(w_2\) from (3) in (2) and using the relation \(W_2 = -\frac{\partial \psi_2}{\partial r}\) we obtain

\[
\nabla^2 \phi_2 = \frac{1}{\alpha_2^2} \frac{\partial^2 \phi_2}{\partial t^2}, \quad \nabla^2 \psi_2 = \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2}, \quad \left[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right],
\]

and

\[
\alpha_2 = \left( (\lambda_2 + 2\mu_2)/\rho_2 \right)^{1/2}, \quad \beta_2 = (\mu_2/\rho_2)^{1/2}.
\]

Now for the liquid part, the equations of motion, under the influence of gravity, may be written in terms of the potential function \(\phi_1\) as (Lamb 1945, p. 556)

\[
\frac{\partial^2 \phi_1}{\partial t^2} = \alpha_1^2 \nabla^2 \phi_1 + g \frac{\partial \phi_1}{\partial z}.
\]
In the above, \(a_1, \alpha_2\) are the velocities of compressional waves in liquids and solids respectively, and \(\beta_2\) is the velocity of distortional waves in the solid. The displacements may be expressed in terms of the potentials \(\phi_1, \phi_2\) and \(\psi_2\) as

\[
q_1 = \frac{\partial \phi_1}{\partial r}, \quad w_1 = \frac{\partial \phi_1}{\partial z}, \quad \phi_1 = \frac{\partial \phi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial z^2} - \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2}
\]

\[
q_2 = \frac{\partial \phi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial z^2}, \quad w_2 = \frac{\partial \phi_2}{\partial z} + \frac{\partial^2 \psi_2}{\partial z^2} - \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2}
\]

3. General theory and boundary conditions

We take \(z = 0\) as the free surface of the liquid, \(z = H\) as the surface of separation of the two media and \(z\)-axis vertically downwards. The source lies at the point \((0, 0, h)\) in the liquid part.

**Boundary conditions:** The boundary conditions are (Ewing et al. 1957, p. 158)

\[
\phi_1 = 0, \quad \text{at } z = 0,
\]

\[
\omega_1 = \omega_2, \quad \text{at } z = H,
\]

\[
(p_{rr})_1 = 0, \quad \text{at } z = H,
\]

\[
(p_{zz})_1 = (p_{zz})_2, \quad \text{at } z = H,
\]

where

\[
(p_{rr})_2 = \mu_2 \left( \frac{\partial q_2}{\partial z} + \frac{\partial w_2}{\partial r} \right),
\]

\[
(p_{zz})_1 = \lambda_1 \nabla^2 \phi_1, \quad (p_{zz})_2 = \lambda_2 \nabla^2 \phi_2 + 2\mu_2 \frac{\partial w_2}{\partial r}.
\]

4. Solution of the problem

4.1 Case I – The waves are smaller than ordinary earthquake Rayleigh waves

In this case we consider (6) for the liquid part. But the gravity terms in the equations for the solid part of the system are omitted by Scholte (1943) and so we use (4) for the solid part. Now we follow Pekeris (1948) in dividing the liquid layer into two parts by the plane \(z = h\) so that the potential \(\phi_1\) is represented by two different expressions in the form given by Ewing et al. (1957, p. 174)

\[
\phi'_1 = \int_0^\infty A(K)J_0(Kr) \sin \tilde{\lambda}_1 z \exp \left\{ i\omega r - g z / 2\alpha_1^2 \right\} dK, \quad 0 \leq z \leq h,
\]

\[
\phi''_1 = \int_0^\infty [B(K) \sin \tilde{\lambda}_1 z + C(K) \cos \tilde{\lambda}_1 z] \exp \left\{ i\omega r - g z / 2\alpha_1^2 \right\} J_0(Kr) dK,
\]

\[
h \leq z \leq H,
\]

where

\[
\tilde{\lambda}_1 = \sqrt{\left( \nu_1^2 - g^2 / 4\alpha_1^4 \right)}, \quad \nu_1^2 = K_{\alpha_1}^2 - K^2, \quad K_{\alpha_1} = \omega / \alpha_1.
\]
Here the condition (8) at the free surface \( z = 0 \) is satisfied by the assumed form of (13). Also \( \phi_1' \) and \( \phi_1'' \) satisfy the following conditions

\[
\phi_1' = \phi_1'' \quad \text{at} \quad z = h, \tag{16}
\]

and

\[
\frac{\partial \phi_1'}{\partial z} - \frac{\partial \phi_1''}{\partial z} = 2 \exp(i\omega t) \int_0^\infty J_0(Kr) dK, \quad \text{at} \quad z = h. \tag{17}
\]

For the solid part

\[
\phi_2 = \int_0^\infty Q_2(K) \exp[-i\bar{v}_2^z]J_0(Kr) dK, \quad \psi_2 = \int_0^\infty S_2(K) \exp[-i\bar{v}_2^z]J_0(Kr) dK, \tag{18}
\]

where

\[
\bar{v}_2^z = K_{\alpha_2}^2 - K^2, \quad \bar{v}_2^\prime = K_{\beta_2}^2 - K^2, \quad K_{\alpha_2} = \omega/\alpha_2, \quad K_{\beta_2} = \omega/\beta_2. \tag{19}
\]

Using (13) and (14) in (16) and (17) and substituting the values of \( \phi_1'' \), \( \phi_2 \) and \( \psi_2 \) in the boundary conditions (9), (10) and (11), we obtain

\[
\exp \left[ -\frac{gH}{2\alpha_1^2} \right] \left[ \tilde{\lambda}_1 \cos \tilde{\lambda}_1 H - \left( g/2\alpha_1^2 \right) \sin \tilde{\lambda}_1 H \right] B(K) + i\bar{v}_2 Q_2(K) \exp[-i\bar{v}_2 H] - K^2 S_2(K) \exp[-i\bar{v}_2^z H] = 2 \sin \tilde{\lambda}_1 h \left[ \sin \tilde{\lambda}_1 H + \left( g/2\tilde{\alpha}_1^2 \right) \cos \tilde{\lambda}_1 H \right] + C^0, \tag{20}
\]

\[
2i\bar{v}_2 Q_2(K) \exp[-i\bar{v}_2 H] + (K^2 - K_{\alpha_2}^2) S_2 \exp[-i\bar{v}_2^z H] = 0, \tag{21}
\]

\[
\rho_1 \exp \left\{ -\frac{gH}{2\alpha_1^2} \right\} \left\{ \omega^2 \sin \tilde{\lambda}_1 H + g \left[ \tilde{\lambda}_1 \cos \tilde{\lambda}_1 H - \left( g/2\alpha_1^2 \right) \sin \tilde{\lambda}_1 H \right] \right\} B(K) + (2\mu_2 K^2 - \rho_2 \omega^2) Q_2 \exp[-i\bar{v}_2 H] - 2\mu_2 K^2 i\bar{v}_2 S_2 \exp[-i\bar{v}_2^z H] = -\frac{2\rho_1}{\tilde{\lambda}_1 C^0} \exp \left[ -\frac{g(h - H)}{2\alpha_1^2} \right] \left\{ \omega^2 \cos \tilde{\lambda}_1 H - g \left[ \tilde{\lambda}_1 \sin \tilde{\lambda}_1 H + \left( g/2\alpha_1^2 \right) \cos \tilde{\lambda}_1 H \right] \right\} \sin \tilde{\lambda}_1 h, \tag{22}
\]

where

\[
C^0 = \left[ 1 - \left( g/\tilde{\alpha}_1 \right) \sin \tilde{\lambda}_1 h \cos \tilde{\lambda}_1 h \right]. \tag{23}
\]

Determining the functions \( B \), \( Q_2 \) and \( S_2 \) from (20) to (22) and then evaluating \( A \) from (16) and (17) we have in this case

\[
\phi_1' = 2 \exp(i\omega t) \int_0^\infty \frac{\exp[-g\varepsilon/2\alpha_1^2]J_0(Kr) K \sin \tilde{\lambda}_1 z}{\tilde{\lambda}_1 \Delta(K) C^0} \left\{ \left( \frac{\rho_2^2}{\beta_2} \right) \left[ 4K^2 \bar{v}_2' \right] + \rho_1 \omega^2 i\bar{v}_2 \right\} \sin \tilde{\lambda}_1 (H - h) + \frac{K\beta_2}{\rho_2} \sin \tilde{\lambda}_1 (H - h) \tag{24}
\]

\[
+ \left( 2K^2 - K_{\beta_2}^2 \right) \left[ \tilde{\lambda}_1 \cos \tilde{\lambda}_1 (H - h) \right] \left( \frac{\rho_2^2}{\beta_2} \right) \left[ 4K^2 \bar{v}_2' \right] \sin \tilde{\lambda}_1 (H - h) \tag{25}
\]

\[
+ \frac{\rho_1 \omega^2 i\bar{v}_2}{\beta_2} \sin \tilde{\lambda}_1 (H - h) \tag{26}
\]

\[
+ \left( 2K^2 - K_{\beta_2}^2 \right) \left[ \tilde{\lambda}_1 \cos \tilde{\lambda}_1 (H - h) \right] \left( \frac{\rho_2^2}{\beta_2} \right) \left[ 4K^2 \bar{v}_2' \right] \sin \tilde{\lambda}_1 (H - h) \tag{27}
\]
Propagation of waves

\[ y = \sin \lambda (H - h) + \lambda g \cos \lambda (H - h) - \frac{g}{2\alpha^2} \sin \lambda (H - h) \]

\[ \phi' = 2 \exp(i\omega t) \int_0^\infty \frac{J_0(Kr)K}{\lambda_1 \Delta(K)} C^0 \Delta \times \left\{ \begin{array}{c}
\rho_1 \omega^2 j^2 \beta_2^2 \left[ 4K^2 \nu_1 \nu_2' + (2K^2 - K_0^2) \right] \\
+ (2K^2 - K_0^2)^2 \left[ \lambda_1 \cos \lambda_1 (H - z) - \frac{g}{2\alpha^2} \sin \lambda_1 (H - z) \right] \\
- \frac{g\omega^2}{4\lambda_1 \alpha^2} \sin \lambda_1 \cos \lambda_1 (H - z) \end{array} \right\} dK, \quad 0 \leq z \leq h \] (24)

\[ \phi = -2 \exp(i\omega t) \int_0^\infty \frac{J_0(Kr)K}{\lambda_1 \Delta(K)} C^0 \Delta \times \left\{ \begin{array}{c}
\rho_1 \omega^2 j^2 \beta_2^2 \left[ 4K^2 \nu_1 \nu_2' + (2K^2 - K_0^2) \right] \\
+ (2K^2 - K_0^2)^2 \left[ \lambda_1 \cos \lambda_1 (H - z) - \frac{g}{2\alpha^2} \sin \lambda_1 (H - z) \right] \\
- \frac{g\omega^2}{4\lambda_1 \alpha^2} \sin \lambda_1 \cos \lambda_1 (H - z) \end{array} \right\} dK, \quad h \leq z \leq H, \] (25)

\[ \phi'' = -4 \exp(i\omega t) \int_0^\infty \frac{J_0(Kr)K}{\lambda_1 \Delta(K)} C^0 \Delta \times \left\{ \begin{array}{c}
\rho_1 \omega^2 j^2 \beta_2^2 \left[ 4K^2 \nu_1 \nu_2' + (2K^2 - K_0^2) \right] \\
+ (2K^2 - K_0^2)^2 \left[ \lambda_1 \cos \lambda_1 (H - z) - \frac{g}{2\alpha^2} \sin \lambda_1 (H - z) \right] \\
- \frac{g\omega^2}{4\lambda_1 \alpha^2} \sin \lambda_1 \cos \lambda_1 (H - z) \end{array} \right\} dK, \] (26)

where

\[ \Delta = \left[ \lambda_1 \cos \lambda_1 H - \left( g/2\alpha^2 \right) \sin \lambda_1 H \right] \left[ 4K^2 \nu_1 \nu_2' + (2K^2 - K_0^2) \right] \rho_2 \beta_2^2 + \left[ \omega^2 \sin \lambda_1 H + g \left( \lambda_1 \cos \lambda_1 H - \left( g/2\alpha^2 \right) \sin \lambda_1 H \right) \right] \rho_1 \nu_2 \omega^2 / \beta_2^2 \] (28)

The integrals can be represented by the sum of branch line integrals and residues. The residues correspond to the pole \( K = K_n \) given by the roots of

\[ \Delta(K) = 0 \] (29)

4.2 Case (ii) – The waves are longer than ordinary earthquake Rayleigh waves

In this case we consider (6) for the liquid part as before. The equation of motion for the solid part is now also affected by the gravitational field. If \( g \) be the acceleration due to gravity then the components of body forces are \( X = 0, Z = g \). We shall assume that the initial stress due to gravity is hydrostatic in nature. The state of initial stress are (Biot 1965, pp 44–45)

\[ s_{11} = s_{33} = s, \quad s_{13} = 0, \] (30)

where \( s \) is a function of depth. The equilibrium conditions of the initial stress field are

\[ \frac{\partial s}{\partial x_1} = 0, \quad \frac{\partial s}{\partial x_3} + \rho_2 g = 0 \] (31)

The dynamical equations of the two-dimensional problem under the initial stress field are (Biot 1965, pp 273–281)
where
\[ s_{jk} = 2 \mu e_{jk} + \lambda e_{\delta_{jk}}, \]
\[ e_{jk} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1} \right) e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \]  

Equation (32) in cylindrical co-ordinates and in terms of potentials \( \phi_2 \) and \( \psi_2 \) may be written as
\[ \left( \nabla^2 - \frac{1}{\alpha_2^2} \frac{\partial^2}{\partial r^2} \right) \phi_2 + \left( \frac{g}{\alpha_2^2} \right) \left( \frac{\partial^2 \psi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_2}{\partial r} \right) = 0, \]  
and
\[ \left( \nabla^2 - \frac{1}{\beta_2^2} \frac{\partial^2}{\partial r^2} \right) \psi_2 - \left( \frac{g}{\beta_2^2} \right) \phi_2 = 0 \]  
Now we seek solutions for \( \phi_2 \) and \( \psi_2 \) in the form given by (18) and we obtain from (34) and (35)
\[ \phi_2 = \int_0^\infty \left[ Q'_2(K) \exp[-i\lambda'_2 z] + Q'_2(K) \exp[-i\lambda'_2 z] \right] J_0(Kr) dK, \]
\[ \psi_2 = \int_0^\infty \left[ a'_2 Q'_2(K) \exp[-i\lambda'_2 z] + a'_2 Q'_2(K) \exp[-i\lambda'_2 z] \right] J_0(Kr) dK, \]  
where
\[ \lambda'_2, \lambda''_2 = K^2 - \frac{1}{2} \left[ K^2_{\alpha_2} + K^2_{\beta_2} \pm \left( (K^2_{\alpha_2} - K^2_{\beta_2})^2 + \frac{4g^2K^2}{\alpha_2^2\beta_2^2} \right)^{1/2} \right], \]
\[ a'_2 = -\frac{\alpha_2^2}{-\frac{g^2}{K^2} + \lambda''_2 + K^2_{\alpha_2}}, \text{ and } a''_2 = -\frac{\alpha_2^2}{-\frac{g^2}{K^2} + \lambda''_2 + K^2_{\alpha_2}}. \]  
Substituting the values of \( \phi'_2, \psi_2 \) and \( \psi_2 \) from (25), (36) and (37) in the boundary condition we obtain as before
\[ \exp \left[ -\frac{gH}{2\alpha_1^2} \left[ \lambda_1 \cos \lambda_1 H - \left( \frac{g}{2\alpha_1^2} \right) \sin \lambda_1 H \right] B(K) + b'_2 Q'_2 \exp[-i\lambda'_2 H] \right. \]
\[ + b'_2 Q'_2 \exp[-i\lambda'_2 H] = \left( 2 \sin \lambda_1 h \left( \frac{1}{c_0} \right) \left[ \sin \lambda_1 H + \left( \frac{g}{2\lambda_1^2} \right) \cos \lambda_1 H \right] \right. \]
\[ \left. \times \exp \left[ \frac{g(h - H)}{2\alpha_1^2} \right], \right. \]
\[ c'_2 Q'_2 \exp[-i\lambda'_2 H] + c''_2 Q''_2 \exp[-i\lambda'_2 H] = 0 \]
Propagation of waves

\[
\exp \left[ \frac{-gH}{2\alpha_t^2} \right] \rho_1 \left[ \omega^2 \sin \lambda_1 H + g \left( \lambda_1 \cos \lambda_1 H - \left( \frac{g}{2\alpha_t^2} \right) \sin \lambda_1 H \right) \right]
\]

\[+ d'_2 \Delta^2 \exp[-i\lambda'_2 H] + d''_2 \Delta^2 \exp[-i\lambda''_2 H] = -2\rho_1 \frac{1}{\lambda_1} \frac{1}{c} \]

\[\times \left\{ \omega^2 \sin \lambda_1 H - g \left[ \lambda_1 \sin \lambda_1 H + \left( \frac{g}{2\alpha_t^2} \right) \cos \lambda_1 H \right] \right\} \sin \lambda_1 h \exp \left[ \frac{g(h - H)}{2\alpha_t^2} \right]. \quad (41)\]

where

\[b'_2 = i\lambda'_{2} + d'_2 (\lambda'_{2}^2 - K_{\beta_{2}}^2), \quad b''_2 = i\lambda''_{2} + d''_2 (\lambda''_{2}^2 - K_{\beta_{2}}^2), \]

\[c'_2 = 2i\lambda'_{2} + d'_2 (2\lambda'_{2}^2 - K_{\beta_{2}}^2), \quad c''_2 = 2i\lambda''_{2} + d''_2 (2\lambda''_{2}^2 - K_{\beta_{2}}^2), \quad (42)\]

\[d'_2 = -\lambda'_{2}K^2 - \rho_2\alpha^2_{2}\lambda'^2_2 + 2\mu_2\alpha^2_{2}\lambda''_2 [\lambda'^2_2 - K_{\beta_{2}}^2] \quad \text{and} \quad d''_2 = -\lambda''_{2}K^2 - \rho_2\alpha^2_{2}\lambda''^2_2 + 2\mu_2\alpha^2_{2}\lambda''_2 [\lambda''^2_2 - K_{\beta_{2}}^2]. \]

Solving (39) to (41) we obtain the values of \(B, Q'_2\) and \(Q''_2\). Using these values, we obtain from (13), (14), (36) and (37) the following expressions

\[\phi'_2 = 2 \exp[\omega t] \int_0^\infty \exp[g(h - H)/(2\alpha_t^2)] \Delta^0 \exp \left[ \frac{-g(z)}{2\alpha_t^2} \right] J_0(Kr) dK, \quad (43)\]

\[\phi''_1 = 2 \exp[\omega t] \int_0^\infty \exp[g(h - H)/(2\alpha_t^2)] \Delta^0 \exp \left[ \frac{-g(z)}{2\alpha_t^2} \right] J_0(Kr) dK, \quad (44)\]

\[\phi_2 = -2 \exp[\omega t] \int_0^\infty J_0(Kr) \Delta^0 \exp \left[ \frac{g(h - H)}{2\alpha_t^2} \right] \rho_1 \omega^2 \left[ c'_2 \exp[-i\lambda'_2 z] - c''_2 \exp[-i\lambda''_2 z] \right] dK, \quad (45)\]

\[\psi_2 = -2 \exp[\omega t] \int_0^\infty J_0(Kr) \Delta^0 \exp \left[ \frac{g(h - H)}{2\alpha_t^2} \right] \rho_1 \omega^2 \left[ c'_2 d'_2 \exp[-i\lambda'_2 z] - c''_2 d''_2 \exp[-i\lambda''_2 z] \right] dK, \quad (46)\]
where

$$\Delta' = [\bar{\lambda}_1 \cos \bar{\lambda}_1 H - (g/2\alpha_1^2) \sin \bar{\lambda}_1 H][c'_2 d'_2 - d'_2 c'_2]$$
$$\quad + \bar{\rho}_1 [\omega^2 \sin \bar{\lambda}_1 H + g(\bar{\lambda}_1 \cos \bar{\lambda}_1 H - (g/2\alpha_1^2) \sin \bar{\lambda}_1 H)][b'_2 c'_2 - c'_2 b'_2]$$  \hspace{1cm} (47)

Again the integrals (43) to (46) can be expressed as the sum of branch line integrals and residues. The residues correspond to the pole $K = K_n$ given by the roots of the equation $\Delta'(K) = 0$. Now the normal mode of solution for the expressions given by (24) to (27) may be written in the final form after expanding $C_n^{-1}$ and retaining only the first two terms,

$$\phi_2 = \frac{2}{\pi} \left( \frac{2\pi}{r} \right)^{1/2} \sum_n \frac{1}{(K_n)^{1/2}} \exp \left[ i \left( \omega t - K_n r - \frac{\pi}{4} \right) \right] \Theta_2(K_n) \sin$$
$$\times \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} + 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \exp \left[ -K_n (\zeta - H) \left( 1 - \frac{c^2}{\alpha_1^2} \right)^{1/2} \right]$$  \hspace{1cm} (48)

$$\psi_2 = \frac{2}{\pi} \left( \frac{2\pi}{r} \right)^{1/2} \sum_n \frac{1}{(K_n)^{1/2}} \exp \left[ i \left( \omega t - K_n r - \frac{\pi}{4} \right) \right] \Xi_2(K_n) \sin$$
$$\times \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} + 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \exp \left[ -K_n (\zeta - H) \left( 1 - \frac{c^2}{\beta_2^2} \right)^{1/2} \right]$$  \hspace{1cm} (49)

where

$$\Theta_2(K_n) = -\frac{\rho_1}{\rho_2} \frac{c^2}{\beta_2^2} \exp \left[ \frac{g(h-H)}{2\alpha_1^2} \right] \left( \frac{2 - c^2/\beta_2^2}{c^2/\alpha_1^2 - 1 - g^2/4\alpha_1^4 K_n^2} \right) \left( \frac{1}{M} \right)$$
$$\times \left[ 1 + \frac{g/\alpha_1^2}{(c^2/\alpha_1^2 - 1 - g^2/4\alpha_1^4 K_n^2)^{1/2}} \sin \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} + 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \right] \cos \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} + 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \right\} ,$$  \hspace{1cm} (50)

$$\Xi_2(K_n) = -2H \frac{\rho_1}{\rho_2} \frac{c^2}{\beta_2^2} \exp \left[ \frac{g(h-H)}{2\alpha_1^2} \right] \left( \frac{1 - c^2/\beta_2^2}{c^2/\alpha_1^2 - 1 - g^2/4\alpha_1^4 K_n^2} \right) \left( \frac{1}{M} \right)$$
$$\times \left[ 1 + \frac{g/\alpha_1^2}{(c^2/\alpha_1^2 - 1 - g^2/4\alpha_1^4 K_n^2)^{1/2}} \sin \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} + 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \right] \cos \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} + 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \right\} ,$$  \hspace{1cm} (51)

and

$$\phi_1 = \phi''_1 = \frac{2}{H} \left( \frac{2\pi}{r} \right)^{1/2} \sum_n \frac{1}{(K_n)^{1/2}} \exp \left[ i \left( \omega t - K_n r - \frac{\pi}{4} \right) \right] \theta_1(K_n)$$
$$\times \sin \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} \sin \left\{ K_n h \left( \frac{c^2}{\alpha_1^2} + 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right\} ,$$  \hspace{1cm} (52)
where

\[ \theta_1(K_n) = \frac{\rho_1}{\rho_2} \cdot \frac{c^4}{\beta_2^5} \cdot \frac{K_n H (1 - c^2/\alpha_2^2)^{1/2}}{(c^2/\alpha_1^2 - 1 - g^2/4\alpha_1^4 K_n^2)} \times \sin \left( K_n H \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right) \div \cos \left( K_n H \left( \frac{c^2}{\alpha_1^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right)^{1/2} \right) \],

\[ M = \frac{\rho_1}{\rho_2} \frac{c^4}{\beta_2^5} \left[ \sin(K_n H \tilde{\lambda}_1^0) - \frac{(1 - c^2/\alpha_2^2)^{1/2} \cos(K_n H \tilde{\lambda}_1^0)}{\tilde{\lambda}_1^0} \right]
+ \frac{g}{\omega^2} \left( \frac{1 - c^2}{\alpha_2^2} \right)^{1/2} \left\{ \cos(K_n H \tilde{\lambda}_1^0) - K_n H \sin(K_n H \tilde{\lambda}_1^0) - \frac{\tilde{\lambda}_1^0 K_n^2}{(1 - c^2/\alpha_2^2)^{1/2}} \right\}
- 4 \left\{ \left( \frac{1 - c^2}{\alpha_1^2} \right)^{1/2} \left( \frac{1 - c^2}{\beta_2^4} \right)^{1/2} + \left( \frac{1 - c^2}{\alpha_2^2} \right)^{1/2} \right\}
- 2 \left( \frac{1 - c^2}{\beta_2^4} \right)^{1/2} \left( \frac{1 - c^2}{\beta_2^4} \right)^{1/2}\cos(K_n H \tilde{\lambda}_1^0) - \frac{g}{2\alpha_2^4 K_n} \sin(K_n H \tilde{\lambda}_1^0) \right\}
+ \left\{ \left( \frac{1 - c^2}{\alpha_2^2} \right)^{1/2} \left( \frac{1 - c^2}{\beta_2^4} \right)^{1/2} - 1 \right\} \left( \frac{1 - c^2}{\beta_2^4} \right)^{1/2}
\times \left( K_n H \sin(K_n H \tilde{\lambda}_1^0) + \frac{\cos(K_n H \tilde{\lambda}_1^0)}{\tilde{\lambda}_1^0} \right) \left( \frac{g}{2\alpha_2^4} - 1 \right) \right] . \]

The period equation (29) may now be written as

\[ \sin(\tilde{\lambda}_1 H + (g/\omega^2) \cos(\tilde{\lambda}_1 H - (g^2/2\alpha_2^2 \alpha_1^2) \sin(\tilde{\lambda}_1 H)) = \frac{\rho_1 \beta_2^4}{\rho_2 c^4} \cos(\tilde{\lambda}_1 H - (g/2\alpha_2^2 \alpha_1^2) \sin(\tilde{\lambda}_1 H)
\times \left( \frac{c^2/\alpha_1^2 - 1 - g^2/4\alpha_1^4 K_n^2}{(1 - c^2/\alpha_2^2)^{1/2}} \right) \left[ 4 \left( \frac{1 - c^2}{\alpha_2^2} \right)^{1/2} \left( \frac{1 - c^2}{\beta_2^4} \right)^{1/2} - (2 - \frac{c^2}{\beta_2^4})^2 \right] . \]

In all the above cases as \( g \to 0 \), the results obtained are in agreement with the corresponding results of the classical problem. Similar solutions may be written for the expressions \( \phi_2, \psi_2, \phi'_1 \) and \( \phi''_1 \) in (43) to (46).

5. Numerical calculation

Equation (29), namely \( \Delta (K) = 0 \), where \( \Delta (K) \) is defined in (28) may be written using (15) and (19) and approximating \( \cos x \equiv 1 \) and \( \sin x \equiv x \) (i.e. considering the arguments of sine
and cosine are small) as
\[
\frac{\rho_2}{\rho_1} \left[ 1 - \frac{gH}{2\alpha_1^2} \right] \left[ \left( 2 - \frac{c^2}{\beta_2^2} \right)^2 - 4 \left( 1 - \frac{c^2}{\alpha_2^2} \right)^{1/2} \left( 1 - \frac{c^2}{\beta_2^2} \right)^{1/2} \right] \\
= \left( 1 - \frac{c^2}{\alpha_2^2} \right)^{1/2} \frac{c^2}{\beta_2^2} \frac{KH}{\rho_2} + \frac{g}{\beta_2^2} \left( 1 - \frac{gH^2}{2\alpha_1^2} \right),
\]
(57)
in which \( c = \omega/K \) is the phase velocity. Now for the sedimentary ocean floor (Ewing et al 1957, p. 176)
\[
\rho_2 = 2\rho_1, \quad \alpha_2 = \sqrt{3}\beta_2, \quad \beta_2 = 15\alpha_1,
\]
and since
\[
\frac{gH^2}{2\alpha_1^2} = \frac{gH}{2\beta_1^2} \frac{\beta_2^4}{\alpha_1^2} = 1125GKH,
\]
(59)
where \( G = g/\beta_2^2K \) is the gravity parameter.
Using (58) and (59), (57) may be written as
\[
2(1 - 125GKH) \left[ \left( 2 - \frac{c^2}{\beta_2^2} \right)^2 - 4 \left( 1 - \frac{c^2}{\alpha_2^2} \right)^{1/2} \left( 1 - \frac{c^2}{\beta_2^2} \right)^{1/2} \right] \\
= \left( 1 - \frac{c^2}{\alpha_2^2} \right)^{1/2} \frac{c^2}{\beta_2^2} \frac{KH}{\rho_2} + G(1 - 125GKH)
\]
(60)
Equation (60) is now dimensionless in \( c^2/\beta_2^2 \) for a particular value of gravity parameter \( G \) and \( KH \) and from which a real root of \( c^2/\beta_2^2 \) may be evaluated for different values of \( G \) and \( KH \). Similar calculations may be drawn in case of granitic and basaltic ocean floors.

6. Discussion

In both cases there is dispersion of waves due to gravity. If \( g \to 0 \) in case (1) we obtain \( \lambda_1 \to \nu_1 \) and the result thus obtained is in good agreement with the classical problem as studied by Ewing et al (1957, p 261) Also if \( \rho_1 \to 0 \) and \( g \to 0 \) in \( \Delta(K) = 0 \), we easily obtain the equation of Rayleigh waves in classical elasto-kinetics as
\[
\left( 2 - \frac{c^2}{\beta_2^2} \right)^2 = 4 \left( \frac{c^2}{\alpha_2^2} - 1 \right)^{1/2} \left( \frac{c^2}{\beta_2^2} - 1 \right)^{1/2}.
\]
(61)
Now if \( \rho_1 \to 0 \) in \( \Delta'(K) = 0 \) as given in (47) we easily obtain the Rayleigh surface waves under the influence of gravity as
\[
c_2' d_{2'}^2 - d_2' c_2'^2 = 0,
\]
(62)
where \( c_2', d_2', c_{2''} \) and \( d_2'' \) have been defined in (42). Also the changes in the above potentials produced by a varying depth \( z \) are represented by the factor \( \sin \left\{ K_n z ((c^2/\alpha_2^2) - 1 - \left( g^2/4\alpha_1^2 K_n^2 \right)^{1/2}) \right\} \) in (52) and the factor \( \sin \left\{ K_n h ((c^2/\alpha_2^2) - 1 - \left( g^2/4\alpha_1^2 K_n^2 \right)^{1/2}) \right\} \) depends on the depth of the source. From (29) and (60) and table 1 we may conclude that the phase
velocity $c$ increases with increasing value of the gravity parameter $G$ for a particular value of $KH$. It is to be noted that this investigation may rise into importance when the degree of accuracy required is considerable.

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Surface waves in fibre-reinforced anisotropic elastic media

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Abstract. The aim of this paper is to investigate surface waves in anisotropic fibre-reinforced solid elastic media. First, the theory of general surface waves has been derived and applied to study the particular cases of surface waves – Rayleigh, Love and Stoneley types. The wave velocity equations are found to be in agreement with the corresponding classical result when the anisotropic elastic parameters tend to zero. It is important to note that the Rayleigh type of wave velocity in the fibre-reinforced elastic medium increases to a considerable amount in comparison with the Rayleigh wave velocity in isotropic materials.

Keywords. Fibre-reinforced medium, surface waves, Rayleigh waves; Love waves; Stoneley waves.

1. Introduction

Surface waves have been well recognized in the study of earthquake waves, seismology, geophysics and geodynamics. A good amount of literature is to be found in the standard books of Bullen (1965), Ewing \textit{et al} (1957), Rayleigh (1885), Love (1911), Stoneley (1924) and Jeffreys (1959), regarding surface waves in classical elasticity. Sengupta and his research collaborators have also studied surface waves (Acharya & Sengupta 1978; Pal & Sengupta 1987; Mukherjee & Sengupta 1991; Das & Sengupta 1992; Das \textit{et al} 1994).

In most previous investigations, the effect of reinforcement has been neglected. The idea of continuous self-reinforcement at every point of an elastic solid was introduced by Belfield \textit{et al} (1983). The characteristic property of a reinforced concrete member is that its components, namely concrete and steel, act together as a single anisotropic unit as long as they remain in the elastic condition, i.e., the two components are bound together so that there can be no relative displacement between them.

In this paper the authors study the propagation of surface waves in fibre-reinforced anisotropic elastic solid media leading to particular cases such as Rayleigh waves, Love waves and Stoneley waves along with numerical results. The results reduce to corresponding classical results when the reinforced elastic parameters tend to zero and the medium becomes isotropic.
2. Formulation of the problem

The constitutive equations for a fibre-reinforced linearly elastic anisotropic medium with respect to a preferred direction \( \alpha \) are (Belfield et al 1983)

\[
\tau_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu_T \epsilon_{ij} + \alpha (a_k a_m \epsilon_{km} \delta_{ij} + \epsilon_{kk} a_i a_j) + 2(\mu_L - \mu_T) (a_i a_k \epsilon_{kj} + a_j a_k \epsilon_{ki}) + \beta (a_k a_m \epsilon_{km} \epsilon_{ij}) ,
\]

where \( \tau_{ij} \) are components of stress, \( \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \) are components of strain, \( \lambda, \mu_T \) are elastic parameters; \( \alpha, \beta, (\mu_L - \mu_T) \) are reinforced anisotropic elastic parameters; \( u_i \) are the displacement vectors components and \( \tilde{\alpha} = (a_1, a_2, a_3) \), where \( a_1^2 + a_2^2 + a_3^2 = 1 \). If \( \tilde{\alpha} \) has components that are \( (1, 0, 0) \) so that the preferred direction is the \( x_1 \) axis, \( \lambda \) simplifies, as given below.

\[
\begin{align*}
\tau_{11} &= (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta) \epsilon_{11} + (\lambda + \alpha) \epsilon_{22} + (\lambda + \alpha) \epsilon_{33}, \\
\tau_{12} &= (\lambda + \alpha) \epsilon_{11} + (\lambda + \mu_T) \epsilon_{22} + \lambda \epsilon_{33}, \\
\tau_{13} &= 2\mu_T \epsilon_{23}, \\
\tau_{12} &= 2\mu_T \epsilon_{12},
\end{align*}
\]

(2)

The equations of motion in absence of body forces are

\[
\begin{align*}
\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} &= \rho \frac{\partial^2 u_1}{\partial t^2}, \\
\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} &= \rho \frac{\partial^2 u_2}{\partial t^2}, \\
\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2},
\end{align*}
\]

(3)–(5)

where \( \rho \) is the density of the elastic medium. Using (2)–(5) and assuming all derivatives with respect to \( x_3 \) vanish, the equations of motion become

\[
\begin{align*}
(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta) \frac{\partial^2 u_1}{\partial x_1^2} + (\alpha + \lambda + \mu_L) \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \mu_T \frac{\partial^2 u_1}{\partial x_1^2} + \mu_T \frac{\partial^2 u_1}{\partial x_2^2} &= \rho \frac{\partial^2 u_1}{\partial t^2}, \\
\mu_L \frac{\partial^2 u_2}{\partial x_1^2} + (\alpha + \lambda + \mu_L) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + (\lambda + 2\mu_T) \frac{\partial^2 u_2}{\partial x_2^2} &= \rho \frac{\partial^2 u_2}{\partial t^2}, \\
(\mu_L - \mu_T) \frac{\partial^2 u_3}{\partial x_1^2} + \mu_T \left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) u_3 &= \rho \frac{\partial^2 u_3}{\partial t^2}
\end{align*}
\]

(6)–(8)

To examine dilatational and rotational disturbances, we introduce two displacement potentials \( \phi \) and \( \psi \) by the relations

\[
u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \quad u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}
\]

(9)

The component \( u_3 \) is associated with purely distortional movement. We note that \( \phi, \psi \) and \( u_3 \) are respectively associated with P waves, SV waves and SH waves. The symbols have their usual significances.
3. General theory and boundary conditions

The propagation of general surface waves is examined here for a fibre-reinforced elastic solid semi-infinite medium $M$ covered by another fibre-reinforced elastic medium $M_1$ ($M_1$ above $M$ and mechanical properties different from $M$ and which is welded in contact with $M$ to prevent any relative motion or sliding during the disturbance). We consider an orthogonal Cartesian co-ordinate system $\alpha x_1 x_2 x_3$ with the origin $O$ at the common plane boundary surface and $\alpha x_2$ directed normally into $M$. We consider the possibility of a wave travelling in the direction $\alpha x_1$ in such a manner that (a) the disturbance is largely confined to the neighbourhood of the boundary and (b) at any instant all particles in any line parallel to $\alpha x_3$ have equal displacements. On account of (a) the wave is a surface wave and on account of (b) all the partial derivatives with respect to $x_3$ vanish.

Now using (9) in (6) we obtain the following wave equation in $M$ satisfied by $\phi$ and $\psi$ as

$$
\begin{aligned}
(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta) \frac{\partial^2 \phi}{\partial x_1^2} + (\alpha + \lambda + 2\mu_L) \frac{\partial^2 \phi}{\partial x_2^2} &= \rho \frac{\partial^2 \phi}{\partial t^2}, \\
(\alpha + 3\mu_L + \beta - 2\mu_T) \frac{\partial^2 \psi}{\partial x_1^2} + \mu_L \frac{\partial^2 \psi}{\partial x_2^2} &= \rho \frac{\partial^2 \psi}{\partial t^2},
\end{aligned}
$$

and similar relations in $M_1$ with $\rho, \lambda, \alpha, \mu_L, \beta$ replaced by $\rho_1, \lambda_1, \alpha_1, \mu_{L1}, \beta_1$. The general solutions for $\phi$ and $\psi$ must satisfy (7).

3.1 Boundary conditions

The boundary conditions for the titled problem are

(i) the component of displacement at the boundary surface between the medium $M$ and $M_1$ must be continuous at all times and places,

(ii) The stress components $\tau_{21}, \tau_{22}$ and $\tau_{23}$ must be continuous across the interface of $M$ and $M_1$ at all times and places,

where $\tau_{21}, \tau_{22}$ and $\tau_{23}$ can be written in terms of $\phi$ and $\psi$ in the medium $M$ with the help of (2) and (9) as

$$
\begin{aligned}
\tau_{21} &= \mu_L \left( 2 \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_2^2} \right), \\
\tau_{22} &= \lambda \nabla^2 \phi + \alpha \left( \frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right) + 2\mu_T \left( \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right), \\
\tau_{23} &= \mu_T \frac{\partial \mu_3}{\partial x_2},
\end{aligned}
$$

where $\nabla^2$ is the two dimensional Laplacian operator given by

$$
\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}
$$

Similar relations in $M_1$ with $\mu_L, \lambda, \alpha, \mu_T$ are replaced by $\mu_{L1}, \lambda_1, \alpha_1, \mu_{T1}$.
4. Solution of the problem

We seek harmonic solutions for (8), (10) and (11) in the form (Bullen 1965),

\[ \phi, \psi, u_3 = \{ \tilde{\phi}(x_2), \tilde{\psi}(x_2), \tilde{u}_3(x_2) \} \exp \{ i \omega (x_1 - ct) \}, \]  

(13)

in \( M \) and similar relations in \( M_1 \) with the functions \( \phi, \psi, u_3 \) being replaced by \( \phi_1, \psi_1, u_1 \). This leads us to a particular solution corresponding to a group of simple harmonic waves of wavelength \( 2\pi/\omega \) travelling forward with speed \( c \).

It is convenient to introduce \( h, r, s \) where

\[
\begin{align*}
    h &= \left( \frac{pc^2 - \mu_L}{\mu_T} \right)^{1/2}, \\
    r &= \left( \frac{pc^2 - (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)}{\alpha + \lambda + 2\mu_T} \right)^{1/2}, \\
    s &= \left( \frac{pc^2 - (\lambda + 3\mu_L + \beta - 2\mu_T)}{\mu_L} \right)^{1/2},
\end{align*}
\]

(14)

and similar expressions \( h_1, r_1, s_1 \) for the medium \( M_1 \). The positive value of the square root being taken in each case.

Now substituting from (13) into (8), (10) and (11) we obtain for the medium \( M \)

\[
\begin{align*}
    u_3 &= C \exp \{ i \omega (-hx_2 + x_1 - ct) \}, \\
    \phi &= A \exp \{ i \omega (-rx_2 + x_1 - ct) \}, \\
    \psi &= B \exp \{ i \omega (-sx_2 + x_1 - ct) \},
\end{align*}
\]

(15)

and for the medium \( M_1 \)

\[
\begin{align*}
    u_1^1 &= C_1 \exp \{ i \omega (h_1x_2 + x_1 - ct) \}, \\
    \phi_1 &= A_1 \exp \{ i \omega (r_1x_2 + x_1 - ct) \}, \\
    \psi_1 &= B_1 \exp \{ i \omega (s_1x_2 + x_1 - ct) \}.
\end{align*}
\]

(16)

In the above, for the effect to be essentially a surface one, each expression must diminish indefinitely with increasing distance from the boundary. This will be the case if each expression contains an exponential factor in which the exponent is real and negative. Hence \( h, r, s \) and similarly \( h_1, r_1, s_1 \) are taken to be imaginary.

Using (15) and (16) in the boundary conditions (i) and (ii) given in §(3 1) we obtain,

\[
\begin{align*}
    A + sB &= A_1 - s_1B_1, \\
    -Ar + B &= A_1r_1 + B_1, \\
    C &= C_1, \\
    \mu_L[2rA + (s^2 - 1)B] &= \mu_L[-2r_1A_1 + (s_1^2 - 1)B_1],
\end{align*}
\]

(17)

(18)

(19)

(20)

\[
\begin{align*}
    [(\lambda + \alpha) + r^2(\lambda + 2\mu_T)]\lambda - (2\mu_T - \alpha)sB = & [(\lambda_1 + \alpha_1) + r_1^2(\lambda_1 + 2\mu_{T_1})A_1 \\
    & + (2\mu_{T_1} - \alpha_1)s_1B_1,
\end{align*}
\]

(21)
It follows from (19) and (22) that both $C$ and $C_1$ vanish, thus there is no propagation of displacement $u_3$. Finally we get the wave velocity equation in the common boundary of the media $M$ and $M_1$ by eliminating the constants $A, B, A_1$ and $B_1$ from the equations (17), (18), (20) and (21) as

\[
\begin{vmatrix}
1 & s & -1 & s_1 \\
-r_r & 1 & -r_1 & -1 \\
2\mu_L r & \mu_L (s^2 - 1) & 2\mu_2\mu_{L1} & -(s_1^2 - 1)\mu_{L1} \\
[(\lambda + \alpha) + r^2(\lambda + 2\mu_T)] & -(2\mu_T - \alpha)s & -[(\lambda_1 + \alpha_1) + r_1^2(\lambda_1 + 2\mu_{T1})] & -(2\mu_{T1} - \alpha_1)s_1 \\
\end{vmatrix} = 0.
\]

(23)

5. Particular cases

5.1 Rayleigh waves

The particular case of $M_1$ replaced by vacuum was first examined by Rayleigh (1885). The absence of stress over the free surface enables us to replace the right-hand side of (20) and (21) by zero, giving

\[
2rA + (s^2 - 1)B = 0,
\]

(24)

\[
[(\lambda + \alpha) + r^2(\lambda + 2\mu_T)]A - (2\mu_T - \alpha)s = 0
\]

(25)

Eliminating $A$ and $B$ from (24) and (25) we obtain the Rayleigh type of waves in the fibre-reinforced elastic medium as

\[
(1 - s_1^2)[(\lambda + \alpha) + r^2(\lambda + 2\mu_T)] = 2rs(2\mu_T - \alpha),
\]

(26)

where $r$ and $s$ have been defined in (14).

Now writing $\mu_L = \mu_L - \mu_T + \mu_T$ and making $\alpha$, $\beta$ and $|\alpha_L - \mu_T|$ all tend to zero, (26) reduces to the following form,

\[
\left(2 - \frac{\rho c^2}{\mu_T}\right)^2 = 4 \left(1 - \frac{\rho c^2}{\lambda + 2\mu_T}\right)^{1/2} \left(1 - \frac{\rho c^2}{\mu_T}\right)^{1/2},
\]

(27)

which is the Rayleigh surface wave in isotropic materials

5.1a Numerical calculations for Rayleigh waves

The following values of elastic constants and density are considered (Chattopadhyay 1998)

\[
\begin{align*}
\lambda & = 5.65 \times 10^9 \text{ Nm}^{-2}, & \mu_L & = 5.66 \times 10^9 \text{ Nm}^{-2}, \\
\mu_T & = 2.46 \times 10^9 \text{ Nm}^{-2}, & \alpha & = -1.28 \times 10^9 \text{ Nm}^{-2}, \\
\beta & = 220.90 \times 10^9 \text{ Nm}^{-2}, & \rho & = 7800 \text{ kg m}^{-3}
\end{align*}
\]

Using (26) and the expressions given in (14) we obtain the following value of the wave velocity $c$ as.

\[
\frac{\rho c^2}{\mu_L} \text{ (dimensionless)} = 40.817316,
\]
The Rayleigh wave propagates very rapidly in fibre-reinforced elastic media according to this theory.

### 5.2 Love waves

For the existence of Love waves we consider a layered semi-infinite medium, in which \( M_1 \) is obtained by two horizontal plane surfaces, a finite distance \( H \) apart, and the medium \( M \) remains as before.

Now we investigate the displacement \( u_3 \) in the direction of \( x_2 \)-axis. For the medium \( M \) the solution for the displacement component \( u_3 \) remains the same but for the medium \( M_1 \) we preserve the full solution, since the displacement component \( u_3 \) no longer diminishes with increasing distance from the boundary surface of two media. Hence,

\[
u_3 = C_1 \exp[i\omega(h_1 x_2 + x_1 - ct)] + D_1 \exp[i\omega(-h_1 x_2 + x_1 - ct)], \tag{28}
\]

where \( h_1 \) is now not necessarily imaginary. For \( M \) we still have \( h \) imaginary. In the present case the boundary conditions are:

(i) \( u_3 \) and \( \tau_{33} \) are continuous at \( x_2 = 0 \)

(ii) \( \tau_{23} = 0 \) at \( x_2 = -H \)

Now using (12), (15) and (28) in the above boundary conditions we obtain

\[
C - C_1 - D_1 = 0, \tag{29}
\]

\[
-\mu_T h C - \mu_T h_1 C_1 + \mu_T h_1 D_1 = 0, \tag{30}
\]

\[
\exp[-i\omega H h_1] C_1 - \exp[i\omega H h_1] D_1 = 0. \tag{31}
\]

Eliminating \( C, C_1 \) and \( D_1 \) from the above equations we obtain

\[
\mu_T h_1 \tan(\omega H h_1) + i\mu_T h = 0 \tag{32}
\]

Substituting for \( h \) and \( h_1 \) from (14) into (32) gives the equation for the velocity \( c \) of Love waves, namely

\[
\mu_T \left( \frac{\mu_L - \rho c^2}{\mu_T} \right)^{1/2} - \mu_T \left( \frac{\rho_L c^2 - \mu_{L1}}{\mu_{T1}} \right)^{1/2} \tan \left[ \omega H \left( \frac{\rho_L c^2 - \mu_{L1}}{\mu_{T1}} \right)^{1/2} \right] = 0 \tag{33}
\]

The requirement that \( h \) should be imaginary and hence that by (32) \( h_1 \) real is, by (14), satisfied if

\[
(\mu_{L1}/\rho_1)^{1/2} < c < (\mu_L/\rho)^{1/2} \tag{34}
\]

Equation (33) shows that \( c \) is dependent on the particular value of \( \omega \) and not a fixed constant so that in the present boundary conditions there is dispersion of the general wave form. We see from (33) that if \( \omega \) is small, \( c \to (\mu_L/\rho)^{1/2} \), while if \( \omega \) is large, \( c \to (\mu_{L1}/\rho_1)^{1/2} \).
5 2a Numerical calculations for Love waves. The upper limit for the wave velocity \( c \) given in the inequality (34) for the existence of propagation of Love waves in different elastic solid medium is given below.

Fibre reinforced medium \((\mu_L/\rho)^{1/2}\) (km/s) = 0.851;
Lead \((\mu/\rho)^{1/2}\) (km/s) = 0.69836,
Copper \((\mu/\rho)^{1/2}\) (km/s) = 2.08,
Iron \((\mu/\rho)^{1/2}\) (km/s) = 3.10,
Earth crust (km/s) = 3.2-3.6.

5 3 Stoneley waves

Stoneley investigated a type of surface waves (Stoneley 1924) which are the generalised form of Rayleigh waves propagating at the common boundary of M and M1. The Stoneley waves in fibre-reinforced elastic media along the common boundary of M and M1 are, in fact, the same as the general discussion of surface waves presented at the starting and as such the wave velocity is determined by the root of the frequency equation (23). This equation of course reduces once more to the classical result when the parameters for the fibre-reinforced medium tend to zero.

6. Discussion

It is clear from the above investigation that the surface waves in the fibre-reinforced medium are affected by the reinforced parameters. In particular, the condition for the existence of propagation of Love waves, given by (34), depends upon the reinforced parameter \( \mu_L \) and \( \mu_{L1} \). Also all the results reduce to the classical isotropic results when the anisotropic parameters for the fibre-reinforced medium tend to zero (if necessary writing \( \mu_L = \mu_{LT} \) and considering \( |\mu_L - \mu_T| \rightarrow 0 \)).

From §5 1a, we conclude that the Rayleigh wave velocity in a fibre-reinforced elastic medium is considerably higher than the Rayleigh wave velocity in isotropic media. In this connection, terrestrial Rayleigh wave speed is about 3 km/s (Love 1911, p. 160). From §5 2a, the value of the upper limit of \( c \) given in (34) for the stable propagation of Love waves in the fibre-reinforced medium decreases in comparison with the upper limits in other elastic media except lead.

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The equation of motion in the liquid medium are
\[
\nabla^2 \varphi = \frac{1}{\alpha_1^2} \frac{\partial^2 \varphi}{\partial t^2},
\]
where \( \alpha_1 = \lambda_1 / \rho \) is the velocity of compressional waves in the liquid.

**General Theory and Boundary Condition**

We consider a liquid layer of thickness \( N \) overlying a visco-elastic semispace, the surface of separation being \( z = H \) and \( z \) axis being vertically downwards [the source being present in the viscoelastic medium at the point \((0, 0, H+d = h)\)]. We use here the subscript (1) for the liquid part and subscript (2) for the solid part.

The boundary conditions for the titled problem are
\[
\varphi_1 = 0 \quad \text{at} \quad z=0
\]
(3.1)

Since no tangential stress acts in a perfect liquid and at \( z = H \)
\[
\omega_1 = \omega_2, \quad \begin{pmatrix} \nabla \nabla \varphi_1 \\ \varphi_1 \end{pmatrix} \approx 0, \quad \begin{pmatrix} \nabla \nabla \varphi_2 \\ \varphi_2 \end{pmatrix} \approx \begin{pmatrix} \nabla \nabla \varphi_1 \\ \varphi_1 \end{pmatrix}
\]
(3.2)

where
\[
\omega_1 = \partial \varphi_1 / \partial z, \quad \begin{pmatrix} \nabla \nabla \varphi_1 \\ \varphi_1 \end{pmatrix} = \lambda_1 \nabla^2 \varphi_1 = \frac{\lambda_1}{\alpha_1^2} \frac{\partial \varphi_1}{\partial t^2}
\]

**Solution of the Problem**

In order to satisfy (3.1) the solution of (2.7) can be written as in Ewing et al (1957, pp. 158)
\[
\varphi_1 = \int_A(k) J_0(\kappa r) \sin \nu_1 z dk, \quad \text{for} \quad 0 < Z < H
\]
where \( \nu_1 = k_{a1}^2 - k^2 \quad k_{a1} = \frac{\omega}{\alpha_1} \)

and a time factor \( \exp(\text{i}\omega t) \) is understood in the above expression. As the source in the solid can produce both compressional and distortional waves and the primary disturbance \( \varphi_{20} \) as created by the spherical waves from the source is

\[
\varphi_{20} = \int_{0}^{\infty} \frac{k}{\nu_2} \cdot e^{-\text{i}\omega z} J_0(kr) \, dk
\]

(4.2)

\( \varphi_2 \) and \( \psi_2 \) can be written as

\[
\varphi_2 = \varphi_{20} + \int_{0}^{\infty} Q_2(k) J_0(kr) e^{-\text{i}\nu_2 z} \, dk
\]

(4.3)

\[
\psi_2 = \int_{0}^{\infty} Q_2(k) J_0(kr) e^{-\text{i}\nu_2 z} \, dk
\]

(4.4)

where

\[
\nu_2^* = k_{a2}^2 - k^2 \\
\nu_2^{*2} = k_{a2}^2 - k^2 \\
k_{a2}^* = \frac{\omega^2}{D_T^*} \\
k_{a2}^{*2} = \frac{\omega^2}{D_S^*}
\]

\[
D_T^* = \sum_{k=0}^{n} (i\omega)^k \nu_{kT}^2 \\
D_S^* = \sum_{k=0}^{n} (i\omega)^k \nu_{KS}^2
\]

(4.5)

If \( z < H + d \) we have

\[
\varphi_2 = \int_{0}^{\infty} \frac{k}{\nu_2^*} e^{\text{i}\nu_2^*(z-H-d)} J_0(kr) \, dk + \int_{0}^{\infty} Q_2(k) J_0(kr) e^{-\text{i}\nu_2^* z} \, dk
\]

(4.6)
\[\psi_2 = \int_0^\infty S_2(k)J_0(kr)e^{-iv_2^*z}dk \] (4.7)

Using (4.6) and (4.7) in the boundary condition (3.2) we get, on simplification

\[\bar{v}_1A \cos \bar{v}_1H + iv_2^* Q_2e^{-iv_2^*H} - k^2 Se^{-iv_2^*H} = ke^{-iv_2^*d} \]

\[2iv_2^* e^{-iv_2^*H} + (v_2^* - k^2)e^{-iv_2^*H} = 2ke^{-iv_2^*d} \]

\[\rho_1\omega^2 A \sin \bar{v}_1H + (2k^2 \mu_k^* - \rho_2\omega^2 \eta_k^*)Q_2e^{-iv_2^*H} - 2k^2 \mu_k^* iv_2^* S_2e^{-iv_2^*H} \]

\[= -\left(\frac{2k^2 \mu_k^* - \rho_2\omega^2 \eta_k^*}{iv_2^*}\right)e^{-iv_2^*d} - 2ke^{-iv_2^*d} \] (4.8)

where

\[\mu_k^* = \sum_{k=0}^{\infty} (i\omega)^k \mu_k \quad \eta_k^* = \sum_{k=0}^{\infty} (i\omega)^k \eta_k \] (4.9)

The determinant of the system of equation given by (4.8) is

\[\Delta = \begin{vmatrix} \bar{v}_1 \cos \bar{v}_1H & iv_2^* & -k^2 \\ 0 & 2iv_2^* & -v_2^*2 - k^2 \\ \rho_1\omega^2 \sin \bar{v}_1H & 2k^2 \mu_k^* - \rho_2\omega^2 \eta_k^* & -2k^2 \mu_k^* iv_2^* \end{vmatrix} \] (4.10)
and the values of $A, Q_2, S_2$ in terms of $K$ and other parameters can be found if $\Delta \neq 0$.

Using $\mu^*_k = \rho_2 D_s^* \eta^*_k$, we can write from (4.10)

$$\Delta(k) = \frac{-\rho_1 \omega \cdot 4 \sin \bar{\nu}_1 H \bar{\nu}_2^*}{D_s^*} + \bar{\nu}_1 \rho_2 D_s^* \eta^*_k \left[ 4k^2 \bar{\nu}_2^* \bar{\nu}_2' + \frac{(2k^2 - k_\beta^* \nu^*)}{(2k^2 - k_\beta^* \nu^*)} \cos \bar{\nu}_1 H \right]$$

(4.11)

and we obtain from (4.8)

$$A = \frac{-\rho_2 \omega \cdot 2 \eta^*_k \nu^*_k \left( 2k^2 - k_\beta^* \right)}{\Delta} \cos \bar{\nu}_2 d$$

(4.12)

$$Q_2 = \frac{K}{i \bar{\nu}_2 \Delta} e^{i \bar{\nu}_2 (H-d)} \left[ \frac{\rho_1 \omega \cdot 4 \nu^*_2}{D_s^*} \sin \bar{\nu}_1 H + \nu_1 \rho_2 D_s^* \eta^*_k \left[ 4k^2 \bar{\nu}_2^* \bar{\nu}_2' - \left( 2k^2 - 2k_\beta^* \right) \right] \cos \bar{\nu}_1 H \right]$$

(4.13)

$$S_2 = \frac{-4k \left( 2k^2 \mu^*_k - \rho_2 \omega \cdot 2 \eta^*_k \nu^*_k \right) \bar{\nu}_1 \cos \bar{\nu}_1 H}{\Delta} e^{-i \bar{\nu}_2 d + i \bar{\nu}_2 H}$$

(4.14)

Now we rewrite (4.13) in the form

$$Q_2 = \frac{k}{i \bar{\nu}_2} e^{i \bar{\nu}_2 (H-d)}$$
\[ \frac{2k \left( 2k^2 - k_\beta^2 \right)^2}{i \nu_2 \Delta} \nu_1 e^{i \nu_2 (H-d)} \rho_2 D_s^2 \eta_k^* \cos \nu_1 H \] \quad (4.15)

Hence from (4.3)

\[ \nu_2 = \int_0^\infty \kappa e^{-\nu_2 |z-H-d|} J_0(\kappa r) d\kappa + \int_0^\infty \kappa e^{-\nu_2 |z-H-\alpha|} J_0(\kappa r) d\kappa \]

The second term may be interpreted as a spherical wave emitted by the image of the source in the interface. Its simple form is \( \exp(-i\kappa_0^2 R')/R' \) where

\[ R' = r^2 + (z-H+d)^2 \]

The first two terms in (4.16) may combined as

\[ 2 \int_0^\infty \kappa \cos \nu_2^* (z-H) e^{-i\nu_2^* d} J_0(\kappa r) d\kappa \quad \text{for} \ H \leq z \leq H+d \] \quad (4.17)

and

\[ 2 \int_0^\infty \kappa e^{-i\nu_2 (z-H)} \cos \nu_2^* d J_0(\kappa r) d\kappa \quad \text{for} \ H+d \leq z \leq \infty \] \quad (4.18)
Using (4.1), (4.4), (.12) and (4.14), we obtain in the same way the expressions for \( \varphi_1 \) and \( \psi_2 \).

Now for the displacements \( q_H \) and \( \omega_H \) at the solid bottom \( Z = H \) we make use of \( \varphi_2 \) given by (4.17) and the third integral in (4.16) and \( \psi_2 \) by (4.4) to obtain on putting

\[
\Delta(k) = \nu(k)\rho_2 D_s^{*2} \eta_k \nu_1 \cos \nu_1 H
\]  

(4.19)

The expressions for displacements at the interface are

\[
q_H = -2k^* \int_0^\infty \frac{k^2}{v(k)} \left[ \frac{\rho_1}{\rho_2} \frac{k^*}{\eta_k \nu_1} \cdot \tan \nu_1 H - 2i\nu_2^* \right] e^{-i\nu_2^* J_1(\nu k)} dk
\]

(4.20)

\[
\omega_H = -2k^* \int_0^\infty \frac{k^2}{v(k)} \left[ \frac{2k^2 - k^*}{v(k)} \right] e^{-i\nu_2^* J_0(\nu k)} dk
\]

(4.21)

The integrals can be represented by the sum of branch-like integrals and residues. The residues, correspond to the poles \( K = K_n \) given by the roots of the equation.

\[
v(k) = 0
\]

(4.22)

**Discussion**

Equations (4.20) and (4.21) represent the expression for the two displacements \( q \) and \( w \) at the visco-elastic solid bottom which are obviously affected by the viscous field. Now making \( \eta_k, \lambda_k, \mu_k (k = 2, ..., n) \rightarrow 0 \) we obtain the problem, as a particular case, when the viscosity is of first order including strain rate and stress rate. Also, for the perfectly elastic solid bottom of the ocean, \( \eta_0 = 1 \) and \( \eta_k = \lambda_k = \mu_k = 0 \) \( (k = 1, 2, ..., n) \) and then the equations (4.20) and...
(4.21) tally with the corresponding classical result as studied by Ewing et al. (1954, pp. 160).

Here the amplitude of wave determined by branch line integrals diminish as $r^2$. As we are interested in an approximation which hold for large values of $r$, the terms corresponding to branch points are left out of consideration and only the residues are computed. The asymptotic values of displacement are obtained as follows:

$$q_H = \frac{2}{H^2} \sqrt{\frac{2\pi}{r}} \sum_{n=1}^\infty \frac{1}{\sqrt{k_n}} Q^* (k_n) e^{-i\beta_2 n d} e^{-i(\omega t - k_n + \pi/4)}$$

$$\omega_H = \frac{2}{H^2} \sqrt{\frac{2\pi}{r}} \sum_{n=1}^\infty \frac{1}{\sqrt{k_n}} W^* (k_n) e^{-i\beta_2 n d} e^{-i(\omega t - k_n + \pi/4)}$$

where the phase velocity $C_n$ and the factor $\nu_{2n}$ for each mode are given by

$$c_n = \frac{\omega}{k_n}, \quad \nu_{2n}^2 = k_\alpha^2 - k_n^2$$

and

$$Q^* (k_n) = \frac{k_n^2 H^2}{R^* (k_n) D_s^2} \frac{c_n^2}{\rho_1^2} \frac{1}{\sqrt{1 - c_n^2 / D_T^2}}$$

$$W^* (k_n) = \frac{k_n^2 H^2}{R^* (k_n) D_s^2} \frac{c_n^2}{\rho_2^2} \left(2 - \frac{c_n^2}{\nu_{2n}^2} \right) \sqrt{1 - c_n^2 / D_T^2}$$
The period equation (4.22) can be written in the dimensionless form as

\[
\tan\left(\frac{k_n H}{\sqrt{\frac{c_n^2}{\alpha_1^2} - 1}} \right) = \frac{\rho_1}{\rho_2} \frac{c_n^4}{D_s^{*2}} \left[ \left( 1 + \frac{1 - \frac{c_n^2}{\alpha_1^2}}{\frac{c_n^2}{\alpha_1^2} - 1} \right) \frac{\tan\left(\frac{k_n H}{\sqrt{\frac{c_n^2}{\alpha_1^2} - 1}} \right)}{\sqrt{\frac{c_n^2}{\alpha_1^2} - 1}} \right]
\]

\[
\frac{k_n H\left(\frac{c_n^2}{D_s^{*2}} - 1\right)}{\frac{c_n^2}{\alpha_1^2} - 1} \cdot \sec^2 \left(\sqrt{\frac{c_n^2}{\alpha_1^2} - 1} \right) - 4 \left( 3 - \frac{2c_n^2}{D_s^{*2}} \right) \frac{c_n^2}{\alpha_1^2} - 1
\]

\[
= 4 \left( 1 - \frac{c_n^2}{D_T^{*2}} \right) \sqrt{1 - \frac{c_n^2}{D_s^{*2}}} \left( 2 - \frac{c_n^2}{D_s^{*2}} \right)^2 (5.4)
\]

It defines, as usual, a relationship between the period \(T = 2\pi/C_n k_n = 2\pi/\omega\) and the phase velocity with the elastic constants and visco-elastic constants as parameters. Now as before when the viscosity effect vanishes, the result obtained main tallies with the corresponding classical problem.
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PROPAGATION OF WAVES IN A VISCO-ELASTIC SOLID HALF-SPACE OVERLYING A LIQUID LAYER WHEN A COMPRESSIONAL WAVE SOURCE BEING PRESENT IN THE SOLID SUBSTRATUM

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**ABSTRACT**

The present paper is concerned with the investigation of propagation of waves in a medium which is composed of a visco-liquid layer and of an underlying visco-elastic solid half space involving strain rate and stress rate of nth order when a compressional wave source being present in the solid substratum. The result is in good agreement with the classical problem when the effect of viscosity is ignored.

**Key words** : Visco-elastic medium, compressional wave source, refraction arrival, strain rate and stress rate, spherical waves

**Introduction**

Jeffreys (1926, pp 472-481), Muskat (1933, pp 14-28) and Sommerfeld and others have discussed the wave propagation in the presence of a source. Their results are directly related to an important practical problems viz. that of the
'refraction arrival' from a source to a receiver in seismology of near earthquake and in seismic refraction investigations. It is to be noted that Stoneley (1926, pp 349-356) studied the effect of the ocean on the transmission of Rayleigh waves considering the bottom of the ocean as a solid half space. In all the above investigation, the effect of viscosity, including strain rate and stress rate have been ignored, although this property plays an important role in the behaviour of solids. A lot of literature on the problems of waves and vibrations in a visco-elastic solid will be found in the monographs of Flugge (1967, pp 3-21), Bland (1960, pp 30-75) and Hunter(1960, pp 1-57). Also a good number of papers have been published by Sengupta and his co-workers in this field.

Assuming that the boundaries are all parallel planes, the authors have investigated the axis-symmetric problem in cylindrical co-ordinate system, as a particular case, the authors have deduced the problem when the viscosity is of first order including strain rate and stress rate.

**Basic equations and relations**

In classical elasto-kinetics, the two dimensional basic equations of motion (in terms of stress and displacement) in Cartesian co-ordinate system, and in the absence of body forces, are

\[
\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{13}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.1}
\]

\[
\frac{\partial \sigma_{31}}{\partial x} + \frac{\partial \sigma_{33}}{\partial z} = \rho \frac{\partial^2 \omega}{\partial t^2}
\]

The stress-strain relation according to Voigt (1887) in an isotropic visco-elastic solid medium of higher order involving strain rate and stress rate is

\[
D_{11} \varepsilon_{y} = D_{11} \Delta \varepsilon_{y} + 2D_{12} \varepsilon_{y} \tag{2.2}
\]
where

\[ D_\eta = \sum_{k=0}^{n} \eta_k \frac{\partial^k}{\partial t^k}, \quad D_\lambda = \sum_{k=0}^{n} \lambda_k \frac{\partial^k}{\partial t^k} \quad \text{and} \quad D_\mu = \sum_{k=0}^{n} \mu_k \frac{\partial^k}{\partial t^k} \]

in which \( \eta_0, \lambda_0, \mu_0 \) and \( \eta_k, \lambda_k, \mu_k \) are the elastic parameters and \( \eta_k, \lambda_k, \mu_k \) are the parameters associated with the \( k \)th order viscosity, \( \epsilon_\eta \) represents strain tensor, \( \delta_\eta \) is the Kronecker delta and \( \Delta \) is the dilatation.

Equation of motion in cylindrical co-ordinate system \((r, \theta, z)\) can be written as (Gooddier, J.N. Tinoshonko, S. 1951 pp. 56)

\[
\frac{\partial (r \hat{R} \hat{r})}{\partial t} + \frac{\partial (r \hat{R} \hat{z})}{\partial z} + \frac{\partial (\hat{z} \hat{r})}{r} = \rho \frac{\partial^2 \mathbf{q}}{\partial t^2} \\
\frac{\partial (r \hat{Z} \hat{r})}{\partial t} + \frac{\partial (r \hat{Z} \hat{z})}{\partial z} + \frac{\hat{z} \hat{r}}{r} = \rho \frac{\partial^2 \mathbf{w}}{\partial t^2}
\]

(2.3)

where \( \mathbf{q} \) and \( \mathbf{w} \) are the displacements in the \( r \) and \( z \) direction; the derivative with respect to \( \theta \) does not appear because of the axial symmetry. Now expressing \( \mathbf{q} \) and \( \mathbf{w} \) in terms of the potential \( \varphi \) and \( \psi \) we have

\[ \mathbf{q} = \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \psi}{\partial r \partial z} \quad \text{and} \quad \mathbf{w} = \frac{\partial \varphi}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) \]

(2.4)

The following stresses in the viscous field can be written from the expression of stresses in classical theory of elastivity (Brekhovskikh, 1960, pp. 304) as

\[ D_\eta \approx \frac{\partial^2 \varphi}{\partial z^2} + 2D (D_\lambda \frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial z^2}) \]

(2.5)
\[ D_{\eta} \sum_{i=1}^{\infty} D \mu \frac{\partial}{\partial r} \left( 2 \frac{\partial \psi}{\partial z} + \nabla^2 \psi \right) \]

\[ D_{\eta} \sum_{i=1}^{\infty} D \lambda \nabla^2 \psi + 2D \mu \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} \right) \]

Introducing (2.5) into (2.3) we obtain

\[ \frac{\partial^2 \varphi}{\partial t^2} = \left( \sum_{k=0}^{n} \frac{\partial^k}{\partial t^k} \sum_{k=0}^{n} \eta_k \frac{\partial^k}{\partial t^k} \right) \nabla^2 \varphi \quad (2.6) \]

\[ \frac{\partial^2 \psi}{\partial t^2} = \left( \sum_{k=0}^{n} \frac{\partial^k}{\partial t^k} \sum_{k=0}^{n} \eta_k \frac{\partial^k}{\partial t^k} \right) \nabla^2 \psi \]

where

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \]

\[ V^2_{kT} = \frac{\lambda_k + 2\mu_k}{\rho} \quad V^2_{kS} = \frac{\mu_k}{\rho} \quad (k=0,1,2,\ldots,n) \]
MAGNETO-VISCO-ELASTIC WAVES IN AN INITIALLY STRESSED CONDUCTING LAYER

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ABSTRACT

The present paper is concerned with the investigation of propagation of waves in an initially stressed magneto-visco-elastic layer involving time rate of strain of order $n$. As particular cases, symmetric vibrations, antisymmetric vibrations and the magneto visco-elastic waves, including strain rate of first order in the initially stressed layer, have been studied. The wave velocity equation that have been obtained in different cases are in complete agreement with the corresponding classical problem when the additional fields are withdrawn.

Key Words: Magneto visco-elastic waves/initial stress/strain rate/symmetric and antisymmetric vibrations/Rayleigh waves

1. Introduction

Dilatational and rotational waves in an initially stressed magneto-elastic conducting medium have been investigated by Yu and Tang (1966, pp. 766). Also, the interplay of electromagnetic field with the motion of the deformation of a solid and
the problems of waves and vibrations in visco-elastic solids have been considered by some investigators. It is to be noted that Yu and Tang in their paper, mentioned earlier, considered the three particular cases of initial stress, viz., (i) hydrostatic tension or compression, (ii) uni-axial tension or compression, and (iii) uniform shear stress. The authors here consider the initial stress as hydrostatic tension or compression as the inner parts of the earth are under considerable stress from the weight of the mantle, on it with the initial equilibrium stress being supposed to be approximately of hydrostatic nature. Moreover, as the earth is placed in its own magnetic field and the material medium of the earth may be visco-elastic in nature in some circumstances, the authors have considered both the magnetic and viscous fields in the titled problem, which has not yet been investigated. Though any material does not remain homogeneous and isotropic when it is subjected to initial stress and magnetic field, yet the authors have ignored such small variations in their way of investigation.

2. Basic Equations

The equation of motion for a perfectly conducting elastic solid under the hydrostatic tension or compression in a uniform magnetic field is (Yu and Tang 1966, p. 766)

$$\rho \frac{\partial^2 u_i}{\partial t^2} = -p_0 \frac{\partial^2 u_i}{\partial x_i \partial x_j} + \mu_0 H_0 \left( \frac{\partial H_i}{\partial x_1} - \frac{\partial H_1}{\partial x_i} \right) + \frac{\partial \tau_{ix}}{\partial x_j}$$

(2.1)

$$H_i = H_0 \left( \frac{\partial u_i}{\partial x_1} - \frac{\partial u_1}{\partial x_i} \right)$$

Where \((-p_0)\) is the hydrostatic tension or compression (tension when \(p_0<0\) and compression when \(p_0>0\)), \(\tau_{ix}\) is the stress tensor over the initial stress, \(H_0\) is the intensity of the uniform magnetic field parallel to the \(x_1\) axis, \(\mu_0\) is the magnetic permeability, and \(\rho\) is the density of the material and \(u_i\) is the displacement components \((i = 1, 2, 3)\).

3. General Theory

Let us consider an initially stressed homogeneous, isotropic visco-elastic solid layer of finite thickness \(2H\) including strain rate of higher order and permeated by uniform magnetic field. We introduce a system of orthogonal cartesian axes \(0-x_1x_2x_3\), where the origin \(O\) is any point on the median plane of the infinite plate and \(Ox_3\) being a line drawn vertically downwards. We assume that there exists a plane wave moving with a constant velocity \(c\) in the \(x_1\) - direction and all cause and effect depends on two space.
variables $x_1$, $x_3$ and time $t$. To separate the effect of purely dilational and purely rotational waves, let us introduce two displacement potentials $\phi (x_1, x_3, t)$ and $\psi (x_1, x_3, t)$ as
\[
\begin{align*}
  u_1 &= \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3} \\
  u_3 &= \frac{\partial \phi}{\partial x_3} - \frac{\partial \psi}{\partial x_1}
\end{align*}
\] (3.1)
so that
\[
\nabla^2 \phi = \Delta, \quad \nabla^2 \psi = \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}, \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}
\]
The stress-strain relation for general, isotropic Voigt type visco-elastic medium are (Voigt, 1887)
\[
\tau = D_\lambda \Delta \delta_y + 2D_\mu \epsilon
\] (3.2)
Where
\[
D_\lambda = \sum_{k=0}^{n} \lambda_k \frac{\partial^k}{\partial t^k}, \quad D_\mu = \sum_{k=0}^{n} \mu_k \frac{\partial^k}{\partial t^k}
\]
in which $\lambda_0$, $\mu_0$ are the elastic parameters and $\lambda_k$, $\mu_k$ ($k = 1 \ldots n$) are the parameters associated with $k$-th order visco-elasticity,
\[
e = \frac{1}{2} (u_{11} + u_{22})
\] is the strain tensor, $\delta_y$ is the Kronecker delta, and $\Delta$ is the dilatation.
Now introducing (3.2) in (2.1), we obtain the displacement equations of motion for the solid problem as
\[
\begin{align*}
  \left( D_\lambda + D_\mu \right) \frac{\partial \Delta}{\partial x_1} + D_\mu \nabla^2 u_1 - p_0 \nabla^2 u_1 &= \rho \frac{\partial^2 u_1}{\partial t^2} \\
  \left( D_\lambda + D_\mu \right) \frac{\partial \Delta}{\partial x_3} + D_\mu \nabla^2 u_3 - p_0 \nabla^2 u_3 + K \left( \frac{\partial^2 u_3}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_1 \partial x_3} \right) &= \rho \frac{\partial^2 u_3}{\partial t^2}
\end{align*}
\] (3.3)
where \[K = \mu_c H_0^2\]
Again using (3.1) in (3.3) we have
\[
\begin{align*}
  \frac{\partial^2 \phi}{\partial t^2} &= \left( D_\tau - p_0/\rho \right) \nabla^2 \phi + K/\rho \frac{\partial^2 \psi}{\partial x_1 \partial x_3} \\
  \frac{\partial^2 \psi}{\partial t^2} &= \left( D_\tau - p_0/\rho \right) \nabla^2 \psi + K/\rho \frac{\partial^2 \psi}{\partial x_1^2}
\end{align*}
\] (3.4)
where

\[ D_T = \sum_{k=0}^{n} V_{kT}^2 \frac{\partial^k}{\partial t^k} \quad D_S = \sum_{k=0}^{n} V_{kS}^2 \frac{\partial^k}{\partial t^k} \]  

(3.5)

\[ V_{kT}^2 = \left( \lambda_k + 2\mu_k \right) / \rho \quad V_{kS}^2 = \frac{\mu_k}{\rho} \quad [k = 0, 1, 2, \ldots, n] \]

4. Boundary Conditions

The boundary conditions for the present problem are

\[ \bar{\tau}_{i3} = \bar{\tau}_{33} = 0 \quad \text{in the plane} \quad x_3 = \pm H \]  

(4.1)

since the planes \( x_3 = \pm H \) are free of stresses,

where

\[ \bar{\tau}_{i3} = \rho D_S \left( 2 \frac{\partial^2 \varphi}{\partial x_1 \partial x_3} - \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_3^2} \right) \]  

(4.2)

5. Solution of the problem

We seek solutions of equations (3.4) in the following form

\[ \varphi, \psi = \{ \tilde{\varphi}(x_3), \tilde{\psi}(x_3) \} \exp \left[ i(wt - \eta x_3) \right]. \]  

(5.1)

Introducing (5.1) in (3.4), we obtain in the following

\[ \left( \frac{d^2}{dx_3^2} - h_1^2 \right) \tilde{\varphi} - \frac{\eta K}{\rho \sum_{k=0}^{n} (iw)^k V_{kT}^2 - p_0 / \rho} \frac{d\tilde{\psi}}{dx_3} = 0 \]  

(5.2)

\[ \left( \frac{d^2}{dx_3^2} - h_2^2 \right) \tilde{\psi} - \frac{K\eta^2}{\rho \sum_{k=0}^{n} (iw)^k V_{kS}^2 - p_0 / \rho} \tilde{\psi} = 0 \]

where

\[ h_1^2 = \eta^2 - \frac{w^2}{\sum_{k=0}^{n} (iw)^k V_{kT}^2 - p_0 / \rho} \quad h_2^2 = \eta^2 - \frac{w^2}{\sum_{k=0}^{n} (iw)^k V_{kS}^2 - p_0 / \rho} \]  

(5.3)
The solutions of equations (5.2) can be written in the form
\[ \hat{\phi} = A \sinh(s_1x_3) + B \cosh(s_1x_3) + C \sinh(s_2x_3) + D \cosh(s_2x_3) \]
\[ \hat{\psi} = C_1 \sinh(s_2x_3) + D_1 \cosh(s_2x_3) \] (5.4)

The constants \( C, D_1 \) and \( D, C_1 \) are connected by the following relations
\[ C = \alpha_1 D_1 \quad D = \alpha_1 C_1 \] (5.5)
where
\[ \alpha_1 = \frac{\eta K}{\rho \left[ \sum_{k=0}^{n} (iw)^k V_{kT}^2 - p_0/\rho \right]} \frac{s_2}{s_2^2 - h_1^2} \] (5.6)

and
\[ s_1^2 = h_1^2, \quad s_2^2 = h_2^2 + \frac{K\eta^2}{\rho \left[ \sum_{k=0}^{n} (iw)^k V_{kS}^2 - p_0/\rho \right]} \] (5.7)

Substituting (5.4) in the boundary conditions (4.1), we have
\[ A \ell_1 p_1 - B m_1 p_1 + C_1 \ell_2 p_2 - D_1 m_2 p_2 = 0 \]
\[ A m_1 q_1 - B \ell_1 q_1 + C_1 m_2 q_2 - D_1 \ell_2 q_2 = 0 \]
\[ A n_1 p_1 - B o_1 p_1 + C_1 n_2 p_2 - D_1 o_2 p_2 = 0 \]
\[ A o_1 q_1 - B n_1 q_1 + C_1 o_2 q_2 - D_1 n_2 q_2 = 0 \] (5.8)

where
\[ \ell_1 = 0, \quad \ell_2 = -\left\{ \left( s_1^2 + \eta^2 \right) + 2\eta \alpha_1 s_2 \right\}, \quad m_1 = 2\eta s_1 \]
\[ n_1 = \left[ V_{kT}^{*2} (s_1^2 - \eta^2) + 2 V_{kS}^{*2} \eta^2 \right], \quad n_2 = 0 \] (5.9)
\[ o_1 = 0, \quad o_2 = -\left[ V_{kT}^{*2} (s_2^2 - \eta^2) \alpha_1 + 2 V_{kS}^{*2} \eta \left( \eta \alpha_1 - i s_2 \right) \right] \]
\[ V_{kT}^{*2} = \sum_{k=0}^{n} (iw)^k V_{kT}^2, \quad V_{kS}^{*2} = \sum_{k=0}^{n} (iw)^k V_{kS}^2 \]

and
\[ p_i = \sinh(s_i H), \quad q_i = \cosh(s_i H) \] (5.10)
Eliminating $A$, $B$, $C_1$, $D_1$, for the system of equations (5.8)

\[
\Delta_1 = \begin{vmatrix}
\ell_1 & m_1 & \frac{\tanh s_1 H}{\tanh \ell_2} & m_2 & \frac{\tanh s_2 H}{\tanh \ell_2} \\
\ell_2 & m_1 & \frac{\tanh s_1 H}{\tanh \ell_2} & m_2 & \frac{\tanh s_2 H}{\tanh \ell_2} \\
\end{vmatrix} = 0 \quad (5.11)
\]

After little simplification the determinantal equation (5.11) leads to the transcendental equation

\[
\frac{\tanh s_1 H}{\tanh s_2 H} + \frac{\tanh s_2 H}{\tanh s_1 H} = \frac{m_1^2 o_1^2 + n_1^2 \ell_2}{m_1 n_1 o_2 \ell_2} \quad (5.12)
\]

6. Particular Cases

I. Symmetric vibrations: Symmetric vibrations are characterized by the symmetry of displacements $u_1$, $u_3$ and the stress $\tau_{33}$ with respect to the plane $x_3 = 0$, while the stress $\tau_{13}$ is antisymmetric with respect to the same plane. In this case, we have to put in the expressions (5.4) $A = C_1 = D_1 = 0$ so that

\[
\varphi_1 = \{ B \cosh (s_1 x_3) + D \cosh (s_2 x_3) \} \exp [i (wt - \eta x_1)]
\]

\[
\psi_1 = \{ C \sinh (s_2 x_3) \} \exp [i (wt - \eta x_1)] \quad (6.1)
\]

The boundary conditions are

\[
\tau_{33} = \tau_{13} = 0 \quad (6.2)
\]

Substituting (6.1) in the boundary conditions (6.2), we have

\[
m_1 B \sinh s_1 H - C_1 \ell_1 \sinh s_2 H = 0 \quad (6.3)
\]

\[
n_1 B \cosh s_2 H - C_1 o_2 \cosh s_2 H = 0
\]

Eliminating $B$ and $C_1$ from (6.3), we have

\[
\frac{\tanh (s_1 H)}{\tanh (s_2 H)} = \frac{\ell_2 n_1}{m_1 o_2} \quad (6.4)
\]
II. Antisymmetric vibrations: Let us now consider the particular case when the displacements $u_1, u_2$ and the stress $\tau_{33}$ are antisymmetric with respect to the plane $x_3 = 0$ while $\tau_{13}$ is symmetric with respect to the same plane. In this case, $B = D = C_1 = 0$, so that

$$
\begin{align*}
\varphi_2 &= \{ A \sinh (s_1 x_3) + C \sinh (s_2 x_3) \} \exp [i (wt - \eta x_1)] \\
\psi_2 &= \{ D_1 \cosh (s_2 x_3) \} \exp [i (wt - \eta x_1)]
\end{align*}
$$

(6.5)

Proceeding in the same way as in case I, we get

$$
\frac{\text{tanh } s_1 H}{\text{tanh } s_2 H} = \frac{m_1 o_2}{\ell_2 n_1}
$$

(6.6)

III. Magneto-visco-elastic waves involving strain rate of first order for the initially stressed layer: If we vary $K$ from 0 to 1, $e^{i\lambda_k}, \mu_k (k = 2, \ldots, n) \rightarrow 0$ in the above problem we easily obtain the magneto visco-elastic wave equation for the initially stressed layer including strain rate of first order.

7. Discussions

The determinantal equation (5.11) represents the required wave velocity equation to determine the phase velocity of the waves propagating parallel to $x_1$ axis. It is obvious that this wave propagation essentially depends on magnetic field, initial stress and visco-elastic nature of the material medium with strain rate of higher order. We consider here the following two parts (A and B).

(A) If the lengths of the waves are very small relative to the thickness $2H$ of the layer, the quantities $s_i H$ ($i = 1, 2$) and $\eta H$ are large and we may assume that $\text{tanh } s_i H \approx 1$ and thus from (5.11) we get

$$
\Delta_2 \cdot \Delta_3 = 0
$$

(7.1)

where

$$
\Delta_2 = \begin{vmatrix} \ell_1 + m_1 & \ell_2 + m_2 \\ n_1 + o_1 & n_2 + o_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \ell_1 - m_1 & \ell_2 - m_2 \\ n_1 - o_1 & n_2 - o_2 \end{vmatrix}
$$

(7.2)

From (7.1) we have

$$
either \quad \Delta_2 = 0 \quad or \quad \Delta_3 = 0
$$

(7.3)
The determinantal equation \( \Delta_2 = 0 \) represents the wave velocity equation of Rayleigh type in the vicinity of uppermost plane horizontal boundary, while the equation \( \Delta_3 = 0 \) represent the wave velocity equation in the vicinity of lowermost plane horizontal boundary of the solid. Thus Rayleigh wave velocity equation occurs twice according to the conditions stated before. In the first case we consider the medium extended to infinity at the lower part and in the second case, the medium extended towards infinity at the upper side. It is evident from the expressions (7.2) and (5.9) that the equations \( \Delta_2 = 0 \) and \( \Delta_3 = 0 \) are simply interchangeable by changing \( s_i \) by \( -s_i \) (i = 1, 2), i.e., by changing the direction of the \( x_3 \) axis. This explains the existence of two wave velocity equations. As the determinantal equations \( \Delta_2 = 0 \) and \( \Delta_3 = 0 \) contain \( \eta \), the magneto-visco-elastic initially stressed Rayleigh wave is subjected to dispersion of the general wave form

Now \( \Delta_2 = 0 \) can be written as

\[
\left( 2 - \frac{C^2}{V_{KS}^2} \right)^2 - 4s_1s_2 \frac{s^2}{\eta^2} + \left( 2 - \frac{C^2}{V_{KS}^2} \right) \frac{K}{\rho V_{KS}^2} + \frac{2\alpha_1}{\eta} (s_1 - s_2) \left\{ \frac{s_1s_2 + \eta^2}{\eta^2} \frac{V_{KT}^2}{V_{KS}^2} - 2 \right\} = 0
\]  

(7.4)

When the viscosity effect and strain rate are neglected then \( \lambda_k, \mu_k \to 0 \) (k = 1, n) and we get the following equations from the determinantal equation \( \Delta_2 = 0 \)

\[
m_1 \sigma_2 - n_1 \ell_2 = 0
\]

\[
m_1 = 2\eta s_1, \quad \sigma_2 = -\left\{ \frac{\lambda_0}{\rho} (s_1^2 - \eta^2) \alpha_1 + \frac{2\mu_0}{\rho} (s_2 \alpha_1 - \eta s_2) \right\}
\]

\[
n_1 = \frac{\lambda_0}{\rho} (s_1^2 - \eta^2) + \frac{2\mu_0}{\rho} s_1^2, \quad \ell_2 = -\left\{ (s_2^2 + \eta^2) + 2\eta \alpha_1 s_2 \right\}
\]

(7.5)

with

\[
s_1^2 = \eta^2 - \frac{w^2}{V_{0T}^2 - \rho_0 / \rho}, \quad s_2^2 = \eta^2 - \frac{w^2}{V_{0S}^2 - \rho_0 / \rho} + \frac{K\eta^2}{\rho \left( V_{0S}^2 - \rho_0 / \rho \right)}
\]

\[
\alpha_1 = \frac{\eta K}{\rho \left( V_{0T}^2 - \rho_0 / \rho \right)}, \quad \sigma_2 = \frac{s_2}{s_2^2 - s_1^2}, \quad V_{0T}^2 = \frac{\lambda_0 + 2\mu_0}{\rho}, \quad V_{0S}^2 = \frac{\mu_0}{\rho}
\]
Equation (7.5) represents the magneto elastic Rayleigh waves velocity equation under the initial stress of hydrostatic tension or compression and is in well agreement with the result obtained by Acharya and Sengupta (1978, pp 237) with some change of notations. In the absence of initial stress and magnetic field, i.e., putting $p_0 = 0$, $K = 0$, we get from (7.4)

\[
\left( 2 - \frac{C^2}{V_{KS}^2} \right)^2 - \frac{4 s_1 s_2}{\eta^2} = 0
\]  

(7.6)

where

\[
s_1^2 = \eta^2 - \frac{w^2}{V_{KT}^2}, \quad s_2^2 = \eta^2 - \frac{w^2}{V_{KS}^2}, \quad C = \frac{w}{\eta}
\]

Equation (7.6) represents the Rayleigh wave velocity equation in higher order viscoelastic medium involving strain rate only. Again when the viscosity effect as well as strain rate is neglected we get from (7.6)

\[
\left( 2 - \frac{C^2}{V_{0S}^2} \right)^2 = 4 \sqrt{1 - \frac{C^2}{V_{0T}^2}} \sqrt{1 - \frac{C^2}{V_{0S}^2}}
\]  

(7.7)

which is in fact the classical Rayleigh wave velocity equation.

Now in case of symmetrical vibrations, we get from (6.4)

\[
m_1 \omega_2 - \ell_2 \eta_1 = 0
\]  

(7.8)

Equation (7.8) represents the wave velocity of Rayleigh waves in magneti-viscoelastic initially stressed conducting layer including strain rate only. The result is in agreement with (7.4) obtained earlier. The determinantal equation $\Delta_2 = 0$ can be treated similarly.

(B) when the length of the wave is very large relative to the thickness $2H$ of the layer, the quantities $s_i H$ ($i = 1, 2$) may be regarded as small for which we make $\tanh (S_i H) = s_i H$.

and we obtain from equation (5.12)

\[
\frac{s_1^2 + s_2^2}{s_1 s_2} = \frac{m_1^2 \omega_2^2 + n_1^2 \ell_2^2}{m_1 n_1 \omega_2 \ell_2}
\]  

(7.9)
In the absence of initial stress and magnetic field \((p_0 = 0, K = 0)\), equation (7.9) reduces to the following form

\[
C^2 V_{KT}^2 = 4 V_{KS}^2 \left( V_{KT}^2 - V_{KS}^2 \right)
\]  

(7.10)

which determines the wave velocity in general order visco-elastic layer involving strain rate only. If we neglect the viscosity effect, (7.10) determines the wave velocity in the classical elastic layer (Nowacki 1963 pp 298-302). In case of symmetric vibrations, equations (6.4) reduces to

\[
\frac{s_1}{s_2} = \frac{\ell_2 n_1}{m_1 o_2}
\]

(7.11)

If the layer is initially free of stress and the magnetic field is absent, we get from equation (7.11) the same equation as in (7.10). In case of antisymmetric vibrations, if we approximate the hyperbolic tangents to the first term of their expansion into series, we get from (6.6)

\[
\frac{s_1}{s_2} = \frac{m_1 o_2}{\ell_2 n_1}
\]

(7.12)

Here again, if we neglect the magnetic field and initial stress and proceed as before, we find from (7.12) that the wave velocity \((c = w/n)\) becomes zero. Hence to obtain the classical result, parallel to Rayleigh and Lamb, in visco-elastic medium with strain rate, the hyperbolic tangents have to be approximated to the first two terms of their expansion into series. Making viscous parameters tend to zero, we obtain the classical result due to Rayleigh and Lamb in elastic medium.

Lastly, if we consider the case when the length of the wave is very small compared to the thickness of the layer, equation (6.6) reduces to the equation (7.8) which is wave velocity of Rayleigh surface waves in magneto visco-elastic initially stressed conducting layer including strain rate.

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PROPAGATION OF MONOCHROMATIC WAVES IN AN INFINITE MICROPOLAR PLATE UNDER THE INFLUENCE OF GRAVITY AND HIGHER ORDER VISCO-ELASTICITY: LAMB'S PROBLEM

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Abstract: The subject of the present paper is to investigate the propagation of monochromatic waves in an infinite micropolar visco-elastic plate under the influence of gravity. The plate has been bounded by two infinite plane boundaries \( x_3 = \pm h \) which are free from stresses. The wave velocity equation has been derived in a closed analytic form which is in agreement with the corresponding problem in the absence of gravity and viscosity.

Introduction

In the classical theory of elasticity the state of deformation is characterised by means of a symmetric strain tensor depending on the displacement vector only. The rotation vector is also determined by the same displacement vector. However, in micropolar theory of elasticity the state of deformation depends on two vectors, namely, the displacement vector and rotation vector. Actually, depending on these two vectors the deformation is characterised by means of two asymmetric tensors namely, the strain tensor and curvature twist tensors, both of which are tensors of rank two. Similarly, the state of stress is characterised by means of two asymmetric tensors - force stress tensor and couple stress tensor. Like the classical theory of elasticity the generalised Hooke's law is a set of equations connecting the force stress and couple stress with strain tensor and curvature twist tensor. It is very difficult to solve the general problems of elastic deformations and waves. However, in some particular cases where, the displacement and rotation vectors are chosen in a very restricted manner in conformance with the nature of the problem and its geometry. In such circumstances it is not a formidable task to solve the problems.

The gradual development of micropolar theory of elasticity as evidenced by the work of Nowacki [1, 2], Eringen [3], Eringen & Suhubi [4]. In recent years De and Sengupta

(1)
[5], Acharya and Sengupta [6], Sengupta and Ghosh [7], Sengupta and Chakraborty [8, 9], Sengupta and Das [10] investigated a good number of research problems in micropolar elasticity.

Formulation of the Problem and the Boundary Conditions

We consider a homogeneous micropolar visco-elastic plate of thickness ‘2h’ through which a monochromatic wave propagates along $x_3$-axis under the influence of gravity. Again we consider the boundary of the layer $x_3 = \pm h$ are free from stresses.

The following conditions should be satisfied on these edges.

$$\sigma_{33} = \sigma_{31} = 0, \quad u_{23} = 0 \quad \text{for} \quad x_3 = \pm h \quad (1)$$

Again we assume that the displacement $\vec{u}$ and rotation $\vec{w}$ do not depend on $x_2$, so that,

$$\vec{u} = (u_1, 0, u_3), \quad \vec{w} = (0, \omega_2, 0) \quad (2)$$

Let $\phi(x_1, x_3, t)$ and $\psi(x_1, x_3, t)$ are the displacement potentials and they are connected with the displacements and rotations as follows:

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_3}, \quad u_3 = \frac{\partial \phi}{\partial x_3} - \frac{\partial \psi}{\partial x_1} \quad (3)$$

So that we have,

$$\nabla^2 \phi = \Delta, \nabla^2 \psi = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$$

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} \quad (4)$$

Since we are considering a plane strain problem under the above assumptions, the equations of motion in a micropolar elastic solid medium under the influence of gravity and viscosity may be written as:

$$(D_{\mu} + D_{\alpha}) \nabla^2 u_1 + (D_{\lambda} + D_{\mu} - D_{\alpha}) \epsilon_{11} - 2D_{\alpha} W_{2, 3} + P g u_{3, 1} = p u_1 \quad (5)$$

$$(D_{\mu} + D_{\alpha}) \nabla^2 u_3 + (D_{\lambda} + D_{\mu} - D_{\alpha}) \epsilon_{33} + 2D_{\alpha} W_{2, 1} + P g u_{1, 1} = p i \omega_2$$

$$(D_{\lambda} + D_{\alpha}) \nabla^2 \omega_2 - 4D_{\alpha} \omega_2 + 2D_{\alpha} (a_{1, 3} - u_{3, 1}) = J \omega_2 \quad (5)$$

Introducing (3) and (4) into (5) we obtain the following set of equations satisfied by $\phi$, $\psi$, $u_2$.
\[(D_T \nabla^2 - \frac{\partial^2}{\partial t^2}) \phi + g \frac{\partial \psi}{\partial x_1} = 0\]

\[(D_s \nabla^2 - \frac{\partial^2}{\partial r^2}) \psi \quad g \frac{\partial \phi}{\partial x_1} + D_\rho \omega_2 = 0\]

\[(D_Q \nabla^2 - D_w - \frac{\partial^2}{\partial t^2}) w_2 + \frac{1}{2} D_w \nabla^2 \psi = 0\] (6)

Where

\[D_T = \sum_{l=0}^n v^2 l t \frac{\partial^2}{\partial t^2} , \quad D_s = \sum_{l=0}^n v^2 l s \frac{\partial^2}{\partial t^2} , \quad D_p = \sum_{l=0}^n v^2 l p \frac{\partial^2}{\partial t^2}, \]

\[D_q = \sum_{l=0}^n \frac{\partial^2}{\partial t^2} , \quad D_w = \sum_{l=0}^n v^2 l w \frac{\partial^2}{\partial t^2} , \quad V_{lT}^2 l + 2\mu_l \rho, \]

\[V_{lS}^2 = \frac{u_l + \alpha_l}{\rho} , \quad V_{lP}^2 = 2\alpha_l \rho , \quad V_{lW}^2 = \frac{r_l + \xi_l}{J} , \]

\[V_{lW}^2 = \frac{4\alpha_l}{J} \] (7)

Now from the boundary conditions we get,

\[\sigma_{33} = 2D_\mu (\phi_{33} - \Psi_{33}) + D_\lambda \nabla^2 \phi = 0\]

\[\sigma_{31} = D_\mu (2\phi_{33} - \Psi_{33} + \Psi_{11}) - D_\lambda (\nabla^2 \Psi + 2w_2) = 0\]

\[\mu_{32} = (D_r + D_\lambda) w_{2,3} = 0\] (8)

**Method of Solutions**

We seek the solutions of the form

\[
(\tilde{\phi}, \tilde{\Psi}, \tilde{\omega}_2) = (\tilde{\phi}(x_3), \tilde{\Psi}(x_3), \tilde{\omega}_2(x_3)) e^{i \left( k x_1 - \eta t \right)}
\] (9)

Obviously, the solutions are as follows:

\[
\tilde{\phi} = A_1 \sin h \lambda_1 x_3 + B_1 \cos h \lambda_1 x_3 + C_1 \sin h \lambda_2 x_3 + D_1 \cos h \lambda_2 x_3
\]

\[
\tilde{\Psi} = A_2 \sin h \lambda_1 x_3 + B_2 \cos h \lambda_1 x_3 + C_2 \sin h \lambda_2 x_3 + D_2 \cos h \lambda_2 x_3
\]

\[
(3.1)
\]
\[ \vec{v}_2 = A_2 \sin \lambda_1 x_3 + B_2 \cos \lambda_1 x_3 + C_2 \sin \lambda_2 x_3 + D_2 \cos \lambda_2 x_3 + E_2 \sin \lambda_3 x_3 + F_2 \cos \lambda_3 x_3 \] (10)

where,

\[ \Sigma \lambda_1^2 = \Sigma K_1^2 + p s , \]

\[ \Sigma \lambda_1^2 \lambda_2^2 = \Sigma K_1^2 K_2^2 \frac{g^2}{c_1^2 c_2^2} + p s (K_1^2 + K_2^2) \]

\[ \lambda_1^2 \lambda_2^2 \lambda_3^2 = K_3^2 \left( \frac{K_1^2 K_2^2}{c_1^2 c_2^2} \right) + p s K_1^2 K_2^2 \] (11)

In which,

\[ K_1^2 = K^2 - \frac{\eta^2}{c_1^2}, \quad K_2^2 = K^2 - \frac{\eta^2}{c_2^2}, \quad K_3^2 = K^2 + \frac{\eta^2}{c_4^2}, \]

\[ c_1^2 = \sum_{l=0}^{n} v_{il}^2 (-i \eta)^l, \quad c_2^2 = \sum_{l=0}^{n} v_{il}^2 (-i \eta)^l, \quad c_3^2 = \sum_{l=0}^{n} v_{il}^2 (-i \eta)^l, \]

\[ p_s = \frac{1}{2} \sum_{l=0}^{n} v_{il}^2 (-i \eta)^l \cdot \sum_{l=0}^{n} v_{il}^2 (-i \eta)^l \]

\[ v_1^2 = \sum_{l=0}^{n} v_{il}^2 (-i \eta)^l, \quad p = \sum_{l=0}^{n} v_{il}^2 (-i \eta)^l \] (12)

Also,

\[ A, A_1, A_2, B, B_1, B_2, C, C_1, C_2; \quad D, D_1, D_2; \quad E, E_1, E_2; \quad F, F_1, F_2, F \]

are connected by the relations

\[ A_1 = p_1 A, \quad B_1 = p_1 B, \quad C_1 = p_2 C, \quad D_1 = p_2 D, \quad E_1 = p_3 E, \]

\[ F_1 = p_3 F, \quad A_2 = q_1 A, \quad B_2 = q_1 B, \quad C_2 = q_2 C, \quad D_2 = q_2 D, \]

\[ E_2 = q_3 E, \quad F_2 = q_3 F \] (13)

where,

\[ p_j = i c_1^2 \left( \lambda_j^2 - K_1^2 \right) / k g, \]

(4)
We now consider two cases

Case I : Symmetric vibrations : Symmetric vibrations are characterised by the symmetry of displacement $u_1$ and stresses $\sigma_{31}, \sigma_{33}$ and $\mu_{32}$ with respect to the plane $x_3=0$. In expression [10] we have,

$$A = C = E = B_1 = D_1 = F_1 = B_2 = D_2 = F_2 = 0$$

(15)

In view of the equations [6] and [6] we have

$$B = A_1 \xi_1, \quad D = C_1 \xi_2, \quad F = E \xi_3,$$

$$A_2 = A_1 \eta_1, \quad C_2 = C_1 \eta_2, \quad E_2 = E_1 \eta_3$$

where,

$$\xi_j = \tan h \lambda_j x_3, \quad gix/c_1^2 (K^2 - \lambda_j^2)$$

$$\eta_j = s (k_2^2 - \lambda_j^2) / \lambda_j^2 - K_j^3, \quad j = 1, 2, 3$$

(16)

We obtain a system of homogeneous equations

we get

$$A_1 \xi_1 + C_1 \xi_2 + E_1 \xi_3 = 0$$

$$A_2 \eta_1 + C_2 \eta_2 + E_2 \eta_3 = 0$$

(17)

where,

$$l_j = \sum_{l=0}^{n} (\lambda_{j+l} + 2 \mu_{j+l}) (-i \eta)^l \lambda_j^2 - \sum_{l=0}^{\eta} \lambda_{j+l} (-i \eta)^l K_j^2 \xi_j$$

$$-2 \sum_{l=0}^{n} \mu_{j+l} (-i \eta)^l K \lambda_j$$

$$m_j = 2 \sum_{l=0}^{n} \mu_{j+l} (-i \eta)^l K \lambda_j + \sum_{l=0}^{n} (\mu_{j+l} + \alpha_{j+l}) (-i \eta)^l K \lambda_j^2$$

(5)
\[ + \sum_{l=0}^{n} (\mu_l - \alpha_2) (-i \eta)^l K^2 - 2 \sum_{l=0}^{n} \alpha_l (-i \eta)^l \eta_j \]

\[ n_j = \eta_j \lambda_j, \quad c_j = \cos h \lambda_j h, \quad s_j = \sin h \lambda_j h; \quad j = 1, 2, 3 \quad (18) \]

Eliminating the independent constants \( A_1, C_1, E_1 \) from the set of equations [17] we have

\[ \begin{vmatrix} l_1 c_1 & l_2 c_2 & l_3 c_3 \\ m_1 s_1 & m_2 s_2 & m_3 s_3 \\ n_1 c_1 & n_2 c_2 & n_3 c_3 \end{vmatrix} = 0 \quad (19) \]

**Case II: Antisymmetric vibrations**: Let us now consider another special type of vibration in which the waves are antisymmetric in nature with respect to the plane \( x_3 = 0 \) and in this case, we have only to retain those terms which are opposite to the sign if we replace \( x_3 \) by \(-x_3\). Therefore, from equations [10] we have,

\[ B = D = F = A_1 = C_1 = E_1 = A_2 = C_2 = E_2 = 0 \quad (20) \]

where

\[ A = B_1 \xi_1', \quad C = D_1 \xi_2', \quad E = F_1 \xi_3', \]

\[ B_2 = B_1 \eta_1', \quad D_2 = D_1 \eta_2', \quad F_2 = F_1 \eta_3', \]

and

\[ \xi_j^1 = \cot h \alpha_j x_3 g i k/c_2^2 (k^2 - k_2^2) \]

\[ n_j^1 = s (k^2 - \lambda_j^2)/\lambda_j^2 - K_3^2, \quad j = 1, 2, 3 \quad (21) \]

Expressing the boundary conditions with the function \( \vec{\phi}, \vec{\Psi}, \vec{w}_1 \) we get the following sets of equations in the unknown \( B_1, D_1, F_1 \)

\[ B_1 l_1' s_1 + D_1 l_2' s_2 + F_1 l_3' s_3 = 0 \]

\[ B_1 m_1' c_1 + D_1 m_2' c_2 + F_1 m_3' c_3 = 0 \]

\[ B_1 n_1 s_1 + D_1 n_2 s_2 + F_1 n_3 s_3 = 0 \quad (22) \]
where

\[ l_j' = \left\{ \sum_{l=0}^{n} (\lambda_l + 2\mu_l) (-i\eta)^l \lambda_j^2 - \sum_{l=0}^{\eta} \lambda_l (-i\eta)^l K^2 \right\} \xi_j \]

\[ -2ix \sum_{l=0}^{n} \mu_l (-i\eta)^l \lambda_j. \]

\[ m_j' = 2ix \sum_{l=0}^{n} \mu_l (-i\eta)^l \xi_j \lambda_j + \sum_{l=0}^{n} (\mu_l + \Sigma_l) (-i\eta)^l \lambda_j^2 \]

\[ + \sum_{l=0}^{n} (\mu_l - \alpha_l) (-i\eta)^l K^2 - 2 \sum_{l=0}^{n} \alpha_l (-i\eta)^l \eta_j \quad j = 1, 2, 3 \]

Eliminating the indispensable constants \( B_1, D_1, F_1 \) from the set of equations (22) we get,

\[ \begin{vmatrix}
    l_1' s_1 & l_2' s_2 & l_3' s_3 \\
    m_1' s_1 & m_2' c_2 & m_3' c_3 \\
    n_1 s_1 & n_2 s_2 & n_3 s_3
\end{vmatrix} = 0 \tag{24} \]

If the gravity field and viscous field diminished from both symmetric and anti-symmetric cases mentioned above, the results are in fair agreement with the classical theory of elasticity which were presented by Nowacki [2] Whenever, the micropolar field are removed, one can see the results are in fair agreement with the corresponding Isotropic theory of elasticity.

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SURFACE WAVES IN MICROPOLAR-THERMO-VISCO ELASTIC MEDIA UNDER THE INFLUENCE OF GRAVITY

by

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Abstract: The present paper is concerned with the investigation of surface waves in micropolar-thermo-visco elastic medium under the influence of gravitational field.

Introduction

A series of research works involving the waves and vibrations in micropolar elastic and micropolar visco-elastic solid medium have been studied by Nowacki and others [1-6].

General Theory

Let $M_1$ and $M_2$ are two homogeneous micropolar visco-elastic solid media under the influence of temperature and gravity welded in contact and separated by a plane horizontal boundary extending to infinity and $M_2$ is above $M_1$. We consider the set of orthogonal cartesian axis $o-x_1, x_2, x_3$, the origin $o$ being any point of the plane boundary and $ox_3$ pointing normally into $M_2$. The components $u_j$ and $u_3$ of the displacement vector $u$ and components $w_j, w_3$ of the rotation vector $\omega$ at any point may be expressed in the form

$$u_j = \phi_j + \psi_j, x_3 = \phi_3 - \psi_1, w_j = \Gamma_j + x_3; w_3 = \Gamma_3 - x_1$$

Where $\phi, \psi, \Gamma, x$ are functions of the co-ordinates $x_1, x_2$ and time $t$ and

$$\nabla^2 \psi = u_{1,3} - u_{3,1}, \nabla^2 \Gamma = \nabla', \nabla^2 x = w_{1,3} - w_{3,1}$$

$$\Delta = u_{1,1} + u_{3,3}; \Delta' = w_{1,1} + w_{3,3}; \nabla^2 = (\cdot)_{11} + (\cdot)_{33}$$

The equations of motion in micropolar visco-elastic solid under the influence of temperature are given by

$$(D_\mu + D_\alpha) \nabla^2 \vec{u} + (D_\lambda + D_\mu - D_\alpha) \text{ div } \vec{u} + 2D_\alpha$$

$$\text{rot } \vec{w} - (3D_\lambda + 2D_\mu/\alpha \cdot \text{ grad } \theta = \rho \vec{u}$$

(121)
The equations of motion for micropolar visco-elastic solid under the influence of temperature and gravity for the systems (1) \( (u_1, o, u_3) \) and (2) \( (o, w_2, o) \) are given by

\[
(D_v + D_e) \nabla^2 \mathbf{w} + (D_v + D_\beta - D_e) \text{grad div } \mathbf{w} - 4D_\alpha \mathbf{w} + 2D_\alpha \text{rot } \mathbf{w} = J \mathbf{w}
\]

and

\[
(D_\mu + D_\alpha) \nabla^2 u_1 + (D_\lambda + D_\mu - D_\alpha) \nabla_1 - 2D_\alpha w_{2,3} - (3D_\lambda + 2D_\mu) \alpha_1 + g_3 = u_1
\]

\[
(D_\mu + D_\alpha) \nabla^2 u_3 + (D_\lambda + D_\mu - D_\alpha) \nabla_3 + 2D_\alpha w_{2,1} - (3D_\lambda + 2D_\mu) \alpha_3
\]

\[
- \text{g } p_{1,1} = u_3
\]

\[
(D_v + D_e) \nabla^2 w_2 - 4D_\alpha w_2 + 2D_\alpha (u_{1,3} - u_{3,1}) = Jw_2
\]

\[
k \nabla^2 \theta = \rho C_e \frac{\partial \theta}{\partial t} + T_o (3D_\lambda + 2D_\mu) \frac{\partial}{\partial t} (\nabla^2 \phi)
\]

4 (a)

and

\[
(D_\mu + D_\alpha) \nabla^2 u_2 + 2D_\alpha (w_{1,3} - w_{3,1}) = \rho u_2
\]

\[
(D_v + D_e) \nabla^2 w_1 + (D_v + D_\beta - D_e) \Delta_1 - 4D_\alpha w_1 - 2D_\alpha u_{2,3} = Jw_1
\]

\[
(D_\alpha + D_\epsilon) \nabla^2 w_3 + (D_\alpha + D_\beta - D_\epsilon) \nabla_1 - 4D_\alpha w_3 + 2D_\alpha u_{2,1} = Jw_3
\]

4 (b)

where

\[
D_\mu = \sum_{r=0}^{n} \mu r \frac{\partial \rho}{\partial r}, \quad D_\lambda = \sum_{r=0}^{n} \lambda r \frac{\partial \rho}{\partial r}, \quad D_\alpha = \sum_{r=0}^{n} \alpha r \frac{\partial \rho}{\partial r},
\]

\[
D_\beta = \sum_{r=0}^{n} \beta r \frac{\partial \rho}{\partial r}, \quad D_\epsilon = \sum_{r=0}^{n} \epsilon r \frac{\partial \rho}{\partial r}, \quad D_t = \sum_{r=0}^{n} t r \frac{\partial \rho}{\partial r},
\]

and \( \lambda_0, \mu_0 \) are elastic constants, \( \alpha_0, \beta_0, \nu_0, \epsilon_0 \) are others material constants; \( \lambda_r, \mu_r, \alpha_r, \beta_r, E_r, r, (r = 1, 2, \ldots, n) \) are the parameters representing the effect of viscosity \( \theta \) is the increment of temperature from the reference state \( T_o \), \( \alpha_s \) is the coefficient of linear expansion of solid, \( \rho \) is the density of the material, \( J \) is rotational inertia and \( C_e \) is specific heat at constant strain.

Using (1), (2) in 4 (a) and 4 (b) be obtain

\[
(D_T \nabla^2 - \frac{\partial^2}{\partial t^2}) \phi - D_\alpha \theta - g \psi_1 = 0
\]

\[
(D_\alpha \nabla^2 - \frac{\partial^2}{\partial t^2}) \psi - D_\rho w_2 + g \Phi_1 = 0
\]

(122)
\[
(D_\theta \nabla^2 - D_w - \frac{\sigma}{\partial r^2})w_2 + \frac{1}{2} D_w \nabla^2 \psi = 0
\]
\[
(C^2_3 \nabla^2 - \frac{\partial}{\partial r})\theta - D_u \frac{\partial}{\partial z} (\nabla^2 \phi) = 0
\]

and
\[
(D_s \nabla^2 - \frac{\partial^2}{\partial t^2})u_2 + D_p \nabla^2 \chi = 0
\]
\[
(D_Q \nabla^2 - D_w - \frac{\partial^2}{\partial t^2}) - \frac{1}{2} D_w u_2 = 0
\]
\[
(D_L \nabla^2 - D_w \frac{\partial^2}{\partial t^2})\Gamma = 0
\]

where

\[
V^2_{r_0} = (\mu_r + \alpha_r)/\rho, \quad V^2_{r_p} = +2\alpha_r)/\rho; \quad V^2_{r_T} = (\lambda_r + 2\mu_r)/\rho; \quad \eta^2 = \frac{3\lambda_r + 2\mu_r}{\rho} \frac{\alpha_t}{p}
\]

\[
V^2_{r_Q} = (\gamma_r + \beta_r)/J; \quad V^2_{r_w} = 4\alpha_r/J, \quad \eta^2 = (2\gamma_r + \beta_r)/J, \quad C^2_3 = K/p C^2_E, \quad r = T_0/C^2_E
\]

and

\[
D_s = \sum_{r=0}^n V^2_{r_0} \frac{\partial r}{\partial t} e.t.c
\]

Eliminating \(\phi, \theta, \Psi, w_2\) from 5 (a) we obtain

\[
[ (D_s \nabla^2 - \frac{\partial^2}{\partial t^2}) (D_Q \nabla^2 - D_w - \frac{\partial^2}{\partial t^2}) + \frac{1}{2} D_p D_w \nabla^2 ]
\]

\[
[ (D_T \nabla^2 - \frac{\partial^2}{\partial t^2}) (c^2 \nabla^2 - \partial/\partial t) - r D_w \frac{\partial}{\partial t} \nabla^2 ] + g^2 \frac{\partial^2}{\partial x^2} (C^2_3 \nabla^2 - \frac{\partial^2}{\partial t^2})
\]

\[
\left\{ \left( D_Q \nabla^2 - D_w - \frac{\partial^2}{\partial t^2} \right) \right\}
\]

\[
(\phi, \theta, \Psi, w_2) = 0
\]

Eliminating \(u_2\) and \(\chi\) from 5 (b) we obtain

\[
[ (D_Q \nabla^2 - D_w - \frac{\partial^2}{\partial t^2}) (D_s \nabla^2 - \frac{\partial^2}{\partial t^2}) + \frac{1}{2} D_p D_w \nabla^2 ] (\chi, u_2) = 0
\]
We seek solution of 6 (a), 6 (b) and 5 3 (b) in the form

\[ \phi, \Theta, \Psi, \omega_2, \omega_4, \chi, \Gamma = [ \Phi (x_3), \Upsilon (x_3), \]

\[ \Psi (x_3), \bar{\omega}_2 (x_3), \bar{\omega}_4 (x_3), \Gamma (x_3) ] e^{i \xi (x_1 - \alpha)} \]  \hspace{1cm} (7)

In the medium \( M_1 \) and \( M_2 \), solutions are respectively as

\[ \bar{\Phi} = \Sigma A_j e^{-\lambda_j x_3}, \quad \bar{\Theta} = \Sigma m_j A_j e^{-\lambda_j x_3}, \quad \bar{\Psi} = \Sigma n_j A_j e^{-\lambda_j x_3}, \quad \bar{\Upsilon}_2 = \Sigma \sigma_j A_j e^{-\lambda_j x_3}, \]

\[ \bar{\chi} = E_1 e^{-\lambda_1 x_3} + E_2 e^{-\lambda_2 x_3}, \quad \Gamma = F e^{-\lambda_3 x_3}, \quad \bar{\omega}_2 = m_5 E_1 e^{-\lambda_5 x_3} + m_6 E_2 e^{-\lambda_6 x_3} \]  \hspace{1cm} (8 (a))

and

\[ \bar{\Phi} = \Sigma A_j' e^{-\lambda_j' x_3}, \quad \bar{\Theta} = \Sigma m_j' A_j' e^{-\lambda_j' x_3}, \quad \bar{\Psi} = \Sigma n_j' A_j' e^{-\lambda_j' x_3}, \quad \bar{\Upsilon}_2 = \Sigma \sigma_j' A_j' e^{-\lambda_j' x_3}, \]

\[ \bar{\chi} = E_1' e^{-\lambda_1' x_3} + E_2' e^{-\lambda_2' x_3}, \quad \Gamma = F e^{-\lambda_3' x_3}, \quad \bar{\omega}_2 = m_5' E_1' e^{-\lambda_5' x_3} + m_6' E_2' e^{-\lambda_6' x_3} \]

\[ [j = 1, 2, 3, 4] \]  \hspace{1cm} (8(b))

where \( \lambda_j^2 \) are the roots of the equation

\[ \left\{ \sum_{r=0}^{n} \bar{V}_{r}^2 (-i \eta)^{r} (-\xi^2 + \lambda^2) + \eta^2 \right\} \]

\[ + \frac{1}{2} \sum_{r=0}^{n} \bar{V}_{r}^2 (-i \eta)^{r} \sum_{r=0}^{n} \bar{V}_{r}^2 (-i \eta)^{r} (-\xi^2 + \lambda^2) \]  \hspace{1cm} (124)
\[
\begin{align*}
\eta \left\{ \sum_{r=0}^{\eta} V_{r}^{2} (-i \eta)^{r} (-\xi^{2} + \lambda^{2}) + \eta^{2} \right\} C_{3}^{2} (-\xi^{2} + \lambda^{2}) + i \eta \\
+i \eta \sum_{r=0}^{\eta} \left\{ C_{3}^{2} (-\xi^{2} + \lambda^{2}) + i \eta \right\} \\
+ \eta \sum_{r=0}^{\eta} V_{r}^{2} (-i \eta)^{r} (-\xi^{2} + \lambda^{2}) - g^{2} \xi^{2} \left\{ C_{3}^{2} (-\xi^{2} + \lambda^{2}) + i \eta \right\} \\
\left\{ \sum_{r=0}^{\eta} V_{rQ}^{2} (-i \eta)^{r} (-\xi^{2} + \lambda^{2}) - \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} + \eta^{2} \right\} = 0 \quad 9 (a)
\end{align*}
\]

and \(\lambda_{2}^{2}, \lambda_{6}^{2}\) are the roots of the equation

\[
\begin{align*}
\sum_{r=0}^{\eta} V_{rQ}^{2} (-i \eta)^{r} (-\xi^{2} + \lambda^{2}) - \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} + \eta^{2} \\
+ \frac{1}{2} \sum_{r=0}^{\eta} V_{rp}^{2} (-i \eta)^{r} \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} (-\xi^{2} + \lambda^{2}) \right\} = 0 \quad 9 (b)
\end{align*}
\]

and

\[
\lambda_{j}^{2} = \xi^{2} + \left( \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} - \eta^{2} \right) / \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} \right\}; \eta = \xi c \quad 9 (c)
\]

Also

\[
\eta_{j} = \frac{1}{8 i \xi} \left[ \left\{ \sum_{r=0}^{\eta} V_{r}^{2} (-i \eta)^{r} (-\xi^{2} + \lambda_{j}^{2}) + \eta^{2} \right\} - \left\{ \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} \right\} m_{j} \right]
\]

\[
m_{j} = \frac{-i \eta \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r}}{C_{3}^{2}(\lambda_{j}^{2} - \xi^{2}) + i \eta} \right\} \quad 0_{j} = \frac{-1}{2} \left\{ \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} (\lambda_{j}^{2} - \xi^{2}) \right\} n_{j}
\]

\[
\sum_{r=0}^{\eta} V_{rQ}^{2} (-i \eta)^{r} \left( \lambda_{j}^{2} - \xi^{2} \right) - \sum_{r=0}^{\eta} V_{rw}^{2} (-i \eta)^{r} + \eta^{2}
\]

(125)
\[
\frac{m_l}{r} = \left( \sum_{r=0}^{\eta} V_{rp}^2 \left( -i \eta \right)^r \left( \xi^2 + \lambda_1^2 \right) / \sum_{r=0}^{\eta} V_{rs}^2 \left( -i \eta \right)^r \left( \xi^2 + \lambda_2^2 \right) + \eta^2 \right) \quad [l = 5, 6]
\]

Similar solutions can be derived for the Medium M_2.

3. Boundary Conditions and Solution of the Problem

(A) The boundary conditions for the system (1) are (i) The displacement vector \( \vec{u} \) (\( u_1, o, u_3 \)) and rotation vector \( \vec{w} \) (\( o, w_2, o \)), temperature and its flux at the boundary surface must be continuous at all times and places (ii) The components \( \sigma_{31}, \sigma_{33} \) and \( \mu_{32} \) across the boundary surface are continuous at all times and places.

(B) The boundary conditions for the system (2) are (iii) the displacement \( u_2 \) and the rotations \( w_2, w_3 \) at the boundary surface must be continuous at all times and places (iv) the components \( \sigma_{32}, \mu_{31} \) and \( \mu_{33} \) across the boundary surface are continuous at all times and places.

where

\[
\sigma_{33} = 2D_\mu \left[ \phi_{33} - \psi_{13} \right] + D_\lambda \nabla^2 \phi \cdot (3D_\lambda + 2D_\mu \alpha_c \theta)
\]
\[
\sigma_{31} = D_\mu \left[ 2 \phi \cdot \phi_{13} + \psi \cdot \psi_{13} \right] + D_\alpha \left( \nabla^2 \psi \cdot 2w_2 \right)
\]
\[
\mu_{32} = (D_v + D_\epsilon) \nabla_2,3 : \sigma_{32} = (D_\mu - D_\alpha) u_{2,3} \cdot 2D_\alpha (\Gamma_1 + x_3)
\]
\[
\mu_{31} = D_\gamma \left[ 2 \Gamma_{13} + x_{33} - x_{11} \right] + D_\epsilon \nabla^2 \psi
\]
\[
\mu_{33} = (2D_\gamma + D_\beta) \left[ \Gamma_{33} + x_{13} \right] + D_\beta \left[ \Gamma_{11} + x_{11} \right]
\]

Using (1), (7), 8(a), 8(b), 11 and boundary conditions (A) we obtain eight simultaneous equations from which we get on eliminating the unknown constants \( A_j, A_j', [j = 1,2,3,4] \) the frequency equations for the first system as

\[
\nabla_2 = \left| a_j \right| = 0
\]

using (1), (7), 8(a), 8(b), (11) and boundary conditions (B) we get six simultaneous equations from which we obtain on eliminating the constants \( E_1, E_2, F, E_1', E_2', F' \) the frequency equation for the second system as

\[
\nabla_3 = \left| b_j \right| = 0
\]

where, \( a_{ij} = n_j \lambda_j - i \xi_j, a_{nm} = i \xi_j + \lambda'_m n_m, a_{2j} = \lambda_j + i \xi_j n_j, a_{2m} = \lambda'_m + i \xi_j n_m \)

[\( j = 1,2,3,4 \) (\( m = 5,6,7,8 \) and \( a_{6m}, a_{6m'} \) etc. we known but not given here to same space]

\( (126) \)
\[ b_{11} = m_5, b_{12} = m_6, b_{13} = 0, b_{14} = -m_5', b_{15} = -m_6', b_{16} = 0 \]

\[ b_{21} = \lambda_5 b_{22} = \lambda_6, b_{23} = -i \xi b_{24} = \lambda_5', b_{25} = \lambda_6', b_{26} = i \xi b_{31} = -i \xi = b_{32}, b_{33} = \lambda_7 b_{34} = i \xi = b_{35} b_{36} = \lambda_7' \]

\[ \{ i = 1, j = 5, i = 2, j = 6 \} \{ m = 4, j = 5, m = 5, j = 6 \} \]

Besides, this, less, bumare are not given to save space

4. Particulars Cases

RAYLEIGH WAVES. For the existence of Rayleigh waves the plane boundary is to be a free surface so that \( M_2 \) is replaced by Caucum. Here the boundary conditions are \( \sigma_{33} = \sigma_{31} = \mu_{33} = 0 \) on the plane boundary of \( M_1 \) with an additional thermal boundary condition

\[ \phi_3 + \hbar = 0 \quad on \quad x_3 = 0, \hbar = \text{plank's constant} \quad (14) \]

Using the above boundary conditions we get the wave velocity equation of Rayleigh waves in the present medium for the first system as

\[ \mathbf{\nabla}_4 = \begin{vmatrix} a_{61} & a_{62} & a_{63} & a_{64} \\ a_{71} & a_{72} & a_{73} & a_{74} \\ \lambda_1 a_{31} & \lambda_2 a_{32} & \lambda_3 a_{33} & \lambda_4 a_{34} \\ (h - \lambda_1) a_{41} (h - \lambda_2) a_{42} (h - \lambda_3) a_{43} (h - \lambda_4) a_{44} \end{vmatrix} = 0 \quad (15) \]

where

\[ a_{31} = \theta_1, a_{32} = \theta_2, a_{33} = \theta_3, a_{34} = \theta_4, a_{41} = m_1, a_{42} = m_2, a_{43} = m_3, a_{44} = m_4 \]

For the second system the boundary conditions are \( \sigma_{32} = \mu_3 = \mu_{33} = 0 \) over the plane boundary of \( M_1 \), proceeding above we obtain the wave velocity equations of Rayleigh waves in the present medium for the second system as

\[ \Delta_2 = \begin{vmatrix} b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \\ b_{61} & b_{62} & b_{63} \end{vmatrix} = 0 \quad (16) \]

Love Waves: For the love type of surface waves we assume that \( M_2 \) is bounded by two horizontal plane surface at a finite distance \( H \) apart. The upper plane surface being
free while the lower plane surface forms the medium $M_1$ and extends to infinity is sufficient to consider the component $U_2$ of displacement vector $\mathbf{u}$ and rotation vector $\mathbf{w}$ Here $\mathbf{u}$ and $\mathbf{w}$ in $M_2$ may no longer diminish with the distance on the boundary between $M_1$ and $M_2$, i.e. for the medium $M_2$ we preserve the first-order term.

**Stoneley Waves**: In the classical theory, Stoneley waves are a generalization of Rayleigh waves propagating along the common boundary of $M_1$ and $M_2$. The Stoneley waves for the present problem are determined by the roots of the frequency equations (12) and (13).

**Discussion**

It is important to note that for the present problem Rayleigh waves are all the fields but the Love waves are affected only by the viscous field.

**References**

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Reissner-Sagoci problem in magneto-elasticity

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[ Abstract: The present paper is concerned with the problem of determining the components of stresses and displacements in the interior of a homogeneous, isotropic, conducting elastic semi-infinite solid subjected to a homogeneous axial magnetic field \( \mathbf{H} = (0, 0, H_0) \) when a part of this circular area of the surface of the medium is forced to rotate through an angle about an axis which is normal to the undeformed surface of the medium. In absence of the magnetic field the results are in complete agreement with the corresponding classical problem of Reissner and Sagoci. ]

1. Introduction

For the last four decades a new domain has been developed in which investigations concern the interactions between the strain and electromagnetic field. This new discipline is called magneto-elasticity (Nowacki). In classical theory of elasticity Reissner and Sagoci in their investigations have considered the torsional oscillations produced in a semi-infinite homogeneous isotropic medium by a periodic shear stress applied in an axially symmetric manner to a circular area of the plane surface of the boundary of the medium. The distribution of stresses in the interior of a semi-infinite elastic medium is determined when a load is applied to the surface by means of a rigid disc and it is assumed that the part of the boundary which lies beyond the edge of the disc is free from stress. The solution of the mixed
boundary value problem is obtained by Reissner and Sagoci through the introduction of a certain system of oblate spheroidal co-ordinates. Here the authors have studied the same problem considering the angle of rotation $\varphi$ (Sneddon) under the influence of magnetic field by the use of Hankel transformation to the solution of a pair of dual integrals. The authors have considered the small perturbations characterized by the displacement vector $u[0, u_\varphi(r, s, t), 0]$ and also assume the perturbations to be independent of the angle $\theta$ as studied in the book of Parton and Perlin.

2. Basic equations and relations

Maxwell's equations for an electro-magnetic field can be written in the following form:

\[
\begin{align*}
\text{curl } \vec{H} &= j + \frac{\partial \vec{D}}{\partial t}, \\
\text{div } \vec{B} &= 0, \\
\text{curl } \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\
\text{div } \vec{D} &= \rho_e, \\
\vec{D} &= \varepsilon \vec{E}, \\
\vec{B} &= \mu_0 \vec{H}.
\end{align*}
\]

We supplement these equations by Hooke's law.

In the presence of electromagnetic fields, the equations of motion in absence of body forces

\[
\frac{\partial T_{1,1}}{\partial x_1} = \rho \frac{\partial^2 u_{11}}{\partial t^2},
\]

assume the form

\[
\rho \frac{\partial^2 u_{11}}{\partial t^2} = \frac{\partial T_{1,1}}{\partial x_1} + (j \times \vec{B})_1 + \nu_e \vec{E}_1.
\]

Finally the current in a moving conductor is defined by the generalized Ohm's law

\[
j = \sigma \left[ \vec{E} + \left( \frac{\partial \vec{u}}{\partial t} \times \vec{B} \right) \right] + \nu_e \frac{\partial \vec{u}}{\partial t},
\]

where $\vec{H}$ is the magnetic field vector, $\vec{E}$ is the electric field vector, $j$ is the current density vector, $\nu_e$ is the electric charge...
density, $\mathbf{B}$ is the magnetic induction vector, $\mathbf{D}$ is the electric-induction vector, $\mu_s$ is the magnetic permeability, $\epsilon$ is the electric permittivity, $\rho$ is the density of the medium, $\sigma$ is the electrical conductivity.

3. General theory and boundary conditions

Let us consider a semi-infinite solid $z \geq 0$ subjected to a homogeneous axial magnetic field $\mathbf{H}(0, 0, H_0)$. The medium has infinite conductivity and permeability of the vacuum is $\chi_0 = 4\pi \times 10^{-7} \frac{H}{m} \left( \frac{N}{A} \right)$ (Parton and Perlin). We introduce a cylindrical system of co-ordinates, in which the $z$-axis is directed along the axis of symmetry of the medium. A circular area $r = r_0$ of the surface is forced to rotate through an angle $\varphi$ about an axis which is normal to the undeformed surface of the medium. It is assumed that the region of the surface lying outside the circle $r > r_0$ is free from stress. It has been shown by Reissner that in this case only the circumferential component of the displacement vector is different from zero and that all the stress component vanishes except $\tau_{z\theta}$ and $\tau_{\theta\theta}$ which are given by the relations (Sneddon)

$$\tau_{z\theta} = \mu \frac{\partial u_\theta}{\partial z}, \quad \tau_{\theta\theta} = \mu \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right). \quad (5)$$

The boundary conditions of the problem are (Sneddon)

$$u_\theta = f(r, t), \quad z = 0, \quad r < r_0, \quad \cdots (6).$$

$$\tau_{z\theta} = 0, \quad z = 0, \quad r > r_0. \quad \cdots (7)$$

In the case considered by Reissner and Sagoci, the surface displacement $f(r, t)$ is of the form

$$f(r, t) = r \varphi(t) \quad \cdots (8)$$

and the relation between angle $\varphi$ and the applied torque $T$ may be derived from the equation

$$T = 2\pi \int_0^{r_0} r \tau_{z\theta}(r, 0) r^2 \, dr. \quad \cdots (9).$$
Formulation of the ‘problem

Let us now consider the perturbation of the Maxwell's electro-magnetic field in the form $\vec{H} = \vec{H}_0 + \vec{h}$ and $\vec{E} = \vec{E}_0 + \vec{e}$, where $\vec{h}$ and $\vec{e}$ are perturbations in the magnetic and electric fields respectively. We then linearize the basic equations describing the motion in the presence of elastic, electric and magnetic fields. In the M.K.S. system the linearized system of equations has the form (Suhubi):

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{x_0 H_0}{\mu} \frac{\partial h_\theta}{\partial z} = \frac{1}{0^2} \frac{\partial^2 u_\phi}{\partial z^2}, \quad \ldots \ (10)$$

$$e_r + x_0 H_0 \frac{\partial u_\theta}{\partial t} = 0 \quad \left[ G_1 = \frac{\mu}{\rho} \right], \quad \ldots \ (11)$$

$$\frac{\partial h_\theta}{\partial z} + x_0 \frac{\partial h_\phi}{\partial t} = 0, \quad \ldots \ (12)$$

Here $e(r, 0, 0)$ and $h(0, h_\phi, 0)$ are perturbations arising in the electric and magnetic fields respectively and $G_1$ is the velocity of the transverse waves. To solve the partial differential equations (10), (11) and (12) we introduce the Hankel transform as

$$U = \int_0^\infty \nu u_\theta J_1(\xi r) \, dr,$$

$$V = \int_0^\infty r h_\phi J_1(\xi r) \, dr,$$

$$W = \int_0^\infty r h_\phi J_1(\xi r) \, dr.$$ \quad (13)

Multiplying both sides of the above equations (10), (11) and (12) by $\pi J_1(\xi r)$ and integrating with respect to $r$ from 0 to $\infty$, we obtain

$$\left( \frac{1}{G_1^2} \frac{\partial^2}{\partial z^2} - \frac{1}{\partial z^2} + \xi^2 \right) U = \frac{x_0 H_0}{\mu} \frac{\partial}{\partial z} V, \quad \ldots \ (14)$$

$$W + x_0 H_0 \frac{\partial U}{\partial t} = 0,$$

$$\frac{\partial W}{\partial z} + x_0 \frac{\partial}{\partial t} V = 0.$$
Eliminating $U$, $V$, and $W$ from these equations, we obtain
\[
\left[ \frac{1}{C^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + \xi^2 \right] \left( \frac{\chi_e H^2}{\mu} \frac{\partial^2}{\partial z^2} \right) (U, V, W) = 0. \quad \ldots \quad (15)
\]

For the determination of $u_\theta$, $\varepsilon_\theta$ and $h_\theta$ in terms of the solutions $U$, $V$ and $W$, we use the formulae (Sneddon9)
\[
u_\theta = \int_0^\infty \xi U(z, t, \xi) J_1(\xi r) \, d\xi, \\
\varepsilon_\theta = \int_0^\infty \xi \varepsilon V(z, t, \xi) J_1(\xi r) \, d\xi, \\
h_\theta = \int_0^\infty \xi W(z, t, \xi) J_1(\xi r) \, d\xi. \quad \ldots \quad (16)
\]

The non-vanishing component of the stresses are (Sneddon8).
\[
\tau_{rr} = \mu \int_0^\infty \xi^2 U(z, t, \xi) \left[ J_0(\xi r) - J_2(\xi r) \right] \, d\xi, \quad \ldots \quad (17)
\]
\[
\tau_{zz} = \mu \int_0^\infty \xi^2 U(z, t, \xi) J_1(\xi r) \, d\xi. \quad \ldots \quad (18)
\]

The arbitrary functions introduced in the solution of equation (16) are determined by the boundary conditions. The solution must be such that the displacement and both components of stresses tend to zero as $z \to \infty$. When $z = 0$, the conditions (6) and (7) must be satisfied. Substituting there relations into expressions (6), (7), (16) and (18), we obtain the dual integral equations as
\[
\int_0^\infty \xi U(0, t, \xi) J_1(\xi r) \, d\xi = f(r, t), \quad r < r_0, \quad \ldots \quad (19)
\]
\[
\int_0^\infty \xi \frac{\partial U}{\partial z}(0, t, \xi) J_1(\xi r) \, d\xi = 0, \quad r > r_0. \quad \ldots \quad (20)
\]

5. Solution in the static case

As studied by Sneddon8, we take $\frac{\partial}{\partial t}$ to be identically zero in the equation (15) and in addition we are interested in the-
interaction between elastic and magnetic fields so that (15) reduces to the following form:

\[
\left(1 + \frac{X_0 H_0^2}{\mu}\right) \frac{d^2}{dx^2} - \xi^2 \right)(U, V) = 0, \quad \ldots \quad (21)
\]

where the relationship between \(U\) and \(V\) is given by the first equation of (14).

Remembering the finiteness condition at infinity, the solution can be written as

\[
U = A(m'\xi) e^{-m'\xi z}, \quad \ldots \quad \text{(22)}
\]

\[
V = C(m'\xi) e^{-m'\xi z},
\]

where \(m' = \sqrt{1 + h_0}, \quad h_0 = \frac{X_0 H_0^2}{\mu}\).

The relation between \(A(m'\xi)\) and \(C(m'\xi)\) in static case can be obtained from the first equation of (14) considering \(\frac{\partial}{\partial t}\) to be identically zero as

\[
C(m'\xi) = -\xi u(1 - m^2) \frac{X_0 H_0}{m'} m', \quad A(m'\xi) = \frac{\xi H_0}{m'} A(m'\xi). \quad \ldots \quad (23)
\]

Substituting the value of \(U\) from (22) into the equations (19) and (20), we obtain the dual integral equations

\[
\int_0^\infty \nu^{-1} F(\eta) J_r \left(\frac{\rho_o}{m}, \eta\right) d\eta = g \left(\frac{\rho_o}{m}\right), \quad \ldots \quad (24)
\]

\[
\int_0^\infty F(\eta) J_r \left(\frac{\rho_o}{m}, \eta\right) d\eta = 0, \quad \ldots \quad (25)
\]

where

\[
\nu = \rho_o v_0, \quad m'\xi = \eta v_0, \quad f(r) = \frac{\nu}{m^2} g \left(\frac{\rho_o}{m}\right),
\]

\[
\eta^2 A \left[\frac{\eta}{v_0}\right] = r_0^2 F(\eta). \quad \ldots \quad (26)
\]

We obtain the solution putting \(\alpha = -1, \nu = 1\) in the result obtained by Busbridge considering \(g \left(\frac{\rho_o}{m}\right) = \frac{\rho_o}{\rho_m} \), so that

\[
F(\eta) = \frac{4\psi}{\pi} \left\{ \sin \eta - \cos \eta \right\}, \quad \ldots \quad (27)
\]
giving finally

\[
\begin{align*}
\mathbf{u}_\theta &= \frac{4\varphi r_0}{\pi m'^2} \int_0^\infty \sin \eta - \eta \cos \eta \frac{e^{-\xi \eta}}{\eta^2} J_1 \left( \frac{\rho_0}{m'} \eta \right) \, d\eta, \\
\mathbf{u}_\phi &= \frac{4\varphi r_0}{\pi m'^2} \int_0^\infty \sin \eta - \eta \cos \eta \frac{e^{-\xi \eta}}{\eta^2} J_1 \left( \frac{\rho_0}{m'} \eta \right) \, d\eta,
\end{align*}
\]

(28)

where \( \xi = \frac{2}{r_0} \) and \( \mathbf{u}_\phi \) can be written as

\[
\begin{align*}
\mathbf{u}_\phi &= \frac{4\varphi r_0}{\pi m'^2} \left[ S^1_\phi - C^1_2 \right], \\
\tau_{x\phi} &= -\frac{4\varphi \mu}{\pi} \left[ S^1_\phi - C^1_2 \right],
\end{align*}
\]

(29)

where

\[
\begin{align*}
S^m_n \left( \frac{\rho_0}{m'}, \xi \right) &= \int_0^\infty p^{m-n} \sin \left( \frac{p}{2} \right) J_m \left( \frac{\rho_0}{m'} \eta \right) e^{-\eta} \, d\eta, \\
C^m_n \left( \frac{\rho_0}{m'}, \xi \right) &= \int_0^\infty p^{m-n} \cos \left( \frac{p}{2} \right) J_m \left( \frac{\rho_0}{m'} \eta \right) e^{-\eta} \, d\eta,
\end{align*}
\]

(30)

When \( \xi = 0 \) it can be easily shown that

\[
\begin{align*}
S^1_\phi &= \frac{m'}{2 \rho_0} \left[ \frac{1}{m'^2} - 1 \right]^{\frac{3}{2}} + \frac{\rho_0}{m'} \left[ \frac{1}{2} \tan^{-1} \left( \frac{\rho_0}{m'} - 1 \right) \right] \left[ \frac{\rho_0}{m'} > 1 \right], \\
C^1_2 &= \frac{(\rho_0 - m'^2)^{\frac{1}{2}}}{\rho_0}, \\
S^1_\phi &= \frac{m'}{2 \rho_0} \left[ 1 - \left( 1 - \frac{\rho_0}{m'} \right) \right] \left[ \frac{\rho_0}{m'} < 1 \right], \\
C^1_2 &= \frac{m'}{2 \rho_0} \left[ 1 - \left( 1 - \frac{\rho_0}{m'} \right) \right] \left[ \frac{\rho_0}{m'} < 1 \right].
\end{align*}
\]

(31)

(32)

So

\[
\begin{align*}
\mathbf{u}_\phi &= \frac{4\varphi r_0}{\pi m'} \left[ \frac{1}{\rho_0} \tan^{-1} \left( \frac{\rho_0}{m'} - 1 \right) \right] - \frac{2}{\pi m'^2} \left( \frac{m'^2}{\rho_0} \right)^{\frac{3}{2}} \left[ \frac{\rho_0}{m'} > 1 \right], \\
\tau_{x\phi} &= -\frac{4\varphi H}{\pi} \left[ \left( \frac{1}{\rho_0} \right)^{\frac{3}{2}} \right] \left[ \frac{\rho_0}{m'} > 1 \right].
\end{align*}
\]

(33)

(34)

Using (16) and (22), \( \mathbf{h}_\phi \) can be easily calculated.
6. Discussion

Like all other problems in magneto-elasticity, it has been assumed here that the heat exchange between the parts of the solid is slow. It is obvious from the above expressions that the circumferential component of displacement $u_\theta$ has been affected by the magnetic field. When the magnetic field is weak or absent $h_\epsilon \to 0$, $m' \to 1$ and then

$$u_\theta = \psi r_0 \left[ \frac{1}{\pi} \arctan \left( \frac{\rho_0}{\rho_0 - 1} \right) \right] - \frac{2}{\pi} \left( \frac{1}{\rho_0 - 1} \right)^{1/2} \rho_0 > 1,$$

$$\tau_{\theta\theta} = -\frac{4\psi \mu}{\pi} \left[ \left( \frac{1}{\rho_0^2 - 1} \right)^{-1/2} \right] \rho_0 < 1,$$

which are in agreement with the corresponding result in classical problem as discussed by Reissner and Sagoci and presented in the standard book of Sneddon.

References

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INFLUENCE OF GRAVITY ON THE PROPAGATION OF WAVES IN A MEDIA IN PRESENCE OF A COMPRESSIONAL WAVE SOURCE

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ABSTRACT

The present paper is concerned with the investigation of the influence of gravity on the propagation of waves in a homogeneous classic solid half-space underlying a liquid layer when a compressional wave source is present in the solid substratum. In the absence of the gravitational field, the results are in agreement with the results of corresponding classical problem.

1. INTRODUCTION

Using the theory of plane waves, Stoneley [1] investigated the effect of the ocean on transmission of Rayleigh waves and calculated phase and group velocities for a water layer assumed to be three km thick over a solid substratum. He confined his attention to the longer-period Rayleigh waves and concluded that the effect of the water layer was unimportant. In a note added to that paper, Jeffreys proved from Stoneley’s equation that there existed a maximum of group velocity at some period shorter than those investigated by Stoneley. Scholte [2] while attempting to explain microseism generation by transfer of energy from gravity surface waves to elastic waves in the bottom, considered the combined effect of gravity and compressibility in a layer of water in contact with an elastic solid bottom. Sommerfeld, [3] Jeffreys [4], Muskat [5] and others have discussed wave propagation in two mediums separated by a plane interface for the case where the distance of a point source from the surface of separation is finite.

In the above classical theory the effect of gravity has been neglected. Bromwich [6] was the first who discussed the influence of gravity on the vibrations of an elastic globe treating the force of gravity as type of body force. Then the effect of gravity on Rayleigh waves has been investigated by Biot [7] on the assumption that the force of gravity creates a type of initial stress hydrostatic in nature.

Assuming the boundaries are parallel planes and the waves are smaller than ordinary earthquake Rayleigh waves, the authors have studied the axis symmetric problem of propagation of waves in a liquid layer overlying a solid half-space under the action of gravitational field when the compressional wave source being present in the lower solid substratum.

2. BASIC EQUATIONS AND RELATIONS

The equations of motion for an isotropic elastic solid medium under no body forces in the Cartesian Co-ordinate System are

\[
\begin{align*}
\rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial z^2} \\
\rho \frac{\partial^2 w}{\partial t^2} &= (\lambda + \mu) \frac{\partial^2 w}{\partial z^2} + \mu \frac{\partial^2 w}{\partial x^2}
\end{align*}
\]

(2.1)

The above equations can be written in the Polar Co-ordinate System \((r, \theta, z)\) as [8]

\[
\begin{align*}
(\lambda + 2\mu) \left( \frac{\partial^2 q}{\partial r^2} + \frac{1}{r} \frac{\partial q}{\partial r} - \frac{q}{r^2} + \frac{\partial^2 w}{\partial z^2} \right) + \mu \left( \frac{\partial^2 q}{\partial z^2} - \frac{\partial^2 w}{\partial \theta^2} \right) &= \rho \frac{\partial^2 q}{\partial t^2} \\
(\lambda + 2\mu) \left( \frac{\partial^2 q}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial q}{\partial \theta} + \frac{\partial^2 w}{\partial z^2} \right) - \mu \left( \frac{\partial^2 q}{\partial z^2} - \frac{\partial^2 w}{\partial \theta^2} \right) &= \rho \frac{\partial^2 w}{\partial t^2}
\end{align*}
\]

(2.2)

where \(\lambda, \mu\) are Lamé elastic constants, \(u\) and \(w\) are the displacements in the direction of \(x\) and \(z\) axes, \(\theta\) is the cubical dilatation, and \(q\) and \(w\) are the displacements in the \(r\) and \(z\) directions, respectively.
Now we define two functions $\phi_2$ and $W_2$ as [8],

$$
\dot{e} = \frac{\partial \phi_2}{\partial t} + \frac{\partial W_2}{\partial z}, \quad \dot{w} = \frac{\partial \phi_2}{\partial t} + \frac{1}{r} \frac{\partial (r W_2)}{\partial r}, \quad W_2 = -\frac{\partial \psi_2}{\partial r} \tag{2.3}
$$

using eqn (2.3) in eqn (2.2) we obtain

$$
\nabla^2 \phi_2 = \frac{1}{\alpha_2^2} \frac{\partial^2 \phi_2}{\partial t^2}, \quad \nabla^2 \psi_2 = -\frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial r^2} \tag{2.4}
$$

where

$$
\alpha_2^2 = \left( \lambda_2 + 2\mu_2 \right), \quad \beta_2 = \frac{\mu_2}{\rho_2} \tag{2.5}
$$

Here $\alpha_2$ is the velocity of compressional waves, $\beta_2$ is the velocity of distortional waves and $\rho_2$ is the density of the solid medium.

3. GENERAL THEORY AND BOUNDARY CONDITIONS

We consider a gravitating and compressible liquid layer of width $H$ bounded by the parallel planes $z = 0$ and $z = H$ lying over a solid semi-space. The surface of separation being $z = H$ and the $z$-axis is taken vertically downwards. Henceforth we shall use the subscript (1) for liquid and the subscript (2) for the solid part, respectively. The displacements may be expressed in terms of the potentials $\phi_1$, $\phi_2$, $\psi_2$ as [8]

$$
q_1 = \frac{\partial \phi_1}{\partial t}, \quad w_1 = \frac{\partial \phi_1}{\partial z} \tag{3.1}
$$

$$
q_2 = \frac{\partial \phi_2}{\partial t} + \frac{\partial^2 \psi_2}{\partial z \partial t}, \quad w_2 = \frac{\partial \phi_2}{\partial t} + \frac{\partial^2 \psi_2}{\partial z^2} - \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial r^2} = \frac{\partial \phi_2}{\partial t} + \frac{\partial^2 \psi_2}{\partial z^2} + K \frac{\partial^2 \psi_2}{\partial z^2} \tag{3.2}
$$

where

$$
K \rho_2 = \frac{\alpha_2^2}{\beta_2} \tag{3.3}
$$

A time factor $\exp(\alpha_2 t)$ is understood with the expressions for the potentials.

**Boundary conditions:**

$$
\phi_1 = 0 \quad \text{at} \quad z = 0 \tag{3.4}
$$

$$
w_1 = w_2, \quad (p_x)_1 = 0, \quad (p_x)_2 = (p_x)_2 \quad \text{at} \quad z = H \tag{3.5}
$$

where the components of stresses

$$
(p_{zz})_1 = \lambda_1 \nabla^2 \phi_1, \quad (p_{zz})_2 = \lambda_2 \nabla^2 \phi_2 + 2\mu_2 \frac{\partial \psi_2}{\partial z}
$$

$$
(p_{xx})_2 = \mu_2 \left( \frac{\partial q_2}{\partial z} + \frac{\partial w_2}{\partial t} \right) \tag{3.6}
$$

4. SOLUTION OF THE PROBLEM

Here we consider the case when the waves are smaller than ordinary earthquake Rayleigh waves. Now under the influence of gravity the velocity potential $\phi_1$ for liquid satisfies the eqn [8, 9]

$$
\frac{\partial^2 \phi_1}{\partial t^2} = \alpha_1^2 \nabla^2 \phi_1 + g \frac{\partial \phi_1}{\partial z} \tag{4.1}
$$

Eqn (4.1) may be written in terms of the displacement potential $\phi_1$ as

$$
\frac{\partial^2 \phi_1}{\partial t^2} = \alpha_1^2 \nabla^2 \phi_1 + g \frac{\partial \phi_1}{\partial z} \tag{4.2}
$$

where $\alpha_1$ is the velocity of the compressional waves in the liquid. The gravity terms in the equations for the solid part of the system has been omitted by Scholte [2], so for the solid part we use eqns (2.4).

In order to satisfy the boundary condition (3.4), $\phi_1$ may be written as [8]

$$
\phi_1 = \int_0^\infty A(K)J_0(Kr) e^{-2\alpha_1^2 \sin \theta} z \, dk, \quad (0 \leq z \leq H) \tag{4.3}
$$
where \( \bar{\nu}_1 = \sqrt{v_1^2 - \frac{g^2}{\omega^2}} \) and \( v_1^2 = K_2a_1^2 - K_2^2 \), \( K_{\alpha_1} = \frac{\omega}{\alpha_1} \) (4.4)

We know a source in a solid half-space can produce both compressional and distortional waves. Now an expression for spherical waves emitted by the point source \((0, H + d = h)\) may be written in the form \([8]\)

\[
\phi_0 = \int_0^\infty \frac{K}{\nu_2} e^{-\nu_2 z^2 + \nu_2 h} J_0(\nu_2 r) d\nu_2
\]

where \( \nu_2 = K_2K_{\alpha_2}^2 - K_2^2 \), \( K_{\alpha_2} = \frac{\omega}{\alpha_2} \) (4.5)

The expressions for \( \phi_2 \) and \( \psi_2 \) can be written in the following form \([8]\)

\[
\phi_2 = \phi_0 + \int_0^\infty Q_2(K) J_0(\nu_2 r) e^{-\nu_2 z^2 + \nu_2 h} dK
\]

\( \psi_2 = \int_0^\infty S_2(K) J_0(\nu_2 r) e^{-\nu_2 z^2 + \nu_2 h} dK \) (4.6)

where \( \nu_2 = K_2^2 - K_2^2 \), \( K_{\beta_2} = \frac{\omega}{\beta_2} \)

If \( Z < H + d \) we have

\[
\phi_2 = \int_0^\infty \frac{K}{\nu_2} e^{-\nu_2 (z - H - d)} J_0(\nu_2 r) dK + \int_0^\infty Q_2 J_0(\nu_2 r) e^{-\nu_2 z^2 + \nu_2 h} dK
\]

\( \psi_2 = \int_0^\infty S_2(J_0(\nu_2 r) e^{-\nu_2 z^2 + \nu_2 h} dK \) (4.7)

Using the above expressions for \( \phi_1, \phi_2 \) and \( \psi_2 \) in the boundary conditions given in eqn (3.5) we obtain the following:

\[
\begin{align*}
\frac{\partial H}{\partial t} & = -\frac{gH}{2a} \sin \frac{\omega \rho_1}{2a} + \frac{\omega^2}{2a} \sin \frac{\omega \rho_1}{2a} \cos \frac{\omega \rho_1}{2a} + gH \\
\frac{\partial S_2}{\partial t} & = -2\frac{\omega^2}{\beta_2} \sin \frac{\omega \rho_1}{2a} + \frac{\omega^2}{\beta_2} \sin \frac{\omega \rho_1}{2a} \cos \frac{\omega \rho_1}{2a}
\end{align*}
\]

(4.8)

(4.9)

The values of \( A, Q_2 \) and \( S_2 \) in terms of \( K \) and other parameters can be found if the determinant \( \Delta \) of the above equations is not equal to zero

Now using \( \mu_2 = \rho_2 \beta_2 \) we can write the expression for \( \Delta \) as

\[
\Delta(K) = e^{\frac{gH}{2a \omega}} \left[ \frac{\bar{\nu}_1 \cos \bar{\nu}_1 H - \frac{g}{2a \omega} \sin \bar{\nu}_1 H}{\rho_2 \beta_2} \right] + \rho_1 \left[ \frac{\omega^2 \sin \bar{\nu}_1 H + g \left( \bar{\nu}_1 \cos \bar{\nu}_1 H - \frac{g}{2a \omega} \sin \bar{\nu}_1 H \right)}{\rho_2 \beta_2} \right]
\]

(4.10)

(4.11)

Now \( A, Q_2 \) and \( S_2 \) may be written in the following form

\[
A = -2\rho_2 \omega^2 \left( 2K^2 - K_{\beta_2}^2 \right) \left( \frac{K}{\Delta} \right) e^{-\nu_2 d}
\]

(4.12)

\[
Q_2 = \frac{Ke^{-\nu_2 d} e^{\nu_2 (H - d)}}{\nu_2 \Delta} \left[ \rho_1 \left( \frac{\omega^2 \sin \bar{\nu}_1 H + g \left( \bar{\nu}_1 \cos \bar{\nu}_1 H - \frac{g}{2a \omega} \sin \bar{\nu}_1 H \right)}{\rho_2 \beta_2} \right) \right]
\]

(4.13)

\[
S_2 = \frac{Ke^{-\nu_2 d} e^{\nu_2 (H - d)}}{\nu_2 \Delta} \left[ \rho_1 \left( \frac{\omega^2 \sin \bar{\nu}_1 H + g \left( \bar{\nu}_1 \cos \bar{\nu}_1 H - \frac{g}{2a \omega} \sin \bar{\nu}_1 H \right)}{\rho_2 \beta_2} \right) \right]
\]

(4.14)

(4.15)

(4.16)

(4.17)
\[
S_2 = -4K \frac{(2\mu_2 K^2 - \rho_2 \omega^2)}{\Delta} \exp \left( \frac{gH}{2\alpha_1} \right) \left( n_1 \cos n_1 H - \frac{g}{2\alpha_1} \sin n_1 H \right) e^{-iv_2d + iv_2H} \tag{4.18}
\]

Now substituting the above expressions for \( A, Q_2 \) and \( S_2 \) in eqns. (4.3), (4.7) and (4.8) to obtain \( \phi_1, \phi_2 \) and \( \psi_2 \) as

\[
\phi_1 = -\frac{\rho_2 \omega^2}{\Delta} (2K^2 - K_{p2}^2) e^{-iv_2d - \frac{gH}{2\alpha_1}} \int_0^{\infty} J_0(Kr) \sin n_1 r \, dK \tag{4.19}
\]

\[
\phi_2 = \int_0^{\infty} e^{-iv_2|z-H-d|} J_0(Kr) \sin n_1 r \, dK + \int_0^{\infty} e^{-iv_2(z-H-d)} J_0(Kr) \sin n_1 r \, dK \tag{4.20}
\]

\[
\psi_2 = -\frac{\rho_2 \omega^2}{\Delta} (2K^2 - K_{p2}^2) \exp \left( \frac{gH}{2\alpha_1} \right) \left[ n_1 \cos n_1 H - \frac{g}{2\alpha_1} \sin n_1 H \right] e^{-iv_2d + iv_2H} J_0(Kr) \, dK \tag{4.21}
\]

Time factor \( \exp(\text{not}) \) is understood with every term in the above expressions. The first two terms in eqn. (4.20) may be combined as follows

\[
2 \int_0^{\infty} \frac{K}{iv_2} \cos n_1 H e^{-iv_2d} J_0(Kr) \, dK \quad \text{for } \, H \leq z \leq H + d \tag{4.22}
\]

or,

\[
2 \int_0^{\infty} \frac{K}{iv_2} e^{-iv_2(z-H)} \cos n_1 V_2 J_0(Kr) \, dK \quad \text{for } \, H + d \leq z < \infty \tag{4.23}
\]

From eqns (3.1) and (3.2) we can evaluate the displacements \( q \) and \( w \). Thus for the solid bottom \( z = H \) we make use of \( \phi_2 \) given by eqn. (4.22) and the third integral in eqn. (4.20) and \( \psi_2 \) given in eqn. (4.21)

So obtain the following displacements

\[
q_H = -2K^2 \int_0^{\infty} V(K) \left[ \frac{\rho_1}{\rho_2} \frac{K^2}{n_1} \left( \frac{\sin n_1 H}{\cos n_1 H - \frac{g}{2\alpha_1} \sin n_1 H} + \frac{g}{w^2} \right) \right] e^{-iv_2d} J_1(Kr) \, dK \tag{4.24}
\]

\[
w_H = -2K^2 \int_0^{\infty} V(K) \left[ \frac{\rho_1}{\rho_2} \frac{K^2}{n_1} \left( \frac{\sin n_1 H}{\cos n_1 H - \frac{g}{2\alpha_1} \sin n_1 H} + \frac{w^2}{e} \right) \right] e^{-iv_2d} J_0(Kr) \, dK \tag{4.25}
\]

where

\[
V(K) = 4K^2 v_2 v_2 + (2K^2 - K_{p2}^2)^2 + \frac{\rho_1}{\rho_2} \left[ \frac{\sin n_1 H}{\cos n_1 H - \frac{g}{2\alpha_1} \sin n_1 H} + \frac{w^2}{e} \right] \tag{4.26}
\]

and

\[
\Delta(K) = V(K)\rho_2 \beta_2^2 e^{-\frac{gH n_1}{2\alpha_1} \cos n_1 H - \frac{g}{2\alpha_1} \sin n_1 H} \tag{4.27}
\]

The above integrals can be represented by the sum of branch line integrals and residues. The residues correspond to the pole \( K = K_0 \) given by the roots of the equation

\[
V(K) = 0 \tag{4.28}
\]

Now the amplitudes of waves determined by branch line integrals diminish as \( r^{-2} \). As we are interested in an approximation which hold for large values of \( r \), the terms corresponding to branch points are left out of consideration and only the residues are computed. The asymptotic values of the displacements are obtained as

\[
q_H = \frac{2}{H^2} \sqrt{\frac{2\pi}{r}} \sum_n \frac{1}{\sqrt{K_n}} P(K_n) e^{-iv_2n d} e^{i(\text{not} - K_n r + \pi/4)} \tag{4.29}
\]
where the phase velocity \( c_n \) and \( \bar{v}_{2n} \) for each mode are given by

\[
W(K_n) = \frac{2}{H^2} \sqrt{2\pi} \frac{1}{\sqrt{K_n}} Q(K_n) e^{-i\bar{v}_{2n}d} e^{i(\omega t - K_n r - \gamma_d)}
\]

(4.30)

and

\[
Q(K_n) = \frac{K_n^2}{R(K_n) \beta_2^2} \left[ 2 - \frac{C_n^2}{\beta_2^2} \right] \left[ \frac{1}{\beta_2^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right] W(K_n) - 2 \frac{1}{1 - \frac{C_n^2}{\beta_2^2}} \frac{1}{1 - \frac{C_n^2}{\beta_2^2}}
\]

\[
R(K_n) = \frac{1}{C_n^2 \rho \beta_2^2}
\]

\[
P(K_n) = \frac{K_n^2}{R(K_n) \beta_2^2} \left[ 2 - \frac{C_n^2}{\beta_2^2} \right] \left[ \frac{1}{\beta_2^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right] W(K_n) - 2 \frac{1}{1 - \frac{C_n^2}{\beta_2^2}} \frac{1}{1 - \frac{C_n^2}{\beta_2^2}}
\]

where \( R(K_n) = \frac{1}{C_n^2 \rho \beta_2^2} \left[ 1 + \frac{1 - \frac{C_n^2}{\alpha_2^2}}{\frac{C_n^2}{\alpha_2^2} - 1 - \frac{g^2}{4\alpha_1^4 K_n^2}} \right] \)

\[
W(K_n) = \frac{1}{1 - \frac{C_n^2}{\beta_2^2}} \frac{1}{1 - \frac{C_n^2}{\beta_2^2}}
\]

\[
S(K_n) = \{ W(K_n) \}^2 + 1
\]

The period eqn (4.32) can be written in dimensionless form

\[
W(K_n) = \frac{\rho \beta_2^4}{C_n^4} \left[ 1 - 1 - \frac{g^2}{4\alpha_1^4 K_n^2} \right] \left[ 2 - \frac{C_n^2}{\beta_2^2} \right] \left[ 1 - \frac{C_n^2}{\beta_2^2} \right] \left[ 1 - \frac{C_n^2}{\beta_2^2} \right]
\]

It defines as usual a relationship between the period \( T = \frac{2\pi}{C_n K_n} = \frac{2\pi}{\omega} \) and the phase velocity with the elastic constants of the system as parameter

**5. NUMERICAL CALCULATION**

The period eqn (4.32) may be written in the following dimensionless simpler form approximating

\[
\sin \eta H \equiv \eta H \quad \text{and} \quad \cos \eta H \equiv 1
\]
\[
\left(2 - \frac{c^2}{\beta_2^2}\right)^2 - 4 \sqrt{1 - \frac{c^2}{\alpha_2^2}} \sqrt{1 - \frac{c^2}{\beta_2^2}} \rho_2 = \frac{c^2}{\beta_2^2} \frac{\frac{c^2}{3} KH}{1 - \frac{gH}{2\alpha_2^2}} + G \right]^{\frac{1}{2}} - \frac{c^2}{\alpha_2^2}
\]

(5.1)

where \( c = \frac{c}{K} \) is the phase velocity, \( G = \frac{E}{\rho K^2} \) is the gravity parameter. Now for granaitic ocean bottom [8] in which

\[
\frac{G}{\rho_1} = 2.5 \quad \frac{\alpha_2}{\beta_2} = 2\sqrt{5} \quad \frac{P_2}{\alpha_1} = 2
\]

we can evaluate \( \sqrt{\beta_2} \) from eqn. (5.1) for a particular value of the gravity parameter and KH as given in the following table.

| Table 1: Evaluation of \( \sqrt{\beta_2} \) for granaitic ocean bottom |
|---|---|---|---|
| G | KH | \( \frac{c^2}{\beta_2^2} \) | \( \sqrt{\beta_2} \) |
| 0.2 | 10 | 0.5483978 | 0.740539 |
| 0.3 | 10 | 0.6763711 | 0.822418 |
| 0.4 | 10 | 0.7474303 | 0.864541 |

The phase velocity may also be calculated for the sedimentary ocean bottom and basaltic ocean bottom in a similar manner.

6. DISCUSSION

Now neglecting the gravitational field i.e. making \( g \to 0 \) (\( G \to 0 \)) the period eqn (4.28) reduces to the following form

\[
\tan \left( KH \sqrt{\frac{c^2}{\alpha_2^2} - 1} \right) = \frac{\rho_2}{\rho_1} \frac{\frac{c^2}{\beta_2^2}}{\alpha_2^4} \sqrt{4 \left(1 - \frac{c^2}{\alpha_2^2} \right) \sqrt{1 - \frac{c^2}{\beta_2^2}} - \left(2 - \frac{c^2}{\beta_2^2}\right)^2} \]

(6.1)

which is in complete agreement with the corresponding result as studied by Ewing et al [8] Now using the same results and approximations as done in arts for granaitic ocean bottom we obtain \( \sqrt{\beta_2} = 0.512717173 \). The dispersion of wave occurs and the phase velocity increases in presence of gravitation field \( g \) which is clear from the form of the period eqn. (4.28) and figure (1) It is also clear from figure (1) that the ratio \( \sqrt{\beta_2} \) increases with the increasing value of the gravity parameter \( G \).

The second term in eqn (4.20) may be interpreted as a spherical wave emitted by the image of the source in the interface. It's simple form is \( \exp(-iK_0R') \) where \( R' = r^2 + (z - H + d)^2 \) Now if \( g \to 0, \ \vec{m_1} \to \vec{0} \) and the above problem reduces to the classical one as studied by Ewing, et al [8] In the above limit the asymptotic values of displacements as given in eqn (4.29) and eqn (4.30) are in complete agreement with the corresponding results of Ewing et al [8]
REFERENCES:


Fig. 1.
$\gamma/\beta_2$ is plotted against the gravity parameter $G$ for granitic ocean bottom when $KH = 10$