CHAPTER – 7
REVIEW ON COUPLE STRESS THEORY
OF ELASTICITY
A REVIEW ON ELASTO-DYNAMIC PROBLEMS
IN COUPLE-STRESS THEORY OF ELASTICITY

Introduction

The classical theory of elasticity is based on an ideal model of an elastic continuous medium in which the loading are transmitted through an infinitesimal area element in the body by means of stress vector only. The deformations thus take place is characterized by the symmetric strain tensors. The symmetric stress and strain tensors are interrelated by the generalized Hooke's law. Though the results obtained for homogeneous, isotropic materials with the application of the classical theory of elasticity are in harmony with experiments, in many cases, remarkable discrepancies between theory and experiments have been observed. The discrepancy between the classical theory of elasticity and experiment is striking where the stress concentration takes place in solid elastic medium i.e. in the neighbourhood of holes, notches, grooves, cracks and also in case of ultrasonic waves. The classical theory of elasticity eventually fails in the study of vibration of grain bodies and polymer [90].

It is expected from the mechanical point of view that the forces across a hypothetical plane within the solid should be statistically equivalent to a force and a couple. Such an assumption relays the fact that not only force stresses but also the couple stresses are transmitted through an area element, both of them obviously are asymmetric in nature. The corresponding deformations are also characterized by two asymmetric tensors namely strain tensor and curvature twist tensor. These are the main theme of couple-stress theory of elasticity.

The concept of couple-stress was originally introduced by Voigt [143]. The complete theory of asymmetric elasticity was developed in 1909 by the Cosserat brothers [28]. In this theory, which was non-linear in the beginning, they assumed to each molecule a perfectly rigid trihedron which during the process of deformation underwent not only the displacement but also the rotation. The material medium is termed as Cosserat continuum or micropolar elastic medium. In spite of the novelty of the idea, the work of Cosserat brothers was not duly appreciated during their life time and unnoticed for a good while.
After a long time, the interests of research workers were concentrated on the simplified Cosserat theory of asymmetric elasticity of the so-called Cosserat Pseudo-continuum. By this name we understand a continuum for which the symmetric stresses (force stress and couple-stress) may occur while the displacement of a body are described by a single displacement vector only. The modern derivation of the Cosserat's theory has been given by Truesdell and Toupin [142], Toupin [141] Aero and Kuvshinski [5,6], Groli [60], Mindlin [76], Mindlin and Tiersten [77] and Cohen [26]. Following the linearized form of constitutive equations in couple stress theory studied by Mindlin and Tiersten some problems of elasto-dynamics have been studied in this review paper.

The investigation of propagation of waves (surface waves and waves in a layer) in solid, elastic, homogeneous and isotropic medium is one of the main feature of classical elasto-dynamics. Both the problems have been studied by a good number of investigators [18, 65, 103, 135]. Regarding Lamb problem, it is noted that Sengupta and Ghosh [118] studied the axis symmetric Lamb problem in couple stress theory of elasticity and Nowacki and Nowacki [92, 93] studied the plane and axis symmetric Lamb problem in micropolar theory of elasticity. In classical theory of elasticity, the problem of moving load over a solid elastic semi-space has been studied by Cole and Huth [27], Lamb [69] and Sneddon [131]. The designs of highways or airport runways, as well as the foundation problems in soil mechanics, particularly when the earth-mass supports a moving load over its free plane surface lead to the investigations of the dynamic stress distribution associated with the problems. It is also noted that Nath and Sengupta [81] investigated steady-state response to moving loads in solid elastic media considering the supersonic-supersonic case.

Sengupta and his research collaborators have studied different problems of elastic waves and vibrations in couple stress theory of elasticity. Effect of couple-stress on the surface waves in solid elastic media has been investigated by Sengupta and Ghosh [114]. Sengupta and Ghosh [115] have also discussed the propagation of waves in solid elastic layers studying its different aspects under the influence of couple stress. The engineering significance of the moving load problem encourages themselves [113] in studying the effect of couple stress on the steady-state response to moving loads in a semi-infinite elastic medium. Effect of couple stress on propagation of waves due to a buried line source has
been investigated by Bandyopadhyay [8]. It is mentioned that, Banerjee and Sengupta have investigated the effect of couple stress on the propagation of waves in an elastic layer immersed in an infinite liquid. The effect of couple stresses on the propagation of waves in an elastic solid half space covered by a liquid layer has been considered by Sengupta and Chakravarty [125], Sengupta and Chel [123] published a paper on the oceanic Rayleigh waves with layered substratum in couple-stress theory of elasticity. They [124] also investigated the effect of couple stress on the solutions of the problem of vertical load distributed sinusoidally on the horizontal surface of a semi-infinite solid.

The list of papers, regarding the effect of couple stress in dynamic problems with one or more extra interacting fields (viz. viscous, gravity or thermal field) are very lengthy. Hence we discuss here the developments of another part of asymmetric elasticity namely micropolar theory. In the present day micropolar theory of elasticity has been flourished through the works of Nowacki [89], Nowacki and Nowacki [94,95], Eringen [49,50,52], Eringen and Suhubi [54], Palmov [97] and others. The contributions of Sengupta and his research collaborators are noteworthy in this field also. De and Sengupta [39] studied surface waves and Sengupta and Ghosh [116] discussed propagation of waves in a layer in micropolar elastic medium. Sengupta and Ghosh [117] have also investigated the steady-state response to moving loads in micropolar theory of elasticity. In this connection two research papers entitled ‘Disturbances in a micropolar elastic half space due to axially symmetric source of finite length’ and ‘solution of the source problem for a finite three dimensional source in a micropolar elastic solid’ by Sengupta and Ghosh [119,120] attract our attention. We also mention here ‘some problems in micropolar theory of elasticity (I, II & III)’ by Sengupta and Chakravarty [126-128]. We again exclude the name of the papers having one or more extra interacting fields in micropolar theory of elasticity.

The present review paper concerned with the general linear theory of Cosserat medium. We confine ourselves to the problem of elastic, homogeneous, isotropic and centro-symmetric bodies. Here like the classical, in the linear asymmetric theory of elasticity we assume that the deformations are small and square and product of the deformation are negligible with respect to the linear terms. It is also assumed that the relation between the state of strain and stress are linear and the increase of temperature is inconsiderable. Lastly we shall consider the theory of couple stress as macroscopic theory like the classical theory.
symmetric bodies. Here, like the classical, in the linear asymmetric theory of elasticity, we assume that the deformations are small and square and product of the deformation are negligible with respect to the linear terms. It is also assumed that the relation between the state of strain and stress are linear and the increase of temperature is inconsiderable. Lastly, we shall consider the theory of couple stress as macroscopic theory like the classical theory of elasticity and distances considered within its frame are much greater than the intermolecular distances. Hence it may be assumed that the action radius of the intermolecular forces is negligible and equals to zero.

**Basic Equations and Relations**

We shall consider the following basic equations and relations in studying different problems in this review paper. The states of stresses and couple stresses are given in terms of the Cartesian tensor notation [77]

\[ \tau^s_i = 2\mu\varepsilon_{ij} + \lambda\delta_{ij} \delta_{ij} = 4\eta x_{ij} + 4\eta' x_{ij} \]

where the local strain \( \varepsilon_{ij} \), rotation \( \omega_i \), and curvature twist tensor \( \chi_{ij} \) are expressed in terms of the displacement \( u_i \), as

\[ \varepsilon_{ij} = \left(\frac{u_{ij} - u_{ij}}{2}\right) \quad \omega_i = -\left(\frac{e_{ijk} u_{ij}}{2}\right) \quad \chi_{ij} = \omega_{ij} \]

where \( \lambda, \mu \) are Lamé's constants, \( \delta_{ij} \) is Kronecker delta, \( \eta \) and \( \eta' \) are the material constraints associated with resistance to curvature, \( \varepsilon_{ijk} \) is the unit alternating tensor and comma denotes the partial differentiation with respect to the space co-ordinates.

The deviation of the couple stress tensor \( \mu^D_{ij} \) and symmetric stress \( \tau^s_{ij} \) are given by

\[ \mu^D_{ij} = \mu_{ij} - \left\{ \left( \mu_{ijk} \delta_{ij} \right) / 3 \right\} \quad \tau^s_{ij} = \left\{ \left( \tau_{ij} + \tau_{ij} \right) / 2 \right\} \]

The stress equation of motion, couple stress equation of motion and linearized stress equation of motion are given by the vector form [77] as

\[ \nabla \cdot \tau + \rho \ddot{\mathbf{u}} = \rho \ddot{\mathbf{u}} \]

\[ \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \]

\[ \nabla \cdot \mu + \rho \ddot{\mathbf{c}} + \tau \times \mathbf{I} = 0 \]
\[
\vec{\nabla} \cdot \tau^s + (1/2) \vec{\nabla} \times \vec{\nabla} \cdot \mu \vec{D} + \rho \vec{f} + (1/2) \rho \vec{\nabla} \times \vec{\xi} = \rho \ddot{\vec{u}}
\]  
(5)

where \( I (= \vec{\nabla} \vec{r}) \) is the unit spatial dyadic, \( \vec{f} \) denotes the body force vector, \( \vec{\xi} \) is the body couple vector and dot denotes the time derivative. Inserting (1) in (5) we get the displacement equation of motion as (77)

\[
\mu \nabla^2 \vec{u} + (\lambda + \mu) \vec{\nabla} \cdot \nabla \cdot \vec{u} + \eta \nabla^2 \vec{\nabla} \times \nabla \times \vec{u} + \rho \vec{f} + (1/2) \rho \vec{\nabla} \times \vec{\xi} = \rho \ddot{\vec{u}}
\]  
(6)

The equations of motion given by (4) and (6) may be written in terms of tensor notations as

\[
\tau_{ij,k} + \rho f_i = \rho \ddot{u}_i \\
\varepsilon_{ijk} \varepsilon_{jk} + \mu_{ij,j} + \rho c_i = 0
\]

\[
\begin{bmatrix}
(i, j = 1, 2, 3)
\end{bmatrix} \begin{bmatrix}
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{bmatrix}
\]  
(7)

\[
\mu \nabla^2 u_i + (\lambda + \mu) u_{j,ji} + \eta \nabla^2 (u_{j,ji} - \nabla u_i) + \rho f_i - \frac{\partial}{\partial x} \varepsilon_{ijk} c_{j,k} = \rho \ddot{u}_i
\]  
(8)

We shall deal mainly with the following displacement equations of motion in absence of body forces and body couples [putting \( f_i = c_i = 0 \) (i = 1, 2, 3) in equation (8)] in studying this review paper

\[
\mu \nabla^2 u_i + (\lambda + \mu) u_{j,ji} + \eta \nabla^2 (u_{j,ji} - \nabla u_i) = \rho \ddot{u}_i \quad (i, j = 1, 2, 3)
\]  
(9)

**Effect of Couple Stress on Surface Waves**

Let us consider two homogeneous elastic medium \( M_1 \) and \( M_2 \) welded in contact (or sufficiently rough enough to prevent any sliding on common surface) at their common surface of separation. The two medium are separated by a plane horizontal boundary extended to infinity and \( M_2 \) being above \( M_1 \). As a reference coordinate system, a set of orthogonal cartesian axes \( OX_1X_2X_3 \), the origin \( O \) being any point of the boundary and \( OX_3 \) pointing normally into \( M_2 \), is taken.

Let us consider the possibility of a wave travelling in the direction \( OX_1 \) in such a manner that (a) the disturbance is largely confined to the neighbourhood of the boundary, and (b) at any instance all particles in any line parallel to \( OX_2 \) have equal displacements. Due to (a), the wave is a surface wave; and due to (b), the case we have taken is
analogous to the plane waves. Then the displacement components, $u_1$ and $u_3$, at any point may be expressed in the form

$$u_1 = \Phi_x + \Psi_z, \quad u_3 = \Phi_z - \Psi_x,$$  

so that

$$\nabla^2 \Phi = 0, \quad \nabla^2 \Psi = u_1 \alpha_1 - u_3 \beta_1,$$

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

where $\Phi$ and $\Psi$ are functions of coordinates $x_1$, $x_3$ and time $t$.

The displacement equations of motion are from (9)

$$\mu \nabla^2 u_1 + (\lambda + \mu) u_{j,j} + \eta \nabla^2 (u_{j,j} - \nabla^2 u_1) = \rho \ddot{u}_i \quad (i, j = 1, 2, 3).$$

Then, in presence of couple-stresses we obtain, in view of (10) and (11), the following relations in $M_1$

$$\nabla^2 \Phi - \alpha_1^2 = 0, \quad \nabla^2 \Psi - \ell_1^2 \nabla^4 \Psi = \psi/\beta_1^2, \quad \nabla^2 u_2 - \ell_1^2 \nabla^4 u_2 = \ddot{u}_2/\beta_1^2,$$  

where $\rho_1$, $\lambda_1$, $\mu_1$ denote the properties of the medium $M_1$ and

$$\alpha_1^2 = (\lambda_1 + 2\mu_1)/\rho_1, \quad \beta_1^2 = \mu_1/\rho_1 \quad \text{and} \quad \ell_1^2 = \eta_1^*\beta_1,$$

$\eta_1^*$ being a constant characterizing the effect of couple-stress in the medium $M_1$. Similar relations hold good in $M_2$ with $\rho_1$, $\lambda_1$, $\mu_1$, $\ell_1$, $\alpha_1$, $\beta_1$ replaced by $\rho_2$, $\lambda_2$, $\mu_2$, $\ell_2$, $\alpha_2$, $\beta_2$.

To solve the equations (12) and similar equations for the medium $M_2$, we assume

$$\Phi = f(x_3)e^{i\xi x_1-ct}, \quad \Psi = g(x_3)e^{i\xi x_1-ct}, \quad u_2 = h(x_3)e^{i\xi x_1-ct}. \quad \text{(14)}$$

On substituting equation (14) in (12) and similar equations for the medium $M_2$, and on simplification we get the solutions satisfying conditions for surface waves, as follows:

In the medium $M_1$

$$\Phi = Ae^{-in\xi x_3 + i\xi x_1-ct}, \quad \Psi = [Be^{-in\xi x_3 + C e^{\xi x_3}}]e^{i\xi x_1-ct},$$

$$u_2 = [B e^{-in\xi x_3 + C \xi e^{\xi x_3}}]e^{i\xi (x_1-ct)}; \quad \text{(15)}$$

in the medium $M_2$

$$\Phi = De^{i\xi (\xi x_3 + x_1-ct)}, \quad \Psi = [E e^{in\xi x_2} + Fe^{-\xi x_3}]e^{i\xi x_1-ct},$$

$$u_2 = [E * e^{in\xi x_3} + F e^{-\xi x_3}]e^{i\xi (x_1-ct)}; \quad \text{(16)}$$
where $A, B, C, D, E, F, B^*, C^*, E^*$ and $F^*$ are all arbitrary constants and $r_j, \eta_j (j = 1, 2)$ are all positive imaginaries and $\zeta_i$ are positive real numbers given by
\[
\eta_j = (p_j^2 - \xi^2)^{1/2}, \quad \zeta_j = (q_j^2 + \xi^2)^{1/2}
\]
\[
r_j = \left( \frac{\rho_j c^2}{\lambda_j + 2\mu_j} - 1 \right)^{1/2}, \quad p_j^2 = \frac{1}{2\ell_j^2} \left( 1 + 4 \frac{\sigma_1 \xi^2 c^2 \ell_j^2}{\mu_j} \right)^{1/2} - 1, \quad q_j^2 = \frac{1}{2\ell_j^2} \left( 1 + 4 \frac{\rho_j \xi^2 c^2 \ell_j^2}{\mu_j} \right)^{1/2} + 1
\]

The above system of equations (15) and (16) will lead us to a particular solution corresponding to a group of simple harmonic waves of wave-length $2\pi/\xi$ travelling forward with speed $c$.

To obtain the velocity equations we have now to apply the following boundary conditions:

(i) The displacements at the common boundary surface between $M_1$ and $M_2$ must be continuous at all times and places.

(ii) The rotation components $\omega_1, \omega_2, \omega_3$ where $\omega_1 = -\frac{u_{2,3}}{2}$, $\omega_2 = \nabla^2 \Psi$, $\omega_3 = \frac{u_{3,1}}{2}$ must be continuous on the common boundary

(iii) The stresses $\sigma_{31}, \sigma_{32}, \sigma_{33}$ and couple-stress $\mu_{31}, \mu_{32}, \mu_{33}$, where
\[
\begin{align*}
\sigma_{31} &= \mu_1 (2\Phi_{13} - \Psi_{1,11} + \Psi_{1,33}) - \eta_{11}^* \nabla^4 \Psi, \\
\sigma_{32} &= \mu_1 u_{2,3} - \eta_{13}^* (u_{2,333} + u_{2,311}), \\
\sigma_{33} &= \lambda_1 \nabla^2 \Phi + 2\mu_1 (\Phi_{3,3} - \Psi_{1,13}), \\
\mu_{31} &= -2\eta_{13}^* u_{2,33}, \quad \mu_{32} = 2\eta_{13}^* \nabla^2 \Psi_{1,3},
\end{align*}
\]

and similar expressions for $M_2$, across the boundary surface between $M_1$ and $M_2$ must be continuous at all times and places. Using (10), (15), (16) and the condition (i), we obtain
\[
\xi A - \eta_1 B - i\xi_1 C = \xi D + \eta_2 E + \eta_2 E + i\xi_2 F, \quad (19)
\]
\[
-r_1 A - (B + C) = r_2 D - (E + F), \quad (20)
\]
\[
B^* + C^* = E^* + F^* \quad (21)
\]

Again, substituting (15) and (16) into (19) and using condition (iii), we obtain
\[
2\mu_1\eta^2_1 + \{(\xi^2 - \eta^2_1)\mu_1 - \eta_1^* (\eta^2_1 + \xi^2)\}B + \{(\xi^2 + \xi^2)\mu_1 - \eta_1^* (\xi^2 - \xi^2)\}C = \\
-2\mu_2 r^2 \xi^2 D + \{(\xi^2 - \eta^2_2)\mu_2 - \eta_2^* (\eta^2_2 + \xi^2)\}E + \{(\xi^2 + \xi^2)\mu_2 - \eta_2^* (\xi^2 - \xi^2)\}F;
\]

\[
\{\eta_1^* (\eta^2_1 + \xi^2) + \eta_1 \mu_1\}B^* + \eta_1^* (\xi^2 - \xi^2) + \xi_1 \mu_1\}C^* = \\
-\{\eta_2^* (\eta^2_2 + \xi^2) + \eta_2 \mu_2\}E^* + \xi_2 \mu_2\}F^*;\]

\[
\{\lambda_1 (\xi^2 + 1) + 2\mu_1 \xi_1\} \xi^2 A + 2\mu_1 \xi_1 B + 2\mu_1 \xi_1 C = \\
= \{\lambda_2 (\xi^2 + 1) + 2\mu_2 \xi^2\} \xi^2 D - 2\mu_2 \xi_2 E - 2\mu_2 \xi_2 F;\]

\[
\eta_1 (\eta^2_1 + \xi^2) \eta^* B + \xi_1 (\xi^2 - \xi^2) \eta^* C = -\eta_2 (\eta^2_2 + \xi^2) \eta^* E - \xi_2 (\xi^2 - \xi^2) \eta^* F;\]

\[
\eta_1^* (-\eta_1 B + \xi_1 C) = \eta_2^* (\eta_2 E - \xi_2 F),
\]

and from the condition (ii), we obtain

\[
-\eta_1 B^* + \xi_1 C^* = \eta_2 E^* - \xi_2 F^*,
\]

\[
-(\eta^2_1 + \xi^2) B + (\xi^2 - \xi^2) C = -(\eta^2_2 + \xi^2) E + (\xi^2 - \xi^2) F,
\]

and the other component \( \omega_2 \) contributes the same equation as (21).

From the equations (21), (23), (26) and (27) we find that only possible values of \( B^*, C^*, E^* \) and \( F^* \) are zeroes. Thus there is no propagation of the displacements \( u_2 \).

Again, equation (25) will be satisfied if

\[
C = -\frac{i\mu_1^2 \eta_1}{\xi_1} B + \frac{K}{\xi_1^2 \eta_1} \quad \text{and} \quad F = -\frac{i\mu_2^2 \eta_2^*}{\xi_2^2 \eta_2} E - \frac{K}{\xi_2^2 \eta_2^*},
\]

where \( K \) is a constant which depends on the parameters of the couple-stresses of the media. From the equations (28) and (29), we obtain

\[
K = \nu_1 p_1^2 B - \nu_2 p_2^2 E,
\]

where

\[
\nu_1 = (1 + i/\xi_1) \left( \frac{1}{\xi_1 \eta_1} + \frac{1}{\xi_2 \eta_2} \right), \quad \nu_2 = (1 + \nu/\xi_2) \left( \frac{1}{\xi_1 \eta_1} + \frac{1}{\xi_2 \eta_2} \right)
\]

In view of (31) the equations (19), (20), (22) and (24) can be written as
$$\xi A - \left( \eta_1 + \frac{\eta_1 p_1^2}{q_1} - iv_1 p_1^2 a \right) B - \xi D - \left( \eta_2 + \frac{\eta_2 p_2^2}{q_2} + iv_2 p_2^2 a \right) E = 0,$$

$$r_1 A + \left( 1 - \frac{in_1 p_1^2}{\zeta_1 q_1^2} + v_1 p_1^2 b \right) B + r_2 D - \left( 1 - \frac{in_2 p_1^2}{\zeta_2 q_2^2} + v_2 p_2^2 b \right) E = 0,$$

$$2\mu_1 r_1 \xi^2 A + \left[ \{(\xi^2 - \eta_1^2) \mu_1 - \eta_1^* p_1^2 \} - i((\xi^2 + \zeta_1^2) \mu_1 - \eta_1^* q_1^2) \right] \frac{\eta_1 p_1^2}{\zeta_1 q_1^2}$$

$$+ v_1 p_1^2 c B + 2\mu_2 r_2 \xi^2 D - \left[ \{(\xi^2 - \eta_2^2) \mu_2 - \eta_2^* p_2^2 \} - i((\xi^2 + \zeta_2^2) \mu_2 - \eta_2^* q_2^2) \right] \frac{\eta_2 p_2^2}{\zeta_2 q_2^2},$$

$$+ v_2 p_2^2 c \right] E = 0,$$

$$\{(\lambda_1 + 2\mu_1)(r_1^2 + 1) - 2\mu_1 \} \xi A + \left\{ 2\mu_1 \left( \eta_1 + \frac{\eta_1 p_1^2}{q_1^2} \right) + iv_1 p_1^2 d \right\} B -$$

$$- \{(\lambda_2 + 2\mu_2)(r_2^2 + 1) - 2\mu_2 \} \xi D + \left\{ 2\mu_2 \left( \eta_2 + \frac{\eta_2 p_2^2}{q_2^2} \right) - iv_2 p_2^2 d \right\} E = 0,$$  \hspace{1cm} (32)

where

$$a = \frac{1}{q_1^2 \eta_1} - \frac{1}{q_2^2 \eta_2}, \hspace{1cm} b = \frac{1}{\zeta_1 q_1^2 \eta_1} + \frac{1}{\zeta_2 q_2^2 \eta_2},$$

$$c = \{(\xi^2 + \zeta_1^2) \mu_1 - \eta_1^* (\zeta_1^2 - \xi^2)^2 \} \frac{1}{\zeta_1 q_1^2 \eta_1} + \{(\xi^2 + \zeta_2^2) \mu_2 - \eta_2^* (\zeta_2^2 - \xi^2)^2 \} \frac{1}{\zeta_2 q_2^2 \eta_2},$$

$$d = \frac{2\mu_1}{q_1^2 \eta_1} - \frac{2\mu_2}{q_2^2 \eta_2}.$$

Eliminating A, B, D and E from the equations (32), we obtain the modified wave-velocity equation of the surface waves under the influence of couple-stresses,
where $\alpha_j^2$, $\beta_j^2$ and $\ell_j^2$ are given by (13) and $j = 1, 2$

If the parameters of couple-stress $\ell_1$ and $\ell_2$ vanish, the wave-velocity equation (34) as obtained above is in agreement with the corresponding classical result, which may be written in the simplified form

$$c^4[(\rho_1 - \sigma_2)^2 + (\rho_1 r_2 + \rho_2 r_1)(\rho_1 \eta_1 + \rho_2 \eta_2)] + 2Qc^2[\eta_1 \rho_2 - r_2 \eta_2 \rho_1 + \rho_2 - \rho_1] + Q^2(\eta_1 \eta_2 + 1)(r_2 \eta_2 + 1) = 0,$$

(35)

where $Q = 2(\rho_1 \beta_1^2 - \rho_2 \beta_2^2) = 2(\mu_1 - \mu_2)$ and $r_j = \left(\frac{c^2}{\alpha_j^2} - 1\right)^{1/2}$, $\eta_j = \left(\frac{c^2}{\beta_j^2} - 1\right)^{1/2}$.

**Particular cases**

**Rayleigh waves**: The particular case of the foregoing problem is to investigate the possibility of Rayleigh waves in an elastic solid under the influence of couple-stress. In this particular case the plain boundary must be a free surface so that $M_2$ is replaced by vacuum. In view of (22), (24) and (25) we obtain

$$2\mu_1 \eta_1^2 A + [(\xi^2 - \eta^2)\mu_1 - \eta_1^* p_1^4] - 1(\xi^2 + \eta^2)\mu_1 - \eta_1^* q_1^4] \eta_1 p_1^2 / \xi_1 q_1^2] B = 0,$$  

(36)

$$\{(\lambda_1 + 2\mu_1)(r_1^2 + 1) - 2\mu_1\} \eta A + 2\mu_1 \eta_1 + (\eta_1 p_1^2 / q_1^2)] B = 0.$$  

(37)

Eliminating $A$ and $B$ from (36) and (37), we obtain
\[ 4\mu_1^2 r_1\xi (n_1 + (n_1 p_1^2 / q_1^2)) - \{(\lambda_1 + 2\mu_1) (r_1^2 + 1) - 2\mu_1\}\{(\xi^2 - \eta_1^2)\nu_1 - \eta_1 p_1^4\} - \]
\[ -i\{(\xi^2 + \zeta_1^2)\nu_1 - \eta_1 q_1^4\} \eta_1 p_1^4 / \zeta_1 q_1^2\} = 0. \]  

Equation (38), in view of (17) becomes
\[ \frac{4\left(\frac{c^2}{\alpha_1^2} - 1\right)}{\frac{c^2}{\beta_1^2}} \left(\frac{p_1^2}{\xi^2} - 1\right)^{1/2} \left(1 + \frac{p_1^2}{q_1^2}\right) = \left(2 - \frac{c^2}{\alpha_1^2}\right)^{1/2} \left(2 - \frac{c^2}{\beta_1^2}\right) \left(1 - \frac{\eta_1 p_1^2}{\zeta_1 q_1^2}\right) \]  

Equation (39) determines the velocity of the Rayleigh surface wave under the influence of couple-stress.

If the parameter \( \ell_1 \) of couple-stress be zero, we obtain the classical result:
\[ (2 - (c^2/\beta_1^2))^2 = 4\left(1 - (c^2/\alpha_1^2)\right)^{1/2} \left(1 - (c^2/\beta_1^2)\right)^{1/2}. \]

The equation (39) can be analysed to study the effect of couple-stress on the said classical problem by assuming the couple-stress parameter \( \ell_1 \) to be so small that its cubes and higher powers are always neglected. Thus the approximate form of equation (39) is
\[ \left(2 - \frac{c^2}{\beta_1^2}\right)^2 = 4\left(1 - \frac{c^2}{\alpha_1^2}\right)^{1/2} \left(1 - \frac{c^2}{\beta_1^2}\right)^{1/2} \left(1 + \frac{c^2}{\beta_1^2} \ell_1^2\right) \]  

Squaring both the sides of (40) and re-arranging we obtain
\[ \frac{c^2}{\beta_1^2} \left[\frac{c^6}{\beta_1^6} - 8 \frac{c^4}{\beta_1^4} + 24 \frac{c^2}{\beta_1^2} - \frac{16}{\alpha_1^2}\right] - 16 \left(1 - \frac{c^2}{\alpha_1^2}\right) - 16 \xi^2 \ell_1^2 \frac{c^2}{\beta_1^4} \times \]
\[ \times \left(1 - \frac{c^2}{\alpha_1^2}\right) - 32 \xi \ell_1 \left[1 - \left(\frac{c^2}{\alpha_1^2} + \frac{c^2}{\beta_1^2}\right) + \frac{c^4}{\alpha_1^2 \beta_1^2}\right]\right) = 0 \]  

We assume \( \lambda = \mu \) (i.e. \( v = \frac{1}{3} \)), then \( \alpha_1 = \sqrt{3} \beta_1 \), and we obtain from equation (41)
\[ \left(\frac{c^6}{\beta_1^6} - 8 \frac{c^4}{\beta_1^4} + \frac{56 c^2}{3 \beta_1^2} - \frac{32}{3}\right) - 16 \xi^2 \ell_1^2 \left(3 - \frac{c^2}{\beta_1^2}\right) \left(2 - \frac{c^2}{\beta_1^2}\right) = 0 \]  

The above equation is a cubic equation in \( c^2/\beta_1^2 \) and has three real and positive roots. One of the roots lies between 0.8453 and 2. In order to satisfy the condition of Rayleigh waves the value of \( \xi^2 \ell_1^2 \) must be less than or equal to 0.09, and the acceptable roots for
the real existence of Rayleigh waves have been computed numerically. It is thus observed that the velocity of Rayleigh waves increases due to the influence of couple-stress.

**Numerical result**

In Table 1 the values of \( c/\beta_{1} \) have been computed for different values of \( \xi^{2} \ell_{1}^{2} \) in equation (42), and thus it has been exhibited that the velocity of Rayleigh waves in presence of couple-stresses increases.

**Table 1. Computation of \( c/\beta_{1} \) for different values of the couple-stress parameter**

<table>
<thead>
<tr>
<th>( \xi^{2} \ell_{1}^{2} )</th>
<th>10^{-2}</th>
<th>10^{-3}</th>
<th>10^{-4}</th>
<th>10^{-5}</th>
<th>10^{-6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c/\beta_{1} )</td>
<td>0.92915</td>
<td>0.92288</td>
<td>0.92199</td>
<td>0.919411</td>
<td>0.9194025</td>
</tr>
</tbody>
</table>

**Love waves**: Let us suppose that the medium \( M_{2} \) is bounded by two horizontal plane surfaces at finite distance \( H \) apart, the upper plane being free, while the lower plane surface forms the medium \( M_{1} \). It is sufficient to consider the displacement \( u_{2} \) only. The notable fact here is that the displacement in \( M_{2} \) may no longer diminish with distance from the boundary between \( M_{1} \) and \( M_{2} \), so that for the medium \( M_{2} \) we preserve the full solution as

\[
\mathbf{u}_{2} = \left[ (A_{1} e^{i\eta_{2} x_{3}} + A_{2} e^{-i\eta_{2} x_{3}}) + (B_{1} e^{-\xi x_{3}} + B_{2} e^{\xi x_{3}}) e^{i \xi (x_{1} - at)} \right],
\]

and for the medium \( M_{1} \), the solution is

\[
\mathbf{u}_{2} = \left[ B_{1} e^{i \eta_{2} x_{3}} + C_{1} e^{i \xi x_{3}} \right] e^{i \xi (x_{1} - at)},
\]

where \( \eta_{2} \) is not necessarily imaginary, while for \( M_{1} \) we see that \( \eta_{1} \) is an imaginary and \( \xi_{1}, \xi_{2} \) are positive quantities.

In addition to the three boundary conditions (i), (ii) and (iii) for general surface waves we have the conditions that there shall be no stress and couple-stress across the free surface \( x_{3} = H \). Hence, in view of (43), (44) the boundary conditions may be written as

\[
\begin{align*}
B^{*} + C^{*} &= (A_{1} + A_{2}) + (B_{1} + B_{2}), \\
\eta_{1} B^{*} + \xi_{1} C^{*} &= \eta_{2} (A_{1} - A_{2}) - \xi_{2} (B_{1} - B_{2}), \\
- i\eta_{1} \mu_{1} (1 + \ell_{1}^{2} \rho_{1}^{2}) B^{*} + \mu_{1} \xi_{1} (1 - \ell_{1}^{2} \rho_{1}^{2}) C^{*} &= \\
= i\eta_{2} \mu_{2} (1 + \ell_{2}^{2} \rho_{2}^{2}) (A_{1} - A_{2}) - \mu_{2} \xi_{2} (1 - \ell_{2}^{2} \rho_{2}^{2}) (B_{1} - B_{2}),
\end{align*}
\]
\[ (-\eta_2^2 B^* + \zeta_1^2 C^* \mu_1 \ell_1^2 = (-\eta_2^2 (A_1 + A_2) + \zeta_2^2 (B_1 + B_2) \mu_2 \ell_2^2, \] (48)

\[ \eta_2 (1 + \ell_2^2 p_2^2) (A_1 e^{i \eta_2^2 H} - A_2 e^{-i \eta_2^2 H}) = \zeta_2 (1 - \ell_2^2 q_2^2) (B_1 e^{-\zeta_2^2 H} - B_2 e^{\zeta_2^2 H}) = 0, \] (49)

\[ -\eta_2^2 (A_1 e^{i \eta_2^2 H} + A_2 e^{-i \eta_2^2 H}) + \zeta_2^2 (B_1 e^{-\zeta_2^2 H} + B_2 e^{\zeta_2^2 H}) = 0 \] (50)

From (48), we get

\[ C^* = \frac{K}{\mu_1 \ell_1^2 \zeta_1} + \frac{\eta_2^2}{\mu_1 \ell_1^2 \zeta_1} B^*, \quad B_1 + B_2 = \frac{K}{\mu_2 \ell_2^2 \zeta_2} + \frac{\eta_2^2}{\mu_2 \ell_2^2 \zeta_2} (A_1 + A_2), \] (51)

where \( K \) is an arbitrary constant which depends on the parameters.

Equations (49) and (50) are satisfied if

\[ \left( \frac{\eta_2^2}{\zeta_2^2} \right) A_1 e^{i \eta_2^2 H} = \left( \frac{\eta_2^2}{\zeta_2^2} \right) A_2 e^{-i \eta_2^2 H} = B_2 e^{\zeta_2^2 H} = B_1 e^{-\zeta_2^2 H}, \] (52)

which gives

\[ \frac{A_2 - A_1}{A_1 + A_2} = i \tan \eta_2 H \quad \text{and} \quad B_1 - B_2 = (B_1 + B_2) \tanh \zeta_2 H. \] (53)

In view of (46), (51) and the second equation of (53), we obtain

\[ K = \left(1/a\right) \left\{ bB^* + c (A_1 + A_2) \right\}, \]

where

\[ a = -\left( \frac{\tanh \zeta_2^2 H}{\mu_2 \ell_2^2 \zeta_2} + \frac{1}{\mu_1 \ell_1^2 \zeta_1} \right), \quad b = \eta_2^2 + \frac{\eta_2^2}{\zeta_1}, \quad c = \frac{\eta_2^2}{\zeta_2} \tan \zeta_2 H - \eta_2 \tan \eta_2 H \] (54)

In view of (51), (54), we obtain from (45) and (47)

\[ \mu_1 \left\{ -\eta_1 (1 + \ell_1^2 p_1^2) + \frac{1 - \ell_1^2 q_1^2}{\zeta_1^2} \eta_1^2 + b_2 \right\} \frac{1 + \left( \frac{\eta_2^2}{\zeta_2^2} \right) + c_1 (A_1 + A_2) = \mu_2 \left\{ \eta_2 (1 + \ell_2^2 p_2^2) (A_1 - A_2) - \left( 1 - \ell_2^2 q_2^2 \right) \left( \frac{\eta_2^2}{\zeta_2^2} \right) \tan \zeta_2 H + c_2 \right\} (A_1 + A_2), \] (55)

where

\[ b_1 = \frac{b}{a} \left( \frac{1}{\mu_2 \ell_2^2 \zeta_2} - \frac{1}{\mu_1 \ell_1^2 \zeta_1} \right), \quad c_1 = \frac{c}{a} \left( \frac{1}{\mu_2 \ell_2^2 \zeta_2} - \frac{1}{\mu_1 \ell_1^2 \zeta_1} \right) \]

\[ \mu_1 b_2 = \frac{b}{a} \left\{ \zeta_1 (1 - \ell_1^2 q_1^2) - \zeta_2 (1 - \ell_2^2 q_2^2) \tan \zeta_2 H \right\}, \] (56)

\[ \mu_2 c_2 = \frac{c}{a} \left\{ \zeta_1 (1 - \ell_1^2 q_1^2) - \zeta_2 (1 - \ell_2^2 q_2^2) \tan \zeta_2 H \right\}, \]

Equation (54), in view of the first equation of (52), becomes

\[ \mu_1 \left\{ -\eta_1 (1 + \ell_1^2) + \left( 1 - \ell_1^2 q_1^2 / \zeta_1^2 \right) \eta_1^2 + b_2 \right\} \frac{1 + \left( \frac{\eta_2^2}{\zeta_2^2} \right) + c_1 = \mu_2 \left\{ \eta_2 (1 + \ell_2^2 p_2^2) \tan \zeta_2 H - \left( 1 - \ell_2^2 q_2^2 \right) \left( \frac{\eta_2^2}{\zeta_2^2} \right) \tan \zeta_2 H + c_2 \right\}. \] (57)
The above equation gives the S. H surface waves of velocity under the influence of couple-stress. If the parameters of couple-stresses, \( \ell_1 \) and \( \ell_2 \) vanish, the wave-velocity equation is in agreement with the classical equation of Love waves. Hence equation (57) shows the remarkable effect of couple-stresses on the Love waves.

The modified equation can be analysed to study the effect of couple-stresses on the classical problem by assuming the parameters of couple-stresses, \( \ell_1 \) and \( \ell_2 \), to be small so that their cubes and higher powers are always neglected. Hence, on the above assumption, we obtain the following approximate form of equation (57):

\[
\mu_1[-i\eta_i(1 + (c^2/\beta_i^2)\xi^2\ell_i^2)] = \mu_2[\eta_2(1 + (c^2/\beta_2^2)\xi^2\ell_2^2) \tan\eta_2 H],
\]

(58)

where \( \eta_i = \xi((c^2/\beta_i^2) - (c^4/\beta_i^4)\xi^2 \ell_i^2 - 1) \) \( (i = 1, 2) \)

(59)

Equation (58) yields a real value of \( c \) if \( \eta_1 \) and \( \eta_2 \) are imaginary and real respectively. The requirement that \( \eta_1 \) should be imaginary and \( \eta_2 \) be real is, by (59),

\[
\beta_2 \left(1 + \xi^2 \ell_2^2\right) < c < \beta_1 \left(1 + \xi^2 \ell_1^2\right),
\]

(60)

which shows that for a particular \( \xi \) the range of \( c \) increases under the influence of couple-stresses. In other words, the velocity \( c \) of Love waves increases under the influence of couple-stresses.

**Stoneley waves**: In the classical theory Stoneley waves are a generalized form of Rayleigh waves propagating along the common boundary of \( M_1 \) and \( M_2 \). Hence, Stoneley waves along the common boundary of the media \( M_1 \) and \( M_2 \) under the influence of couple-stresses are determined by the roots of the wave equation (25). When the couple-stress parameters vanish, we get equation (26), which is the frequency equation of Stoneley waves in the classical theory.

**Effect of couple stress on waves in a layer**

Let us introduce a Cartesian frame of reference \( ox_1x_2x_3 \) taking the origin in the middle plane of the elastic layer, the middle plane coincides with the plane \( ox_1x_2 \). We consider the effect of couple-stresses on the propagation of waves in an elastic layer of thickness \( 2h \). The planes bounding the layer are \( x_3 = \pm h \) and are supposed to be free of stress. There exists a plane wave moving with the constant velocity \( c \) in the direction of
Both the longitudinal and transverse waves in the infinitely extended elastic layer would be propagated. The boundary surfaces of the elastic space leads evidently to a distortion of the state of stress, which influences the velocity of propagation of elastic waves. From the nature of the problem the non-zero displacements \( u_1 \) and \( u_3 \) at any point may be expressed in the form

\[
\begin{align*}
  u_1 &= \phi_1 - \psi_3, \\
  u_3 &= \phi_3 + \psi_1
\end{align*}
\]  

(61)

where \( \phi \) and \( \psi \) are displacement potentials, functions of the coordinates \( x_1 \), \( x_3 \) and time.

The displacement equations of motion are

\[
\mu \nabla^2 u_i + (\lambda + \mu) u_{j,j} + \eta \nabla^2 (u_{j,j} - \nabla^2 u_i) = \rho \ddot{u}_i \quad [i, j = 1, 2, 3]
\]  

(62)

The equation of motion (62) in view of (61), yield the following equations

\[
\begin{align*}
  \nabla^2 \phi - c_1^{-2} \phi &= 0 ; \\
  \nabla^2 \psi - c_2^{-2} \nabla^4 \psi &= c_2^{-2} \psi
\end{align*}
\]  

(63)

where

\[
\begin{align*}
  c_1^{-2} &= (\lambda + 2\mu)/\rho, \quad c_2^{-2} = \mu/\rho, \quad \ell^2 = \eta/\mu,
\end{align*}
\]

\( \eta \) is a constant characterizing the existence of couple-stresses and \( \lambda, \mu \) are Lamé's elastic constants. If \( \ell \), the parameter of couple-stress, be zero the classical result of the corresponding problem follows at once.

Our problem here is to seek solutions of the equations (63) subject to the boundary conditions

\[
\sigma_{33} = \sigma_{31} = \mu_{32} = 0 \quad \text{in the plane } x_3 = \pm h
\]  

(64)

The stresses \( \sigma_{ij} \) and couple-stresses \( \mu_{ij} \) are given by [1]

\[
\begin{align*}
  \sigma_{33} &= 2\mu e_{33} + \lambda (e_{11} + e_{33}), \\
  \sigma_{11} &= 2\mu e_{11} + \lambda (e_{11} + e_{33}), \\
  \sigma_{13} + \sigma_{31} &= 4\mu e_{13}, \\
  \mu_{32} &= 4\eta (e_{31,3} - e_{33,1}) \\
  \sigma_{13} - \sigma_{31} &= \mu_{12,1} + \mu_{32,3}, \\
  \mu_{12} &= 4\eta (e_{11,3} - e_{13,1})
\end{align*}
\]  

(65)

On substituting for \( u_1 \) and \( u_3 \) from (61) in (65) we get

\[
\begin{align*}
  \sigma_{11} &= 2\mu (\phi_{11} - \psi_{13}) + \lambda \nabla^2 \phi, \\
  \sigma_{33} &= 2\mu (\phi_{33} - \psi_{13}) + \lambda \nabla^2 \phi, \\
  \sigma_{13} &= \mu (2\phi_{13} - \psi_{33} + \psi_{11}) - \eta \nabla^4 \psi, \\
  \sigma_{31} &= \mu (2\phi_{31} - \psi_{33} + \psi_{11}) + \eta \nabla^4 \psi, \\
  \mu_{32} &= -2\eta \nabla^2 (\psi_{33}), \\
  \mu_{12} &= -2\eta \nabla^2 (\psi_{11})
\end{align*}
\]  

(66)
To solve the equations (63), we assume
\[ \phi(x_1,x_3,t) = f(x_1,x_3)e^{i\omega t}, \quad \psi(x_1,x_3,t) = g(x_1,x_3)e^{i\omega t} \] (67)

Owing to (67), equations (63) take the form
\[ \nabla^2 f + k_1^2 f = 0, \quad \nabla^2 g + k_2^2 g - \ell^2 \nabla^4 g = 0 \] (68)
where
\[ k_1^2 = \left( \frac{\omega^2}{c_1^2} \right), \quad k_2 = \left( \frac{\omega^2}{c_2^2} \right) \]

The solution of the equation (67) are
\[ f = f^*(x_3)e^{-i\alpha x_1}, \quad g = g^*(x_3)e^{-i\alpha x_1} \] (69)

Introducing (69) into (68), we arrive at two ordinary differential equations
\[ \left( \frac{d^2}{dx_3^2} \right) f^* - (\alpha^2 - k_1^2)f^* = 0; \quad \left[ \nabla^2 \left( \frac{d^2}{dx_3^2} \right) - \alpha^2 \right]^2 - \left( \frac{d^2}{dx_3^2} \right) - \alpha^2 + k_2^2 \right] g^* = 0 \] (70)
the solutions of which are
\[ \begin{align*}
    f^* &= A \sinh v_1 x_3 + B \cosh v_1 x_3 \\
    g^* &= C \sinh v_2 x_3 + D \cosh v_2 x_3 + E \sinh v_3 x_3 + F \cosh v_3 x_3
\end{align*} \] (71)
where
\[ v_1 = \sqrt{\alpha^2 - k_1^2}, \quad v_2 = \sqrt{\alpha^2 - p^2}, \quad v_3 = \sqrt{\alpha^2 + q^2} \]
\[ p^2 = \left( \frac{1}{2\ell^2} \right) [(1 + 4k_2^2 \ell^2)^{1/2} - 1], \quad q^2 = \left( \frac{1}{2\ell^2} \right) [1 + (1 + 4k_2^2 \ell^2)^{1/2}] \]

By virtue of equations (71), (69) and (67), we finally obtain
\[ \begin{align*}
    \phi &= (A \sin v_1 x_3 + B \cosh v_1 x_3) e^{i(\omega t - \alpha x_1)} \\
    \psi &= (C \sinh v_2 x_3 + D \cosh v_2 x_3 + E \sinh v_3 x_3 + F \cosh v_3 x_3) e^{i(\omega t - \alpha x_1)}
\end{align*} \] (72)

We consider the particular case
\[ \begin{align*}
    \phi_1 &= B \cosh v_1 x_3 e^{i(\omega t - \alpha x_1)} \\
    \psi_1 &= (C \sinh v_2 x_3 + E \sinh v_3 x_3 + F \cosh v_3 x_3) e^{i(\omega t - \alpha x_1)}
\end{align*} \] (73)

Introducing (73) into (61), it is readily observed that in this case the displacements are symmetric with respect to \( x_3 = 0 \), so are the stresses \( \sigma_{33}, \sigma_{11} \) and couple-stress \( \mu_{32} \).

Besides, the stress \( \sigma_{13} \) and the couple-stress \( \mu_{12} \) are antisymmetric with respect to the plane \( x_3 = 0 \).

Substituting from (73) into the boundary conditions (64), we get,
\[
(p_0^2 - 2\mu_0^2)B\cosh v_1 h + 2i\mu_0(Cv_2 \cosh v_2 h + Ev_3 \cosh v_3 h) = 0
\]
\[
2i\alpha_1 B\sinh v_1 h + C{(v_1^2 + \alpha^2) - \ell^2(v_1^2 - \alpha^2)^2}\sinh v_2 h
+ E{(v_3^2 + \alpha^2) - \ell^2(v_3^2 - \alpha^2)^2}\sinh v_3 h = 0
\]
\[
Cv_2(v_2^2 - \alpha^2)\cosh v_2 h + Ev_3(v_3^2 - \alpha^2)\cosh v_3 h = 0
\]

Since \(\ell\) is small, the system of equations (74) takes the form
\[
\begin{align*}
&\alpha{(c^2/c_2)^2 - 2}\cosh v_1 h]B + 2i(Cv_2 \cosh v_2 h + Ev_3 \cosh v_3 h) = 0 \\
&2iB\alpha_1 \sinh v_1 h + C(2\alpha^2 - k_2^2)\sinh v_2 h + E(2\alpha^2 - k_2^2)\sinh v_3 h = 0 \\
&- Cv_2 p^2 \cosh v_2 h + Ev_3 q^2 \cosh v_3 h = 0
\end{align*}
\]

(75)

These equations lead to the transcendental equation
\[
\frac{\tanh v_1 h}{\tanh v_2 h} = \left\{2 - (c^2/c_1^2)\right\} \frac{\left(2\alpha^2 - k_2^2\right)}{4v_1 v_2 \left(1 + (p^2/q^2)\right) \left(1 + (v_2 p^2/v_3 q^2)\right) \frac{\tanh v_3 h}{\tanh v_2 h}}
\]

(76)

It is readily observed that the wave velocity equation (76), as derived above for the propagation of waves in an elastic layer, contains the couple-stress parameter \(\ell\). Assuming the couple-stress parameter \(\ell\) to be small so that its cubes and higher powers are always neglected, To discuss the result of the wave velocity equation (76) in the following cases

**Case I.** If the length of the wave is large compared with the thickness of the layer \(2h\), the quantities \(v_1 h, v_2 h\) and \(\alpha h\) can be regarded as small and the hyperbolic tangents can be replaced by their arguments. But \(v_3 h\) is not small. So the equation (76) becomes
\[
4v_1^2 \left(1 + (p^2/q^2)\right) = \left\{2 - (c^2/c_1^2)\right\} \frac{\left(2\alpha^2 - k_2^2\right)}{1 + (p^2/q^2) \frac{\tanh v_3 h}{v_3 h}}
\]

(77)

Equation (77) determines the velocity \(c\) of the plane wave under the influence of couple-stress in an infinitely extended layer.

If the parameter \(\ell\) of couple-stress vanishes, we get from equation (77)
\[
4 \left\{1 - (c^2/c_1^2)\right\} - \left\{2 - (c^2/c_2^2)\right\} = 0
\]

i.e. \(c^2 c_1^2 = 4c_2^2(c_1^2 - c_2^2)\) which is in agreement with the classical result of Rayleigh [164] and Lamb [74].
Since the parameter \( \ell \) of couple-stresses is small, we get the following approximate form of the equation (77)

\[
4v^2(1 + k^2/\ell^2) = \left(2 - c^2/c_2^2\right)(2\alpha^2 - k^2)
\]

which can be rewritten as

\[
4\left(1-c^2/c_1^2\right)\left(1+\left(c^2/c_2^2\right)\alpha^2\ell^2\right) = \left(2-c^2/c_2^2\right)^2
\]

we assume \( \lambda = \mu \), (\( v = 1/4 \)), then \( c_1^2 = 3c_2^2 \), and \( c^2/c_2^2 = \frac{8}{3}(1 + \frac{1}{\alpha^2} \ell^2) \)

which shows that the symmetrical mode of vibration \( c/c_2 \) increases due to the presence of couple-stress. On the other hand the velocity \( c \) of the propagation of waves in the elastic layer increases due to the presence of couple-stress.

**Case II.** If, again, the length of the wave is very small compared with the thickness of the layer \( 2h \), the quantities \( v_1h, v_2h, v_3h, \) and \( ah \) are large and we may assume that the ratio of hyperbolic tangents in equation (76) approaches unity and equation (76) becomes

\[
4 \left(1-c^2/c_1^2\right)^{1/2} \left(1-p^2/\alpha^2\right)^{1/2} \left(1+p^2/q^2\right) = \left(2-c^2/c_2^2\right)^2 \left\{1+\left(v_2p^2/v_3q^2\right)\right\}
\]

Equation (79) determines the velocity of the Rayleigh surface wave under the influence of couple-stress. If \( \ell = 0 \), couple-stress vanish and we get the classical result

\[
\left\{2-\left(c^2/c_2^2\right)\right\}^2 = 4 \left\{1-\left(c^2/c_1^2\right)\right\}^{1/2} \left\{1-\left(c^2/c_2^2\right)\right\}^{1/2}.
\]

The couple-stress parameter \( \ell \) being small we get the following approximate form of the equation (79),

\[
(c^2/c_2^2) \left[c^6/c_2^6\right] - 8(c^4/c_2^4) + c^2\left\{24/c_2^2\right\} - \left(16/c_1^2\right) - 16\left\{1-(c_2^2/c_1^2)\right\} - 16(c^2/c_2^2)\alpha^2\ell^2 \times \\
\times \left\{1-(c^2/c_1^2)\right\} - 32\alpha^2\ell^2\left[1 - \left(c^2/c_2^2 + c^2/c_1^2\right) + \frac{c^4}{c_1^2 c_2^2}\right] = 0.
\]

We assume \( \lambda = \mu \) (i.e. \( v = \frac{1}{4} \)), then \( c_1 = \sqrt{3} \ c_2 \), and obtain from the above equation (80)

\[
\left(\frac{c^6}{c_2^6} - 8\frac{c^4}{c_2^4} + \frac{56 c^2}{3 c_2^2} - \frac{32}{3}\right) - \frac{16}{3} \alpha^2\ell^2 \left(3 - \frac{c^2}{c_2^2}\right) \left\{2 - \frac{c^2}{c_2^2}\right\} = 0
\]

The above cubic equation in \( c^2/c_2^2 \) has three real and positive roots. One of the roots lies between 0.8453 and 2. In order to satisfy the condition of Rayleigh waves the value of
\( \alpha^2\varepsilon^2 \) must be less than 0.09 and the acceptable root for the real existence for Rayleigh waves have been computed numerically. It is thus observed that the velocity of Rayleigh waves increases due to the influence of couple-stress.

**Numerical result**

In the following table the values of \( c/c_2 \) have been computed for different values of \( \alpha^2\varepsilon^2 \) in equation (81) and thus it has been exhibited that the velocity of Rayleigh waves in presence of couple-stresses increases and the velocity of Rayleigh waves under the influence of couple-stress, approaches to the classical value 0.9194019 \( c_2 \) with the diminution of the value of the parameter of couple-stress.

**Table 2**: Computation of \( c/c_2 \) for different values of couple-stress parameter

<table>
<thead>
<tr>
<th>( \alpha^2\varepsilon^2 )</th>
<th>10^{-2}</th>
<th>10^{-3}</th>
<th>10^{-4}</th>
<th>10^{-5}</th>
<th>10^{-6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c/c_2 )</td>
<td>0.92915</td>
<td>0.92288</td>
<td>0.92199</td>
<td>0.91941</td>
<td>0.9194025</td>
</tr>
</tbody>
</table>

Let us consider another set of interesting solutions given by

\[
\phi_2 = A \sinh \nu_1 x_3 e^{i(\omega t - \alpha x_1)} , \quad \psi_2 = (D \cosh \nu_2 x_3 + F \cosh \nu_3 x_3) e^{i(\omega t - \alpha x_1)}. \tag{82}
\]

Introducing (82) into (61), it is readily observed that the displacements \( u_1, u_3 \), the stresses \( \sigma_{11}, \sigma_{33} \) and couple-stress \( \mu_{32} \) are antisymmetric with respect to the plane \( x_3 = 0 \) while the stresses \( \sigma_{13}, \sigma_{31} \) and the couple-stress \( \mu_{12} \) are symmetric with respect to this plane.

Introducing (82) into the boundary condition (64) and making use of equations (66), we obtain the following three linear equations for \( A, D \) and \( F \).

\[
\begin{align*}
(p \omega^2 - 2\mu\alpha^2)A \sinh \nu_1 h + 2i\mu\alpha (Dv_2 \sinh \nu_2 h + Fv_3 \sinh \nu_3 h) &= 0, \\
2i\alpha v_1 A \cosh \nu_1 h + D((v_2^2 + \alpha^2) - \ell^2(v_2^2 - \alpha^2)^2) \cosh \nu_2 h &+ F((v_3^2 + \alpha^2) - \ell^2(v_3^2 - \alpha^2)^2) \cosh \nu_3 h = 0, \\
D(v_2^2 - \alpha^2)v_2 \sinh \nu_2 h + F(v_3^2 - \alpha^2)v_3 \sinh \nu_3 h &= 0.
\end{align*}
\tag{83}
\]

Since \( \ell \) is small, elimination of \( A, D, F \) from the equation (83) leads to the equation

\[
\frac{\tanh \nu_1 h}{\tanh \nu_2 h} = 4v_1 v_2 \left[1 + (p^2/q^2)\right]/\left(2\alpha^2 - k_2^2\right)\left(2 - (c^2/c_2^2)\right) \left(1 + \frac{p^2 v_2}{q^2 v_3} \tanh \nu_2 h\right) \tag{84}
\]
Case I. If the length of the wave is very small compared with the thickness of the layer, equation (84) reduces to (79) and yields the velocity of propagation of Rayleigh surface waves under the influence of couple-stress. If the parameter of couple-stress is zero, the classical result follows.

Case II. If the length of the wave be large compared with the thickness of layer, the quantities $v_1h$, $v_2h$ and $\alpha h$ can be regarded as small and the hyperbolic tangents $\tanh v_1h$, $\tanh v_2h$ can be replaced by the first two terms of their expansions into series. Then equation (84) becomes

$$
\frac{v_1 \{1 - (v_1^2 h^2 / 3)\}}{v_2 \{1 - (v_2^2 h^2 / 3)\}} = \frac{4v_1v_2 \{1 + (p^2 / q^2)\}}{(2\alpha^2 - k_2^2) \{2 - (c^2 / c_2^2)\} \left\{1 + \frac{p^2 v_2^2 \{1 - (v_2^2 h^2 / 3)\} h}{q^2 v_3 \tanh v_3 h}\right\}}
$$

If $\ell = 0$ we get, on simplification,

$$(c^2 / c_2^2) = \frac{4}{3} \alpha^2 h^2 \{1 - (c_2^2 / c_1^2)\}, \quad c = (\omega / \alpha)$$

which is the classical result of Rayleigh and Lamb since $\ell$ is small, we obtain the following approximation of equation (85)

$$(c^2 / c_2^2) = \left(4 \alpha^2 h^2 / 3\right) \{1 - (c_2^2 / c_1^2)\} + 4 \alpha^2 \ell^2$$

(86)

which shows that the value of $c / c_2$ increases in presence of couple stress. In other words the velocity $c$ of the propagation of waves in the elastic layer increases due to the presence couple stress.

Axis Symmetric Lamb’s Problem in Couple Stress Theory: We consider a time varying loading $z(r, t) = p(r) \exp(i\omega t)$ acting on the elastic semi-space bounded by the $z = 0$ plane, $z$ axis being pointed into the medium. The loading being axially symmetrical, it produces in the semispace an axis-symmetrical state of stress and deformation and the cylindrical co-ordinates $(r, \phi, z)$ is used to investigate the problem.

We consider the displacement equation of motion in absence of body forces and body couples as

$$
\mu \nabla^2 \ddot{u} + (\lambda + \mu) \nabla \cdot \nabla \dot{u} + \eta \nabla^2 \nabla \times \nabla \times \ddot{u} = \rho \dddot{t}
$$

(87)
we express the displacement $\mathbf{u}$ into its lamellar and solenoidal components

$$\mathbf{u} = \nabla \phi + \nabla \times \mathbf{H}, \quad \nabla \cdot \mathbf{H} = 0$$

(88)

Then the above equation of motion reduces to the following equations

$$c_1^2 \nabla^2 \phi = \ddot{\phi}; \quad c_2^2 (1 - \ell^2 \nabla^2) \nabla^2 \mathbf{H} = \ddot{\mathbf{H}}$$

(89)

where

$$\ell^2 = \frac{n}{\mu}, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}$$

For harmonic waves we insert $\phi = \phi \exp(i\omega t)$ and $\mathbf{H} = \mathbf{H} \exp(i\omega t)$ in (89) to obtain

$$(\nabla^2 + \sigma_1^2) \ddot{\phi} = 0, \quad (\nabla^2 + \beta_1^2)(\nabla^2 - \beta_2^2) \ddot{\mathbf{H}} = 0$$

(90)

where

$$\sigma_1 = \omega/c_1, \quad \beta_1 = 2^{-1/2} \ell^{-1} [(1 + 4\ell^2 \sigma_2^2)^{1/2} - 1]^{1/2}$$

$$\sigma_2 = \omega/c_2, \quad \beta_2 = 2^{-1/2} \ell^{-1} [(1 + 4\ell^2 \sigma_2^2)^{1/2} + 1]^{1/2}$$

Then, considering $\ddot{\mathbf{H}} = \ddot{\mathbf{H}}' + \ddot{\mathbf{H}}''$ the complete solution becomes

$$\ddot{\mathbf{u}} = \nabla \dddot{\phi} + \nabla \times \dddot{\mathbf{H}}' + \nabla \times \dddot{\mathbf{H}}''$$

(91)

where $\dddot{\phi}$, $\dddot{\mathbf{H}}'$ and $\dddot{\mathbf{H}}''$ are governed by the Helmholtz equations

$$(\nabla^2 + \sigma_1^2) \dddot{\phi} = 0, \quad (\nabla^2 + \beta_1^2) \dddot{\mathbf{H}}' = 0, \quad (\nabla^2 - \beta_2^2) \dddot{\mathbf{H}}'' = 0$$

(92)

**Boundary conditions**: The boundary conditions on $z = 0$

$$\tau_{zz} = -p(r) e^{i\omega t}, \quad p_r = p_{\phi} = \mu_{z\phi} = \mu_{zr} = 0$$

(93)

where

$$p_r = \tau_{sr} - \frac{1}{2} \left( \frac{\partial}{\partial r} \mu_{r\phi} + \frac{1}{r} \frac{\partial}{\partial \phi} \mu_{\phi\phi} + \frac{\partial}{\partial z} \mu_{z\phi} + \frac{\mu_{r\phi} + \mu_{\phi r}}{r} - \frac{1}{r} \frac{\partial}{\partial \phi} \mu_{zz} \right)$$

$$p_\phi = \tau_{s\phi} + \frac{1}{2} \left( \frac{\partial}{\partial r} \mu_{r\phi} + \frac{1}{r} \frac{\partial}{\partial \phi} \mu_{\phi\phi} + \frac{\partial}{\partial z} \mu_{z\phi} + \frac{\mu_{r\phi} - \mu_{\phi r}}{r} - \frac{\partial}{\partial \phi} \mu_{zz} \right)$$

we consider the particular case in which the external loading and the displacement vector $\ddot{\mathbf{u}}$ are independent of $\phi$, the body forces and body couples being discarded. In this case $w_r = w_z = 0$, $p_r = \mu_{zr} = 0$.

Hence the boundary conditions reduce to

$$\tau_{sr} - \frac{1}{2} \left( \frac{\partial}{\partial r} \mu_{r\phi} + \frac{\partial}{\partial z} \mu_{z\phi} + \frac{\mu_{r\phi} + \mu_{\phi r}}{r} \right) = 0$$

(94)

$$\tau_{zz} = -p(r) e^{i\omega t}, \quad \mu_{z\phi} = 0$$

$$w_r = w_z = 0, \quad p_r = \mu_{zr} = 0$$
With the boundary conditions (94), we seek solutions of (92). In this particular case \( \overline{H}_r = \overline{H}_z = 0 \). Hence the equations (92) take the following forms

\[
(\nabla^2 + \sigma_1^2) \overline{\phi} = 0, \quad (\nabla^2 - \frac{1}{r^2} + \beta_1^2) \overline{H}_r = 0, \quad (\nabla^2 - \frac{1}{r^2} - \beta_2^2) \overline{H}_z = 0
\]  

(95)

Applying in equations of (95), the Hankel transformation defined by

\[
\phi^*(\alpha, z) = \int_0^\infty r \overline{\phi}(r, z) J_0(\alpha r) \, dr,
\]

\[
\psi^*(\alpha, z) = \int_0^\infty r \overline{\psi}(r, z) J_0(\alpha r) \, dr
\]

we obtain

\[
\phi^* = Ae^{-\alpha z}, \quad \psi_1^* = Be^{-\nu_1 r}, \quad \psi_2^* = ce^{-\nu_2 z}
\]

(97)

where \( \sigma = (\alpha^2 - \alpha_1^2)^{1/2}, \quad \nu_1 = (\alpha^2 - \beta_1^2)^{1/2}, \quad \nu_2 = (\alpha^2 + \beta_2^2)^{1/2} \)

Hence in view of (88), (91) and (97) we obtain

\[
u_t = \exp(\omega t) \int_0^\infty \left\{ -\alpha A e^{-\sigma z} + \alpha (\nu_1 B e^{-\nu_1 r} + \nu_2 C e^{-\nu_2 z}) \right\} \alpha J_1(\alpha r) \, d\alpha
\]

(98)

\[
u_z = \exp(\omega t) \int_0^\infty \left\{ -\alpha A e^{-\sigma z} + \alpha^2 (B e^{-\nu_1 r} + C e^{-\nu_2 z}) \right\} \alpha J_0(\alpha r) \, d\alpha
\]

(99)

Substituting (98) and (99) in the boundary conditions (94) we obtain

\[
2\sigma A - \left\{ (\nu_1^2 + \alpha^2) - \ell^2 (\nu_2^2 - \alpha^2)^2 \right\} \alpha B - \left\{ (\nu_1^2 + \alpha^2) - \ell^2 (\nu_2^2 - \alpha^2)^2 \right\} \alpha C = 0
\]

(100)

\[
(2\alpha^2 - \sigma_2^2) A - 2\alpha^2 (\nu_1 B + \nu_2 C) = -\frac{p^*(\alpha)}{\mu}
\]

\[
\alpha \nu_1 (\alpha^2 - \nu_1^2) B + \alpha \nu_2 (\alpha^2 - \nu_2^2) C = 0
\]

where we have expressed the loading \( p(r) \) by the Hankel integral

\[
P(r) = \int_0^\infty P^*(\alpha) \alpha J_0(\alpha r) \, d\alpha, \quad P^*(\alpha) = \int_0^\infty r P(r) J_0(\alpha r) \, dr
\]

(101)

Solving the system of equations (100) we obtain

\[
A(\alpha) = -\frac{K P^*}{R(\alpha) \mu}, \quad B(\alpha) = -\frac{2\sigma}{R(\alpha) \mu} \frac{P^*}{\mu}, \quad C(\alpha) = \frac{\nu_1}{\nu_2} \frac{\beta_1^2}{\beta_2^2}
\]

(102)
where \( R(\alpha) = K(2\alpha^2 - \sigma_2^2) - 4\alpha^2\nu_1\sigma \left(1 + \frac{\beta_1^2}{\beta_2^2}\right) \), \( K = (2\alpha^2 - \sigma_2^2) \left(1 + \frac{\nu_1\beta_1^2}{\nu_1\beta_2^2}\right) \)

Hence for a concentrated force \( z(r, t) \)

\[
\begin{align*}
  u_r &= \frac{P_0 e^{i\alpha t}}{2\pi \mu} \int_0^\infty \left\{ K\alpha e^{-\alpha z} - 2\sigma\nu_1\alpha \left(e^{-\nu_1 z} + \frac{\beta_1^2}{\beta_2^2} e^{-\nu_2 z}\right) \right\} \frac{\alpha J_1(\alpha r)}{R(\alpha)} d\alpha \\
  u_z &= \frac{P_0 e^{i\alpha t}}{2\pi \mu} \int_0^\infty \left\{ K\alpha e^{-\alpha z} - 2\alpha^2 \left(e^{-\nu_1 z} + \frac{\nu_1}{\nu_2 \beta_2^2} e^{-\nu_2 z}\right) \right\} \frac{\alpha J_0(\alpha r)}{R(\alpha)} d\alpha \\
\end{align*}
\]

Thus the displacements given by (103) under the influence of couple stress being known we can determine the stresses and the couple stresses from equation (1). If \( \ell \) tends to zero the classical results follow [93].

**Effect of couple stress on steady-state response to moving loads:**

We consider a homogeneous, isotropic elastic semi space selecting the origin on the free plane boundary and \( x_2 \)-axis pointing into the medium, the Cartesian co-ordinates of a point of the semi-space \( x_2 \geq 0 \) are supposed to be \( (x_1, x_2, x_3) \), the corresponding displacement components are designated by \( (u_1, u_2, u_3) \) As regards the line load we suppose that on plane boundary of the semi-space there acts the loading \( P\delta(x_1 + ut) \) where \( P \) is a constant and \( \delta(t) \) is the Dirac-delta function

We assume that a plane strain state prevails and the elastic displacements \( u_1, u_2 \), \( u_3 = 0 \) are derivable from the displacement potentials \( \phi(x_1, x_2) \) and \( \psi(x_1, x_2) \) so that

\[
  u_1 = \phi_{,1} - \psi_{,2} \quad , \quad u_2 = \phi_{,2} + \psi_{,1}
\]

Displacement equations of motion in absence of body forces and body couples are

\[
\mu \nabla^2 u_i + (\lambda + \mu) u_j_{,ij} + \eta \nabla^2 (u_{j,ii} - \nabla^2 u_i) = \rho \ddot{u}_i \quad (i, j = 1, 2, 3)
\]

The equations of motion (105) in view of (104) yield the following two equations

\[
\nabla^2 \phi - c_1^{-2} \phi = 0 \quad ; \quad \nabla^2 \psi - \ell^2 \nabla^4 \psi = c_2^{-2} \psi
\]

where \( c_1^2 = (\lambda + 2\mu)/\rho \), \( c_2^2 = \mu/\rho \), \( \ell^2 = \eta/\mu \)
\( \eta \) is a constant characterizing the existence of couple-stresses and \( \lambda, \mu \) are Lame's elastic constants. If \( \ell = 0 \) the couple-stress vanishes and the classical result follows at once.

Expressions for stresses and couple stress in terms of \( \phi \) and \( \psi \) are

\[
\begin{align*}
\tau_{11} &= 2\mu(\phi_{,12} - \psi_{,12}) + \lambda \nabla^2 \phi, \\
\tau_{22} &= 2\mu(\phi_{,12} + \psi_{,12}) + \lambda \nabla^2 \phi, \\
\tau_{12} &= \mu(2\phi_{,12} + \psi_{,11} - \psi_{,22}) - \eta \nabla^4 \psi, \\
\tau_{21} &= \mu(2\phi_{,12} + \psi_{,11} - \psi_{,22}) + \eta \nabla^4 \psi
\end{align*}
\]

(107)

\( \mu_{13} = 2\eta \nabla^2 (\psi_{,1}), \mu_{23} = 2\eta \nabla^2 (\psi_{,2}) \)

Now, in the problem stated above, the load moves in the negative direction of \( x_1 \)-axis at a constant speed \( u \). An observer moving with the load at the same speed would see the load as stationary. We introduce a Galilean transformation \[ x'_1 = x_1 + ut, \quad x'_2 = x_2, \quad t' = t \]

(108)
than the boundary conditions would be independent of \( t' \). For a concentrated line load the boundary conditions are in the moving co-ordinates, are

\[
\begin{align*}
\tau_{22} &= -p\delta(x'_1), \\
\tau_{21} &= \mu_{23} = 0
\end{align*}
\]

(109)

Now, as the response in the elastic semi-space is in a steady state, \( \phi \) and \( \psi \) will be independent of \( t' \) as seen by an observer moving with the load. In other words, \( x_1 \) and \( t \) enter \( \phi \) and \( \psi \) in combination \( x_1 + ut \). Under this assumption, equations (106) become

\[
\begin{align*}
\nabla^2 \phi &= c_1^{-2} u^2 \frac{\partial^2 \phi}{\partial x_1'^2}, \\
\nabla^2 \psi - \ell^2 \nabla^4 \psi &= c_2^{-2} u^2 \frac{\partial^2 \psi}{\partial x_1'^2}
\end{align*}
\]

(110)

On introducing the Mach numbers \[ M_1 = u/c_1, \quad M_2 = u/c_2 \]

(111)

and the parameters,

\[
\begin{align*}
\bar{\beta}_1 &= \sqrt{1 - M_1^2}, & \bar{\beta}_2 &= \sqrt{1 - M_2^2} & \text{if } M_1 < 1, \ M_2 < 1. \\
\beta_1 &= \sqrt{M_1^2 - 1}, & \beta_2 &= \sqrt{M_2^2 - 1} & \text{if } M_1 > 1, \ M_2 > 1.
\end{align*}
\]

(112)

(113)

we obtain the following partial differential equations
\[ \beta_1^2 \frac{\partial^2 \phi}{\partial x_1'^2} + \frac{\partial^2 \phi}{\partial x_2'^2} = 0 \quad \text{if } M_1 < 1 \]  \hspace{1cm} (114)\\
\[ \beta_2^2 \frac{\partial^2 \psi}{\partial x_1'^2} + \frac{\partial^2 \psi}{\partial x_2'^2} - \epsilon^2 \nabla^4 \psi = 0 \quad \text{if } M_2 < 1 \]\\
\[ \beta_1^2 \frac{\partial^2 \phi}{\partial x_1'^2} - \frac{\partial^2 \phi}{\partial x_2'^2} = 0 \quad \text{if } M_1 > 1 \]  \hspace{1cm} (115)\\
\[ \beta_2^2 \frac{\partial^2 \psi}{\partial x_1'^2} - \frac{\partial^2 \psi}{\partial x_2'^2} + \epsilon^2 \nabla^4 \psi = 0 \quad \text{if } M_2 > 1 \]

In view of (110) and (111), the stresses and couple-stresses given by (107) reduce into

\[ \frac{\tau_{11}}{\mu} = (M_2^2 - 2M_1^2 + 2) \frac{\partial^2 \phi}{\partial x_1'^2} - 2 \frac{\partial^2 \psi}{\partial x_1' \partial x_2'} \]  \hspace{1cm} (116)\\
\[ \frac{\tau_{12}}{\mu} = M_2 \frac{\partial^2 \psi}{\partial x_1'^2} - 2 \frac{\partial^2 \psi}{\partial x_1' \partial x_2'} + 2 \frac{\partial^2 \phi}{\partial x_2'^2} \]\\
\[ \frac{\tau_{21}}{\mu} = 2 \frac{\partial^2 \phi}{\partial x_1' \partial x_2'} - (M_2^2 - 2) \frac{\partial^2 \phi}{\partial x_2'^4} \]\\
\[ \mu_{13} = 2\eta \nabla^2 \left( \frac{\partial \psi}{\partial x_1'} \right) \quad ; \quad \mu_{23} = 2\eta \nabla^2 \left( \frac{\partial \psi}{\partial x_2'} \right) \]

The boundary conditions (109) become, at \( x_2' = 0 \)

\[ \left( M_2^2 - 2 \right) \frac{\partial^2 \phi}{\partial x_1'^2} + 2 \frac{\partial^2 \psi}{\partial x_1' \partial x_2'} = -\frac{p}{\mu} \delta(x_1') \]  \hspace{1cm} (117)\\
\[ 2 \frac{\partial^2 \phi}{\partial x_1' \partial x_2'} - (M_2^2 - 2) \frac{\partial^2 \psi}{\partial x_1'^2} = 0 \]\\
\[ \nabla^4 \left( \frac{\partial \psi}{\partial x_2'} \right) = 0 \]

Now, we have to determine the functions \( \phi \) and \( \psi \) which satisfy the differential equations (114) or (115), the boundary conditions on the free surface (117) and the appropriate radiation and finiteness conditions at infinity.

The nature of the solution depends on the Mach numbers \( M_1, M_2 \). Three cases can be distinguished:

(a) \( M_2 > M_1 > 1 \), (Supersonic) (b) \( M_2 > 1 > M_1 \), (Transonic) (c) \( 1 > M_2 > M_1 \) (Subsonic)

Since \( C_1 > C_2 \), so that \( M_2 > M_1 \), the three cases above exhaust all possibilities

**Supersonic Case \([M_1 > 1, M_2 > 1]\).**

In this case, we must solve the equations (115), we take the solutions as
\( \phi(x_1', x_2') = A_1(x_2')e^{i\lambda x_1'} \), \( \psi(x_1', x_2') = A_2(x_2')e^{i\lambda x_1'} \)

From the first and second equation of (115), we have respectively

\[ A_1(x_2') = Ae^{-i\eta_1 x_2'} \quad (119) \]
\[ A_2(x_2') = Be^{-i\eta_2 x_2'} + C e^{-i\eta_3 x_2'} \quad (120) \]

where \( \eta_1 = \beta_1 \lambda \), \( \eta_2 = \sqrt{\lambda^2 - p^2} \), \( \eta_3 = \sqrt{\lambda^2 + q^2} \)

\[ p^2 = \frac{1}{2\epsilon^2} \left[ 1 + (1+\beta_2^2)4\epsilon^2 \lambda^2 \right] - 1 \]
\[ q^2 = \frac{1}{2\epsilon^2} \left[ 1 + 4(1+\beta_2^2)\epsilon^2 \lambda^2 \right] + 1 \quad (121) \]

where \( \lambda, A, B \) and \( C \) are arbitrary constants Hence, the solutions are

\[ \phi = Ae^{i\lambda x_1 - i\eta_1 x_2} \], \( \psi = (Be^{-i\eta_2 x_2} + Ce^{-i\eta_3 x_2'})e^{i\lambda x_2'} \quad (122) \]

which satisfy the radiation condition at infinity only backward running waves are admitted. The other possible solutions, of the form \( e^{i(\lambda x_1 + \eta_1 x_2')} \) are rejected on the basis of the radiation condition because they represent disturbance which originate at infinity and coverage towards the load. We assume a general in the form

\[ \phi(x_1', x_2') = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\lambda)e^{-i\eta_1 x_1' - i\lambda x_1'}d\lambda \]
\[ \psi(x_1', x_2') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{B(\lambda)e^{-i\eta_2 x_2'} + C(\lambda)e^{-i\eta_3 x_2'}\}e^{i\lambda x_2'}d\lambda \quad (123) \]

The boundary conditions (117) give

\[ (i\lambda)^2(M_2^2 - 2)A(\lambda) - 2i\lambda\{\eta_2 B(\lambda) + \eta_3 C(\lambda)\} = P(\lambda) \]
\[ 2\eta_1 A(\lambda) + (M_2^2 - 2)(B(\lambda) + C(\lambda)\lambda = 0 \]
\[ C(\lambda) = \frac{-\eta_2(\eta_2^2 - \lambda^2)}{\eta_3(\eta_3^2 - \lambda^2)}B(\lambda) \quad (124) \]

where \( P(\lambda) = -P/\mu \) is the Fourier transform of \(-P/\mu)\delta(x_1')\).

Hence, from (124) we get on simplification
\[ A(\lambda) = (M_2^2 - 2) \left( 1 + \frac{\eta_2 p_2^2}{\eta_3 q_2^2} \right) \lambda \frac{P^* (\lambda)}{\Delta^*} \]
\[ B(\lambda) = -2\eta_1 P^*(\lambda) / \Delta^* \]  

where \( \Delta^* = (M_2^2 - 2)^2 \left( 1 + \frac{\eta_2 p_2^2}{\eta_3 q_2^2} \right) \eta_1^2 + 4\eta_1 \eta_2 \left( 1 + \frac{p_2^2}{q_2^2} \right) \)  

In view of (125) and (126), (123) becomes

\[ \phi(x'_1, x'_2) = \frac{M_2^2 - 2}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{\alpha} \frac{P^*(\lambda)}{1} e^{i\lambda(x_1 - x_2)} d\lambda \right\} \left\{ \frac{2\eta p_2}{\Delta^*} e^{i\lambda(x_1' - x_2')} \right\} d\lambda \]  

\[ \psi(x'_1, x'_2) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\eta \lambda P^*(\lambda)}{\Delta^*} \left\{ e^{i\lambda x_1' - \eta x_2'} - \frac{\eta_2 p_2^2}{\eta_3 q_2^2} e^{i\lambda x_1' - \eta_2 x_2'} \right\} d\lambda \]

Since the parameter of the couple-stresses \( \ell^2 \) is small, we may neglect its cubes and higher powers. Then the approximate forms of \( \phi \) and \( \psi \) are as follows.

\[ \phi(x'_1, x'_2) = \frac{M_2^2 - 2}{4\pi \Delta} \left\{ \alpha \frac{P^*(\lambda)}{1} e^{i\lambda(x_1 - x_2)} d\lambda + \Delta \ell^2 \int_{-\infty}^{\infty} \frac{2\eta \lambda P^*(\lambda)}{1} e^{i\lambda(x_1' - x_2')} d\lambda \right\} \]  

\[ \psi(x'_1, x'_2) = \frac{2\beta_1}{2\pi \Delta} \int_{-\infty}^{\infty} \frac{e^{i\lambda(x_1' - x_2')}}{1} d\lambda \left\{ \frac{1}{1 - \alpha} + \frac{2\ell^2 \Delta \Delta_1}{i\lambda^2} + \frac{(1 + \beta_2^2)^2}{2\beta_2} \ell^2 \Delta_1 e^{i\lambda(x_1' - x_2')} \right\} e^{i\lambda(x_1' - x_2')} d\lambda \]

where \( \Delta = (M_2^2 - 2)^2 + 4\beta_1 \beta_2 \), \( \Delta_1 = \frac{2\beta_1 (1 + \beta_2^2)}{\beta_2 \Delta} (1 - \beta_2^2) \)

In view of (104), (116), (128) and (129), we obtain

\[ u_1 = \frac{P}{\mu \Delta} \left\{ (2 - M_2^2) H(x_1' - x_2') + 2\beta_1 \beta_2 H(x_1' - x_2') \right\} + \frac{P}{\mu \Delta} \ell^2 \left[ \Delta_1 (2 - M_2^2) F_1(x_1' - x_2') \right. \]  

\[ - \left( 2\beta_1 \beta_2 \Delta_1 - \frac{2\beta_1}{2\beta_2} (1 + \beta_2^2)^2 \right) F_2(x_1' - x_2') - \beta_1 (1 + \beta_2^2)^2 x_2' \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') \right\} \]  

\[ u_2 = -\frac{P}{\mu \Delta} \left\{ \beta_1 (2 - M_2^2) H(x_1' - x_2') - 2\beta_1 H(x_1' - x_2') \right\} + \frac{P \beta_1}{\Delta \mu} \ell^2 \left[ (2 - M_2^2) \Delta_1 \right. \]  

\[ F_1(x_1' - x_2') + 4\Delta_1 F_2(x_1' - x_2') + \frac{(1 + \beta_2^2)^2}{\beta_2} \ell^2 x_2' \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') \right\} \]
\[ \tau_{11} = \frac{P}{\Delta} \left[ (2 - M_2^2) \{ 2 + (\lambda/\mu) M_2^2 \} \delta(x_1' - x_2') + \frac{4 \beta_1 \beta_2}{\Delta} \delta(x_1' - x_2') \} + \right. \\
+ \frac{P}{\Delta} \varepsilon^2 \left[ \Delta_1 \frac{\partial}{\partial x_2'} F_1(x_1' - x_2') \left( 2 + \frac{\lambda}{\mu} M_1^2 \right) (2 - M_2^2) - 4 \beta_1 \beta_2 \left\{ 2 \Delta_1 \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') \right. \\
+ \frac{(1 + \beta_2^2)^2}{2 \beta_2} x_2' \frac{\partial^2}{\partial x_2'} F_2(x_1' - x_2') \left. \right\} + \frac{4 \beta_1 (1 + \beta_2^2)^2}{2 \beta_2} \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') \right] \\
\tau_{22} = \frac{P}{\Delta} \left[ (2 - M_2^2) \left( 2 \beta_1^2 + \frac{\lambda}{\mu} M_1^2 \right) \delta(x_1' - x_2') - 4 \beta_1 \beta_2 \delta(x_1' - x_2') \right. \\
+ \frac{P}{\mu} \varepsilon^2 \left[ \Delta_1 (2 - M_2^2) \frac{\partial}{\partial x_1'} F_1(x_1' - x_2') \left( 2 \beta_1^2 + \frac{\lambda}{\mu} M_1^2 \right) \beta_1^2 + \\
+ 4 \beta_1 \beta_2 \left\{ 2 \Delta_1 \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') \right. \\
+ \frac{(1 + \beta_2^2)^2}{2 \beta_2} x_2' \frac{\partial^2}{\partial x_2'} F_2(x_1' - x_2') \left. \right\} - \\
\left. \frac{4 \beta_1 (1 + \beta_2^2)^2}{2 \beta_2} \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') \right\] \\
\tau_{12} = -\frac{2P}{\Delta} \beta_1 (2 - M_2^2) \left[ \delta(x_1' - x_2') - \delta(x_1' - x_2') \right] - \frac{2P}{\mu} \varepsilon^2 \beta_1 (2 - M_2^2) \Delta_1 F_1(x_1' - x_2') + \\
+ \beta_1 \left\{ 2 \Delta_1 \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') + \frac{(1 + \beta_2^2)^2}{2 \beta_2} x_2' \frac{\partial^2}{\partial x_2'} F_2(x_1' - x_2') \right\} (2 - M_2^2) \right] \\
where \quad F_1 = \delta^{(1)}(x_1' - x_2') = -(x_1' - x_2')^{-1} \delta(x_1' - x_2') \\
\frac{\partial}{\partial x_1'} F_1 = \delta^{(2)}(x_1' - x_2') = 2(x_1' - x_2')^{-2} \delta(x_1' - x_2') \\
\frac{\partial^2}{\partial x_1'^2} F_1 = \delta^{(3)}(x_1' - x_2') = \frac{6}{x_1' - x_2'}(x_1' - x_2')^{-3} \delta(x_1' - x_2'), \quad (i = 1, 2) \\
In case of the above relations (128 - 132), there is a term containing the parameter of couple-stress \( \ell \) with the classical part. Hence the displacements and stresses are slightly changed due to the effect of couple stresses. 

The difference of shear stresses \\
\[ \tau_{12} - \tau_{21} = -2 \eta \nabla^4 \psi = -4 \frac{P}{\Delta} \beta_1 (1 + \beta_2^2)^2 \varepsilon^2 \frac{\partial}{\partial x_1'} F_2(x_1' - x_2') \]
is significant along the line $x_1 = \beta_2 x_2$ due to influence of couple-stresses.

It is also found that the difference of shear stresses increases with the increasing value of the parameter of couple-stress $'\lambda'$ for a constant speed of the moving loads.

**Subsonic Case \( M_1 < 1, \ M_2 < 1 \).**

We have to find $\phi$ and $\psi$ satisfying the partial differential equations (114). We take the solution as

$$
\phi = A_1(\lambda_1 x_2')e^{i\lambda x_1'}, \quad \psi = B_1(\lambda_1 x_2')e^{i\lambda x_1'} \tag{133}
$$

From the first and second equation of (114), we get,

$$
A_1(\lambda_1 x_2') = Ae^{-\eta_1 x_2'}, \quad B_1(\lambda_1 x_2') = Be^{-\eta_2 x_2'} + Ce^{-\eta_3 x_2'} \tag{134}
$$

where $\eta_1 = |\lambda| \beta_1$, $\eta_2 = \sqrt{(\lambda^2 - p_2^2)}$, $\eta_3 = \sqrt{\lambda^2 + q^2}$,

$$
p^2 = \frac{1}{2\lambda^2} \left\{ \sqrt{\{1 + 4(1-\beta_2^2)\lambda^2\lambda^2\} - 1} \right\}, \quad q^2 = \frac{1}{2\lambda^2} \left\{ 1 + \sqrt{\{1 + 4(1-\beta_2^2)\lambda^2\lambda^2\}} \right\} \tag{135}
$$

Hence, we obtain the solution as

$$
\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\lambda)e^{i\lambda x_1'-\eta_1 x_2'}d\lambda, \quad \psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ B(\lambda)e^{-\eta_2 x_2'} + c(\lambda)e^{-\eta_3 x_2'} \} e^{i\lambda x_1'}d\lambda \tag{136}
$$

On substituting (136) into (117), the boundary conditions at $x_2' = 0$ reduce to

$$
\begin{align*}
A(\lambda)(M_2^2 - 2)i\lambda - 2\{\eta_2 B(\lambda) + \eta_3 C(\lambda)\} &= P^*(\lambda) \\
2\eta_1 A(\lambda) + (M_2^2 - 2)\{B(\lambda) + C(\lambda)\}i\lambda &= 0 \\
C(\lambda) &= (\eta_2 p^2 / \eta_3 q^2) B(\lambda)
\end{align*} \tag{137}
$$

Hence $A(\lambda) = (M_2^2 - 2)P^*(\lambda)\{1 + (\eta_2 p^2 / \eta_3 q^2)\}i\lambda / \Delta^*$; $B(\lambda) = -2\eta_1 P^*(\lambda) / \Delta^*$ \(\Delta^* = -\lambda^2(M_2^2 - 2)\{1 + (\eta_2 p^2 / \eta_2 q^2)\} + 4\eta_1 \eta_2 \{1 + (p^2 / q^2)\}\)

Therefore, (136) in view of (138) becomes
\[
\phi(x_1', x_2') = \frac{M_2^2 - 2}{2\pi} \int_{-\infty}^{\infty} \left\{ 1 + \frac{n_2 p^2}{n_3 q^2} \right\} \lambda \frac{\Phi(\lambda)}{\Delta} e^{i\lambda x_1' - n_1 x_2'} d\lambda \\
\psi(x_1', x_2') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ e^{-n_2 x_2'} + \frac{n_2 p^2}{n_3 q^2} e^{-n_3 x_3'} \right\} \frac{2\eta_1 \Phi(\lambda)}{\Delta} e^{i\lambda x_1'} d\lambda
\]

(139)

Since the couples-stress parameter '\( \lambda' \) is small, we may neglect its cubes and higher powers. Hence the approximate forms of \( \phi \) and \( \psi \) are

\[
\phi(x_1', x_2') = \frac{-K_1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(\lambda)}{i\lambda} e^{i\lambda x_1' - \beta_1 |x_1'| x_2'} d\lambda - \frac{K'}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(\lambda)}{i\lambda} e^{i\lambda x_1' - \beta_2 |x_1'| x_2'} d\lambda
\]

and,

\[
\psi(x_1', x_2') = \frac{K_2}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(\lambda)}{i\lambda} e^{i\lambda x_1' - \beta_1 |x_1'| x_2'} d\lambda - \frac{1 - \beta_2^2}{2\beta_2} K_2 \lambda^2 x_2' \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(\lambda)}{i\lambda} e^{i\lambda x_1' - \beta_2 |x_1'| x_2'} d\lambda
\]

\[
- \frac{2\beta_1}{2 - M_2^2} K' \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi(\lambda)}{i\lambda} e^{i\lambda x_1' - \beta_2 |x_1'| x_2'} d\lambda
\]

(141)

where

\[
K_1 = \frac{2 - M_2^2}{(2 - M_2^2)^2 - 4\beta_1 \beta_2} \quad , \quad K_2 = \frac{2\beta_1}{(2 - M_2^2)^2 - 4\beta_1 \beta_2}
\]

\[
K' = (2 - M_2^2)(1 - \beta_2^2)(1 + \beta_2^2) 2\beta_1 \beta_2 \{(2 - M_2^2)^2 - 4\beta_1 \beta_2\}^2
\]

The displacement components and stresses are found to be

\[
u_1 = \frac{P K_1}{\mu} \left[ \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \frac{\beta_1 x_2'}{x_1'} \right] - \frac{P}{\mu} \beta_2 K_2 \left[ \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \frac{\beta_2 x_2'}{x_1'} \right] + \frac{P}{\pi \mu} \lambda^2 \left[ \frac{2\beta_1 K'}{x_2'} \left( \frac{x_2'}{x_1' + \beta_2 x_2'} \right)^2 + \right.
\]

\[
+ K^2 (1 - \beta_2^2) x_1' x_2' \times \left[ \frac{3\beta_2^2 x_2' - x_1'}{(\beta_2^2 x_2' + x_1')^3 - \frac{1}{(x_1' + \beta_2 x_2')^3}} \right] + \frac{4\beta_1 \beta_2^2 K'}{x_2'} \left( \frac{x_2'}{x_1' + \beta_2 x_2'} \right)^2 \right] \]

(142)

\[
u_2 = \frac{P}{2\pi \mu} \left\{ K_2 \log(x_1' + \beta_2 x_2') - K_1 \beta_1 \log(x_1' + \beta_1 x_2') \right\} - \frac{P}{\pi \mu} \lambda^2 \left[ \frac{\beta_1 K'}{(x_1' + \beta_1 x_2')^2} - K_2 (1 - \beta_2^2) x_1' \left( \frac{\beta_2^2 x_2' - 3x_1'}{(x_1' + \beta_2 x_2')^3} \right) + \frac{2\beta_1 \beta_2^2 K'}{2 - M_2^2} \left( \frac{x_2'}{x_1' + \beta_2 x_2'} \right)^2 \right] \]

\[
\tau_{11} = \frac{P}{\pi} \left[ \frac{M_2^2 - 2M_1^2 + 2}{x_1'} + \frac{K_1 \beta_2 x_2'}{x_1' + \beta_1 x_2'} - \frac{2K_2 \beta_2^2 x_2'}{x_1' + \beta_2 x_2'} \right] + \frac{P}{\pi \mu} \lambda^2 \left[ (M_2^2 - 2M_1^2 + 2) \beta_1 K' \times \right.
\]

\[
\left. - \right] 
\]

(143)
The above equations (142), (143) show the significant influence of couple-stresses on the classical responses. It is interesting to calculate the stress-difference

\[ \tau_{12} - \tau_{21} = -2\eta \nabla^4 \psi = -4K_2(1 - \bar{P}_2^2)P \lambda^2 \frac{x_1^3 - 3\bar{P}_2x_1'x_2'}{(x_1'^2 + \bar{P}_2^2x_2'^2)^3} \]  

(144)

which shows that the difference of shear-stresses depends on the parameter \( \lambda' \) for the moving loads.

\[ \tau = - \frac{(1 - \bar{P}_2^2)(x_1^3 - 3\bar{P}_2x_1'x_2')}{(x_1'^2 + \bar{P}_2^2x_2'^2)^3} \]

where \( \tau = \frac{\tau_{12} - \tau_{21}}{4K_2P} \)  

(145)

and investigate the effects of couple-stresses along the line \( x_1' = x_2' \). In this case,

\[ \tau = - (1 - 3\bar{P}_2^2)(1 - \bar{P}_2^2)\lambda^2/(1 + \bar{P}_2^2)^3 \]

and \( \tau = 0 \) if \( \bar{P}_2^2 = \frac{1}{3} \). If \( \frac{1}{3} < \bar{P}_2^2 < 1 \), \( \tau \) increases or decreases as the parameter of couple-stress \( \lambda' \) increases or decreases. Hence, the effect of couple-stresses on the difference of...
shear-stresses is appreciable when the speed of the loads lies between 0 and 0.8164 $C_2$. It is also found that the effect of couple-stresses on the difference of shear-stresses increases as the speed of the loads increases and attains maximum when speed becomes nearly equal to 0.6324 $C_2$.

**Transonic Case $[M_1 < 1, M_2 > 1]$.**

In this case the load is moving at a speed slower than the longitudinal wave speed but faster than the shear wave speed and the solution to the applicable differential equations (i.e., first equation of (114) and the second equation (115) may be taken in the form

\[
\begin{align*}
\phi &= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\lambda) e^{-\eta_1 x_2} e^{i\lambda x_2} d\lambda, \\
\psi &= \frac{1}{2\pi} \int_{-\infty}^{\infty} B(\lambda) e^{-\eta_2 x_2} + C(\lambda) e^{-\eta_3 x_2} e^{i\lambda x_2} d\lambda
\end{align*}
\]  

(146)

where $\eta_1 = \beta_1/\lambda$, $\eta_2 = \sqrt{\lambda^2 - p^2}$, $\eta_3 = \sqrt{\lambda^2 + q^2}$

\[
p^2 = \frac{1}{2\lambda^2} [1 + (1 + (\beta_2^2 + 1) 4\lambda^2)^{1/2} - 1], \quad q^2 = \frac{1}{2\lambda^2} [1 + (1 + (1 + \beta_2^2) 4\lambda^2)^{1/2}]
\]

The boundary conditions (117) can now be written as

\[
(\lambda^2 - 2)A(\lambda)\lambda - 2(\eta_2 B(\lambda) + \eta_3 C(\lambda)) = P^*(\lambda)
\]

(147)

\[
2\eta_1 A + (\lambda^2 - 2)\{B(\lambda) + C(\lambda)\} i\lambda = 0
\]

C(\lambda) = (\eta_2 p^2/\eta_3 q^2) B(\lambda)

Hence, $A(\lambda) = [(\lambda^2 - 2)\{1 + (\eta_2 p^2/\eta_3 q^2) i\lambda P^*(\lambda)\}] / \Delta^*$; $B(\lambda) = -2\eta_1 P^*(\lambda) / \Delta^*$

(148)

where $\Delta^* = -\lambda^2 (\lambda^2 - 2)^2 [1 + (\eta_2 p^2/\eta_3 q^2)] + 4\eta_1 \eta_2 (1 + (p^2/q^2))$

In view of (148), (146) becomes

\[
\begin{align*}
\phi &= \frac{\lambda^2 - 2}{2\pi} \int_{-\infty}^{\infty} \left(1 + \frac{\eta_2 p^2}{\eta_3 q^2}\right) \frac{\lambda P^*(\lambda)}{\Delta^*} e^{i\lambda x_1 - \eta x_2} d\lambda, \\
\psi &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta_1 \frac{P^*(\lambda)}{\Delta^*} \left(e^{-\eta x_2} + \eta_2 p^2 e^{-\eta x_2}\right) e^{i\lambda x_1} d\lambda
\end{align*}
\]  

(149)

As in the previous cases we may evaluate the expressions for displacements and stresses.
Effect of Couple-stress on moving loads

To illustrate the procedure for the solution of the dynamical problem, we consider a homogeneous isotropic, elastic semispace $x_2 \geq 0$ having origin on the free plane boundary and $x_2$-axis pointing into the medium. The loads may move along the free plane boundary $x_2 = 0$. We assume that a plane strain state prevails and the elastic displacements $u_1, u_2$ ($u_3 = 0$) are derivable from the displacements potentials $\phi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ so that

$$u_1 = \phi_1 + \psi_2 ; \quad u_2 = \phi_2 - \psi_1$$

(150)

The equations of motion (105) in view of (150) yield the following equations

$$\nabla^2 \phi - c_1^2 \phi^* = 0$$

(151)

$$\nabla^2 \psi - \lambda^2 \nabla^4 \psi = c_2^2 \psi^*$$

(152)

where $c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$, $\lambda^2 = \eta/\mu$, $\nabla(\cdot)^2 = (\cdot)_{,11} + (\cdot)_{,22}$ in which $\eta$ being a parameter characterizing the effect of couple stresses and $\lambda$, $\mu$ are Lamé's elastic constants. If $\lambda \to 0$, the couple stress vanishes and we get the classical result. Also the stresses and couple stresses may be written from (1) and (150) in terms of $\phi$, $\psi$, $\eta$ as

$$\tau_{11} = 2\mu(\phi_{,11} + \psi_{,12}) + \lambda \nabla^2 \phi, \quad \mu_{13} = 2\eta \nabla^2(\psi_{,1})$$

$$\tau_{22} = 2\mu(\phi_{,22} - \psi_{,12}) + \lambda \nabla^2 \phi, \quad \mu_{23} = 2\eta \nabla^2(\psi_{,2})$$

(153)

$$\tau_{12} = \mu(2\phi_{,12} - \psi_{,11} + \psi_{,22}) - 2\eta \nabla^4 \psi, \quad \tau_{21} = \mu(2\phi_{,12} - \psi_{,11} + \psi_{,22}) + 2\eta \nabla^4 \psi$$

Now we replace the time variable $t$ by a space like variable $\tau$ define by $\tau = c_1 t$ and obtain from the equations (151) and (152)

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial^2 \tau^2},$$

(154)

$$\nabla^2 \psi - \lambda^2 \nabla^4 \psi = \beta^2 \frac{\partial^2 \psi}{\partial \tau^2},$$

where

$$\beta^2 = \frac{c_1^2}{c_2^2} = (\lambda + 2\mu)/\mu$$

(155)

To find the solutions of the wave equations (154) we introduce two dimensional Fourier Transform.
\[
\overline{\phi}(\xi, x_2, \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x_1, x_2, \tau) e^{i(\xi x_1 + \zeta \tau)} dx_1 dt,
\]
\[
\overline{\psi}(\xi, x_2, \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x_1, x_2, \tau) e^{i(\xi x_1 + \zeta \tau)} dx_1 dt.
\]

(156)

Multiplying both sides of the equations (154) by \(e^{i(\xi x_1 + \zeta \tau)}\) and integrate over the entire \(x_1\)-plane, we find that the functions \(\overline{\phi}(\xi, x_2, \zeta)\) and \(\overline{\psi}(\xi, x_2, \zeta)\) satisfy the following differential equations

\[
\frac{d^2 \overline{\phi}}{dx_2^2} = (\xi^2 - \zeta^2) \overline{\phi} + \left[ \frac{\lambda^2}{2} \frac{d^4}{dx_2^4} - (2\xi^2\lambda^2 + 1) \frac{d^2}{dx_2^2} + \left( \xi^2 + \lambda^2\xi^4 - \beta^2\zeta^2 \right) \right] \overline{\psi} = 0
\]

(157)

To satisfy the finiteness condition as \(x_2 \to \infty\) the solutions of the equations (157) are assumed in the form

\[
\overline{\phi}(\xi, x_2, \zeta) = A \exp\left[ -\left( \xi^2 - \zeta^2 \right)^{1/2} x_2 \right],
\]

(158)

\[
\overline{\psi}(\xi, x_2, \zeta) = B \exp\left[ -\left( \xi^2 - \beta^2\zeta^2 \right)^{1/2} x_2 \right] + C \exp\left[ -\left( \xi^2 + \beta^2\zeta^2 + \frac{1}{\lambda^2} \right)^{1/2} x_2 \right]
\]

(159)

in which the terms involving \(\lambda^3\) and higher powers are neglected in the series expansion.

Since the parameter of the couple stresses \(\lambda\) is small once the function \(\overline{\phi}(\xi, x_2, \zeta)\) and \(\overline{\psi}(\xi, x_2, \zeta)\) have been determined the corresponding wave functions \(\phi(x_1, x_2, \tau)\) and \(\psi(x_1, x_2, \tau)\) are given by Fourier's inversion theorem for two dimensional transform from which we can derive the following relations

\[
\frac{\tau_{11}}{2\mu} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ i\xi \left\{ (\xi^2 - \beta^2\zeta^2)^{1/2} Be^{-\left( \xi^2 - \beta^2\zeta^2 \right)^{1/2} x_2} + (\xi^2 + \beta^2\zeta^2 + \frac{1}{\lambda^2})^{1/2} Ce^{-\left( \xi^2 + \beta^2\zeta^2 + \frac{1}{\lambda^2} \right)^{1/2} x_2} \right\} \right] d\xi d\zeta
\]

\[
\frac{\tau_{22}}{2\mu} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ i\xi \left\{ (\xi^2 - \beta^2\zeta^2)^{1/2} Be^{-\left( \xi^2 - \beta^2\zeta^2 \right)^{1/2} x_2} + (\xi^2 + \beta^2\zeta^2 + \frac{1}{\lambda^2})^{1/2} Ce^{-\left( \xi^2 + \beta^2\zeta^2 + \frac{1}{\lambda^2} \right)^{1/2} x_2} \right\} \right] d\xi d\zeta
\]
$$\tau_{12} = \frac{1}{2\mu} \int \int \left[ i\xi (\xi^2 - \zeta^2) \frac{1}{\lambda^2} + \frac{1}{2} (\xi^2 - \lambda^2 \xi^4) \psi + \frac{1}{2} \left( (\xi^2 - \beta^2 \zeta^2) (1 + \lambda^2 \beta^2 \zeta^2) \right) Be^{-(\xi^2 - \beta^2 \zeta^2)^2 x_2}ight]$$

$$\left( \xi^2 + \beta^2 \zeta^2 + \frac{1}{\lambda^2} \right) \lambda^2 \beta^2 \zeta^2 C e^{-i(\xi x_1 + \zeta \tau)} d\xi d\zeta$$

$$\frac{\tau_{21}}{2\mu} = \frac{1}{\sqrt{2\pi}} \int \int \left[ i\xi (\xi^2 - \zeta^2) \frac{1}{\lambda^2} + \frac{1}{2} (\xi^2 + \lambda^2 \xi^4) \psi + \frac{1}{2} \left( (\xi^2 - \beta^2 \zeta^2) (1 - \lambda^2 \beta^2 \zeta^2) \right) Be^{-(\xi^2 - \beta^2 \zeta^2)^2 x_2}ight]$$

$$\left( \xi^2 + \beta^2 \zeta^2 + \frac{1}{\lambda^2} \right) \lambda^2 \beta^2 \zeta^2 C e^{-i(\xi x_1 + \zeta \tau)} d\xi d\zeta$$

$$\frac{\mu_{23}}{2\eta} = \frac{1}{\sqrt{2\pi}} \int \int \left[ (\xi^2 - \beta^2 \zeta^2) \frac{1}{\lambda^2} \lambda^2 \beta^2 \zeta^2 C e^{-i(\xi^2 - \beta^2 \zeta^2)^2 x_2} - (\xi^2 - \beta^2 \zeta^2) \frac{1}{\lambda^2} \beta^2 \zeta^2 + \frac{1}{\lambda^2} \right]$$

$$\left. e^{-i(\xi x_1 + \zeta \tau)} d\xi d\zeta \right)$$

Solution

(i) When a variable pressure is applied to the boundary: Consider a variable pressure $p(x_1, \tau)$ which is applied to the boundary $x_2 = 0$. The boundary conditions are

$$\tau_{22} = -p(x_1, \tau), \quad \tau_{12} = 0, \quad \mu_{23} = 0 \quad \text{on} \quad x_2 = 0.$$  \hspace{1cm} (161)

using equation (160) and the boundary conditions (161) the following equations are obtained

$$\left( \xi^2 - \frac{1}{2} \beta^2 \zeta^2 \right) A - i\xi \left[ (\xi^2 - \beta^2 \zeta^2) B + (\xi^2 + \beta^2 \zeta^2 + \frac{1}{\lambda^2} \right] C = -\tilde{p}(\xi, \zeta) / 2\mu, \hspace{1cm} (162)$$

$$i\xi (\xi^2 - \zeta^2)^2 + \frac{1}{2} [2\xi^2 - \beta^2 \zeta^2 + \nu_2] B + \frac{1}{2} \nu_1 C = 0$$ \hspace{1cm} (163)

$$\left( \xi^2 - \beta^2 \zeta^2 \right)^2 \beta^2 \zeta^2 B - \left( \xi^2 + \beta^2 \zeta^2 + \frac{1}{\lambda^2} \right)^2 \left[ \beta^2 \zeta^2 + \frac{1}{\lambda^2} \right] C = 0$$ \hspace{1cm} (164)

where $\tilde{p}(\xi, \zeta)$ is the Fourier transform of $p(x_1, \tau)$ and

$$\nu_1 = \xi^2 - \lambda^2 \xi^4 - \lambda^2 \beta^2 \zeta^2 - \lambda^2 \beta^4 \zeta^4 - \beta^2 \zeta^2, \quad \nu_2 = \lambda^2 \beta^2 \zeta^2 \xi^2 - \xi^4 \lambda^2 - \lambda^2 \beta^4 \zeta^4.$$
From (164), a relation between $B$ and $C$ is obtained as

$$C = \gamma_1 B,$$  \hspace{1cm} (165)

where

$$\gamma_1 = \{(\xi^2 - \beta^2 \xi^2)^{1/2} \beta^2 \xi^2 / (\xi^2 + \beta^2 \xi^2 + \frac{1}{\lambda^2})\}$$  \hspace{1cm} (166)

Substituting this value in equations (162) and (166) one gets

$$A = \frac{\bar{P}(\xi, \zeta)}{f^* + g^*} \left[ \frac{1}{2\mu} - \frac{1}{2} \left[ 2\xi^2 - \beta^2 \xi^2 + \nu_2 + \gamma_1 \nu_1 \right] \right]$$  \hspace{1cm} (167)

$$B = -\frac{\bar{P}(\xi, \zeta)}{f^* + g^*} \left[ \frac{i(\xi^2 - \zeta^2)^2}{2\mu} \right]$$  \hspace{1cm} (168)

where

$$f^* = f_0 - \frac{1}{2} (\xi^2 - \beta^2 \xi^2) (\nu_2 + \gamma_1 \nu_1), \quad g^* = g_0 + \gamma_1 \xi^2 (\xi^2 - \zeta^2)^2 \left( \frac{1}{2} \beta^2 \zeta^2 + \frac{1}{\lambda^2} \right)$$  \hspace{1cm} (169)

Now using (160), (165) and (166) following relations are obtained

$$\tau_{11} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{\bar{P}}{f^* + g^*} \left[ \left( \frac{1}{2} \beta^2 \xi^2 \right) \xi^2 + \frac{1}{2} \beta^2 \xi^2 \right] \right\} \left\{ \left( \frac{1}{2} \beta^2 \zeta^2 \right) \xi^2 + \frac{1}{2} \beta^2 \zeta^2 \right\} e^{-\left(\xi^2 - \zeta^2\right)^2 x_2}$$

$$+ \left\{ \frac{1}{2} \beta^2 \xi^2 \xi^2 + \frac{1}{2} \beta^2 \zeta^2 \xi^2 \right\} e^{-\left(\xi^2 - \zeta^2\right)^2 x_2}$$

$$- \frac{1}{2} \left( \xi^2 + \frac{1}{2} \beta^2 \xi^2 \right) (\nu_2 + \gamma_1 \nu_1) \right\} \right\} e^{-i(\xi_1 + \zeta_1)} d\xi d\zeta$$  \hspace{1cm} (170)

$$\tau_{22} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left( \frac{1}{2} \beta^2 \xi^2 \right) \xi^2 + \frac{1}{2} \beta^2 \zeta^2 \right\} \right\} \left\{ \left( \frac{1}{2} \beta^2 \xi^2 \right) \xi^2 + \frac{1}{2} \beta^2 \zeta^2 \right\} e^{-\left(\xi^2 - \zeta^2\right)^2 x_2}$$

$$+ \left\{ \frac{1}{2} \beta^2 \xi^2 \xi^2 + \frac{1}{2} \beta^2 \zeta^2 \xi^2 \right\} e^{-\left(\xi^2 - \zeta^2\right)^2 x_2}$$

$$- \frac{1}{2} \left( \xi^2 + \frac{1}{2} \beta^2 \xi^2 \right) (\nu_2 + \gamma_1 \nu_1) \right\} \right\} e^{-i(\xi_1 + \zeta_1)} d\xi d\zeta$$  \hspace{1cm} (171)
\[
\tau_{12} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p \text{d} \xi}{f^* + g^*} \left[ \left( \frac{1}{\xi^2 - \zeta^2} \right)^{\frac{1}{2}} \left( \frac{1}{\xi^2 - \beta^2 \zeta^2} \right) \left( e^{-\frac{\xi^2 - \zeta^2}{2} x_2} - e^{-\frac{\xi^2 - \zeta^2}{2} x_2} \right) \right] \\
+ \left\{ \frac{1}{2} \left( \xi^2 - \zeta^2 \right)^{\frac{1}{2}} (v_2 + \gamma_1 v_1) e^{-\frac{\xi^2 - \zeta^2}{2} x_2} - v_2 e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} \right\} e^{-i(\xi x_1 + \zeta \xi)} d\xi d\zeta
\]

(172)

\[
\tau_{21} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p \text{d} \xi}{f^* + g^*} \left[ \left( \frac{1}{\xi^2 - \zeta^2} \right)^{\frac{1}{2}} \left( \frac{1}{\xi^2 - \beta^2 \zeta^2} \right) \left( e^{-\frac{\xi^2 - \zeta^2}{2} x_2} - e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} \right) \right] \\
+ \left\{ \frac{1}{2} \left( \xi^2 - \zeta^2 \right)^{\frac{1}{2}} (v_2 + \gamma_1 v_1) e^{-\frac{\xi^2 - \zeta^2}{2} x_2} + v_2 e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} \right\} e^{-i(\xi x_1 + \zeta \xi)} d\xi d\zeta
\]

(173)

\[
\mu_{23} = -\frac{\ell^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p \text{d} \xi}{f^* + g^*} \left( \frac{1}{\xi^2 - \zeta^2} \right)^{\frac{1}{2}} \left( \frac{1}{\xi^2 - \beta^2 \zeta^2} \right)^{\frac{1}{2}} \left( \frac{1}{\ell^2} \right)^{\frac{1}{2}} \gamma_1 e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} - e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} \}
\]

(174)

and the components of displacement vector are given by

\[
u_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p \text{d} \xi}{f^* + g^*} \left[ \left( \frac{1}{\xi^2 - \zeta^2} \right)^{\frac{1}{2}} \left( \frac{1}{\xi^2 - \beta^2 \zeta^2} \right) \left( e^{-\frac{\xi^2 - \zeta^2}{2} x_2} - e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} \right) \right] \\
- \left\{ \frac{1}{2} (v_2 + \gamma_1 v_1) e^{-\frac{\xi^2 - \zeta^2}{2} x_2} - \gamma_1 (\xi^2 + \beta^2 \zeta^2 + \frac{1}{\ell^2})^2 \right\} e^{-i(\xi x_1 + \zeta \xi)} d\xi d\zeta
\]

(175)

\[
u_2 = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p \text{d} \xi}{f^* + g^*} \left[ \left( \frac{1}{\xi^2 - \zeta^2} \right)^{\frac{1}{2}} \left( \frac{1}{\xi^2 - \beta^2 \zeta^2} \right) \left( e^{-\frac{\xi^2 - \zeta^2}{2} x_2} - e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} \right) \right] \\
- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p \text{d} \xi}{f^* + g^*} \left[ \left( \frac{1}{\xi^2 - \zeta^2} \right)^{\frac{1}{2}} \left( \frac{1}{\xi^2 - \beta^2 \zeta^2} \right) \left( e^{-\frac{\xi^2 - \zeta^2}{2} x_2} - e^{-\frac{\xi^2 - \beta^2 \zeta^2}{2} x_2} \right) \right] \\
- \gamma_1 (\xi^2 + \beta^2 \zeta^2 + \frac{1}{\ell^2})^2 \right\} e^{-i(\xi x_1 + \zeta \xi)} d\xi d\zeta
\]
In the above expressions for stresses and displacements (except \( \mu_{23} \)) the first two terms stand for the classical case. Now from relation (169) the expression for \( f^0 + g^0 \) can be written as

\[
f^0 + g^0 = -(\xi^2 - \frac{1}{2} \beta^2 \xi^2)^2 + \xi^2 (\xi^2 - \xi^2)^2 \xi^2 (\xi^2 - \beta^2 \xi^2)^2 \]

(177)

(ii) When a pulse of pressure moving uniformly along the boundary:

In this section we shall determine the stress setup in the interior of the semi-infinite elastic medium when a pulse of pressure of shape \( p = \chi(x_1) \) move with uniform velocity \( v \) along the boundary \( x_2 = 0 \). Boundary conditions are given by

\[
p(x_1, \tau) = \chi(x_1 - vt) = \chi(x_1 - \beta_1 \tau), \quad \beta_1 = v/c_1
\]

(178)

Now

\[
\tilde{p}(\xi, \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(x_1 - \beta_1 \tau) e^{i(\xi_1 + \zeta \tau)} dx_1 d\tau
\]

(179)

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(u) e^{i\xi u} du \int_{-\infty}^{\infty} e^{i\zeta \tau} \delta(\zeta + \beta_1 \xi) d\tau = 2\chi(\xi) \delta(\zeta + \beta_1 \xi)
\]

where \( \chi(\xi) = \frac{1}{2} \int_{-\infty}^{\infty} \chi(u) e^{i\xi u} du \)

(180)

In case if \( \chi(u) \) is an even function of \( u \), then

\[
\chi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(u) \cos(\xi u) du
\]

(181)

Remembering that for any function \( X(\xi^2, \zeta^2) \)

\[
\int_{-\infty}^{\infty} \chi(\xi^2) \delta(\zeta + \beta_1 \xi) e^{-i\xi \tau} d\zeta = X(\xi^2, \beta_1^2 \xi^2),
\]

(182)

the following relations are obtained from relations (170) to (174)

\[
+ \frac{1}{2} (\gamma_1 v_1) e^{i(\xi_1^2 - \zeta_1^2) x_1} - \gamma_1 \xi_1^2 e^{i(\xi_1^2 + \beta_1^2 \xi_1^2) x_1} \int_{-\infty}^{\infty} e^{-i(\xi_1 x_1 + \zeta_1 \tau)} d\xi d\zeta
\]

(176)
The displacement components may be written from relations (175) and (176) as

\[
\tau_{11} = 2 \int_{\xi_0}^{\infty} \frac{x(\xi) \cos(\xi(x_1 - vt))}{\theta \phi} \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} - \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} d\xi
\]

\[
+ \left( \gamma_1^* + \frac{1}{\lambda^2 \xi^2} \right) \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} + \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} d\xi
\]

\[
\tau_{22} = -2 \int_{\xi_0}^{\infty} \frac{x(\xi) \cos(\xi(x_1 - vt))}{\theta \phi} \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} + \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} d\xi
\]

\[
+ \left( \gamma_1^* + \frac{1}{\lambda^2 \xi^2} \right) \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} + \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} d\xi
\]

\[
\tau_{12} = 2 \int_{\xi_0}^{\infty} \frac{x(\xi) \sin(\xi(x_1 - vt))}{\theta \phi} \left\{ (1 - \beta_1^2)^2 \left( \frac{1}{2} \beta_2^2 \right) e^{-(\beta_2^2)^2 x_2 \xi} - e^{-(\beta_2^2)^2 x_2 \xi} \right\} d\xi
\]

\[
+ \left( \gamma_1^* + \frac{1}{\lambda^2 \xi^2} \right) \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} + \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} d\xi
\]

\[
\tau_{21} = 2 \int_{\xi_0}^{\infty} \frac{x(\xi) \sin(\xi(x_1 - vt))}{\theta \phi} \left\{ (1 - \beta_1^2)^2 \left( \frac{1}{2} \beta_2^2 \right) e^{-(\beta_2^2)^2 x_2 \xi} - e^{-(\beta_2^2)^2 x_2 \xi} \right\} d\xi
\]

\[
+ \left( \gamma_1^* + \frac{1}{\lambda^2 \xi^2} \right) \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} + \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} d\xi
\]

\[
\mu_{23} = -2\lambda^2 \int_{\xi_0}^{\infty} \frac{x(\xi) \sin(\xi(x_1 - vt))}{\theta \phi} \left\{ (1 - \beta_1^2)^2 \left( \frac{1}{2} \beta_2^2 \right) e^{-\left(\beta_2^2\right)^2 x_2 \xi} \right\} d\xi
\]

\[
+ \left( \gamma_1^* + \frac{1}{\lambda^2 \xi^2} \right) \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} + \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} d\xi
\]

The displacement components may be written from relations (175) and (176) as

\[
u_1 = \frac{1}{\mu} \sqrt{\int_{\xi_0}^{\infty} \frac{x(\xi) \sin(\xi(x_1 - vt))}{\theta \phi} \left\{ \phi^* e^{-(\beta_2^2)^2 x_2 \xi} - \frac{\beta_2^2}{2} e^{-\frac{1}{2} (1 - \beta_1^2)^2 x_2 \xi} \right\} \xi d\xi}
\]
\[
\tau_{11} = P \left\{ \frac{2}{\pi} \int_0^\infty \cos[\xi(x_1 - vt)] \left[ \left( \phi^* \right)^* \right] e^{-\left(\frac{\beta^2}{2}\right)^2 \xi^2} - \left(1 - \beta_1^2 + \frac{\beta_2^2}{\lambda^2 \xi^2} \right) e^{-\left(\frac{\beta^2}{2}\right)^2 \xi^2} \right\}
\]

(iii) In case of a point force moving uniformly over the boundary:

The simplest example of this kind is in which the stress is produced by the application to the boundary of a point force of magnitude \( P \), whose point of application moves with uniform velocity \( v \). The form of the pressure pulse in this case is \( \chi(x_1) = P\delta(x_1) \) so that \( \chi(\xi) = P / 2 \).

From relation (183) the following results are obtained

\[
\tau_{11} = -\frac{1}{\xi^2} \left\{ \frac{1}{2} (v_2^* + \gamma_1 v_1^*) e^{-\left(\frac{\beta_1^2}{2}\right)^2 \xi^2} + \gamma_1^* \left(1 - \beta_1^2 \right)^2 \left(1 + \frac{\beta_2^2}{\lambda^2 \xi^2} \right)^2 e^{-\left(\frac{\beta_1^2}{2}\right)^2 \xi^2} \right\}
\]
\[\tau_{22} = -P \frac{2 \omega \cos[\xi(x_1 - vt)]}{\pi^2} \left[ \phi^* e^{-(1 - \beta^2_1)x_2^2} + \theta^* e^{-(1 - \beta^2_2)x_2^2} \right] + \gamma_1^* \left( 1 + \beta^2_2 + \frac{1}{\ell^2 x^2} \right)\]

\begin{align*}
(1 - \beta^2_1) \frac{1}{2} e^{-(1 + \beta^2_2 + \frac{1}{\ell^2 x^2})^2 x_2} - \frac{1}{2} (1 - \beta^2_2) \left( \nu_2^* + \gamma_1^* \nu_1^* \right) \left( e^{-\left(1 - \beta^2_1\right)^2 x_2^2} - e^{-\left(1 - \beta^2_2\right)^2 x_2^2} \right) \right] \, d\xi.
\end{align*}

\[\tau_{12} = P \frac{2 \omega \sin[\xi(x_1 - vt)]}{\pi^2} \left[ (1 - \beta^2_1) \frac{1}{2} (1 - \beta^2_2) \left( e^{-\left(1 - \beta^2_1\right)^2 x_2^2} - e^{-\left(1 - \beta^2_2\right)^2 x_2^2} \right) \right]

\begin{align*}
+ \frac{1}{2} (1 - \beta^2_1) \left( \nu_2^* + \gamma_1^* \nu_1^* \right) e^{-\left(1 - \beta^2_1\right)^2 x_2^2} - \nu_2^* e^{-\left(1 - \beta^2_2\right)^2 x_2^2} - \gamma_1^* \nu_1^* e^{-\left(1 + \beta^2_2 + \frac{1}{\ell^2 x^2}\right)^2 x_2^2} \right] \, d\xi.
\end{align*}

\[\tau_{21} = P \frac{2 \omega \sin[\xi(x_1 - vt)]}{\pi^2} \left[ (1 - \beta^2_1) \frac{1}{2} (1 - \beta^2_2) \left( e^{-\left(1 - \beta^2_1\right)^2 x_2^2} - e^{-\left(1 - \beta^2_2\right)^2 x_2^2} \right) \right]

\begin{align*}
+ \gamma_1^* (\nu_1^* - 2 - \beta^2_2 + \frac{2}{\ell^2 x^2}) e^{-\left(1 + \beta^2_2 + \frac{1}{\ell^2 x^2}\right)^2 x_2^2} \right] \, d\xi.
\end{align*}

\[\mu_{23} = -\ell^2 P \frac{2 \omega \xi \sin[\xi(x_1 - vt)]}{\pi^2} \left[ (1 - \beta^2_1) \frac{1}{2} (1 - \beta^2_2) \left( e^{-\left(1 - \beta^2_1\right)^2 x_2^2} - e^{-\left(1 - \beta^2_2\right)^2 x_2^2} \right) \right]

\begin{align*}
+ \left( \beta^2_2 + \frac{1}{\ell^2 x^2} \right) \left( 1 + \beta^2_2 + \frac{1}{\ell^2 x^2} \right) \gamma_1^* e^{-\left(1 + \beta^2_2 + \frac{1}{\ell^2 x^2}\right)^2 x_2^2} \right] \, d\xi.
\end{align*}

\[u_1 = \frac{P}{2 \mu} \left[ \frac{2 \omega \sin[\xi(x_1 - vt)]}{\pi^2} \phi^* - \frac{1}{\xi} \left( \phi^* e^{-(1 - \beta^2_1)^2 x_2^2} - \left(1 - \beta^2_2\right) e^{-(1 - \beta^2_2)^2 x_2^2} \right) \right] \, d\xi

\begin{align*}
+ \frac{1}{\xi} \left( \nu_2^* + \gamma_1^* \nu_1^* \right) \left( e^{-\left(1 - \beta^2_1\right)^2 x_2^2} - \gamma_1^* \left(1 - \beta^2_1\right)^2 \left( 1 + \beta^2_2 + \frac{1}{\ell^2 x^2} \right)^2 \right) \right] \, d\xi.
\end{align*}

\[u_2 = -\frac{P}{2 \mu} \left[ \frac{2 \omega \cos[\xi(x_1 - vt)]}{\pi^2} \left(1 - \beta^2_1\right)^2 \frac{1}{\xi} \left( e^{-\left(1 - \beta^2_1\right)^2 x_2^2} - \left(1 - \beta^2_2\right) e^{-(1 - \beta^2_2)^2 x_2^2} \right) \right]

\begin{align*}
+ \left( \frac{1}{\xi} \left(1 - \beta^2_1\right)^2 \left(1 + \beta^2_2 + \frac{1}{\ell^2 x^2} \right)^2 \right) \right] \, d\xi.
\end{align*}
Using relation (184) the following results may be written as

\[
\tau_{11} + \tau_{22} = P \sqrt{2/\pi} \int_0^\infty \frac{2 \cos[\xi(x_1 - v_1 t)]}{\theta^* + \phi^*} \left[ \left(1 - \frac{\beta_2^2}{2}\right) \xi^2 \right] e^{-\left(1 - \beta_1^2\right)^2 \frac{1}{2} x_2^2} + \frac{1}{2} \left(\beta_1^2 - \beta_2^2\right) \xi^2 \right] d\xi 
\]

(185)

\[
\tau_{21} - \tau_{12} = P \sqrt{2/\pi} \int_0^\infty \frac{2 \sin[\xi(x_1 - v_1 t)]}{\theta^* + \phi^*} \left[ 2 \nu v_i e^{-\left(1 - \beta_1^2\right)^2 \frac{1}{2} x_2^2} + \gamma v_1 (2v_1 - 4 \nu^2) \right] d\xi 
\]

(186)

Now using relation (185) and considering \(1 - \beta_1^2 > 0\), \(1 - \beta_2^2 > 0\) and neglecting higher powers of \(\lambda\) the following results are obtained

\[
\tau_{11} + \tau_{22} = P \sqrt{2/\pi} \left(1 - \frac{\beta_2^2}{2}\right) \left(\beta_1^2 - \beta_2^2\right) \xi^2 \cos \theta + \frac{P \sqrt{2/\pi} \left(\beta_1^2 - \beta_2^2\right)}{2} \cos \theta 
\]

\[
\left[ \lambda^2 (\beta_2^2 - 1 - \beta_2^4) 2r^3 \cos 3\theta + \lambda^3 (1 - \beta_2^2) \beta_2^2 6r^4 \cos 4\theta \right] \]

(187)

where \(r^2 = (1 - \beta_1^2) x_2^2 + x_1^2\), \(\tan \theta = x_1 / (1 - \beta_1^2) x_2\), \(x_1 = x_1 - v_1 t\)

In the above relation (187) the following Fourier Cosine integral transform has been used \(F_c[e^{-ax}x^{n-1} \xi] = \sqrt{2/\pi (n-1)!} r^{-n} \cos n\theta \quad [n > 0, a > 0]\)

where \(r = (a^2 + \xi^2)^{1/2}\) and \(\theta = \tan^{-1} \xi / a\)

It is again mentioned that the first term corresponds to the classical case whether the second term is the contribution of couple stress. If \(\lambda \to 0\) the effect of couple stress vanishes and the result for classical case is obtained, which in agreement with the result of Cole and Huth and Sneddon. It is also clear from relation (186) that the difference of stresses tend to zero as \(v_2 \to 0\) and \(v_1 \to 0\) when \(\lambda \to 0\).

It is obvious from relation (187) that neither the value of the sum of the stresses \(|\tau_{11} + \tau_{22}|\) is dimensionless for classical case (first term) nor the increment (second
Graph 1: Variation of $|\tau_{11} + \tau_{22}|$ at different values of the ratio of velocities in classical case.

Graph 2: Variation of increment of $|\tau_{11} + \tau_{22}|$ at different values of the ratio of velocities due to presence of couple stress.

Table 1: Values of $|\tau_{11} + \tau_{22}|$ in classical case and increment of sum of the stresses due to couple stress.

<table>
<thead>
<tr>
<th>Ratio of Velocities</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical</td>
<td>3.997449</td>
<td>2.776266</td>
<td>2.079931</td>
<td>1.562021</td>
<td>1.234368</td>
</tr>
<tr>
<td>Increment</td>
<td>1.001624</td>
<td>1.001386</td>
<td>1.001164</td>
<td>1.000779</td>
<td>1.000411</td>
</tr>
</tbody>
</table>

$C_1 = 5000 \text{ m/sec}, \quad C_2 = 3000 \text{ m/sec}, \quad v_1 = 50 \text{ m/sec}, \quad x_2 = 5 \text{ m}$

Standard value of the velocity $v$ is 160 m/sec.
term) due to couple stress is dimensionless. Hence an attempt has been made to explain the variations in both cases through the ratio of two terms corresponding to two different values of the velocity \( v \). Hence the term multiplied with \( l^3 \) is omitted in the relation (187) and the values are considered for numerical calculation.

**Discussion:**

The Rayleigh and Love type of surface waves are increased under the action of couple-stress. As the couple stress parameters \( l \) is small, the change in the wave velocity \( C \) is also small. The waves in the layer are also affected by couple stress as studied in this review paper. The stress distribution due to the presence of moving load in the solid elastic semispace may be considered worthy for numerical calculations as the improper integrals have been evaluated and they are written in terms of different functions.

But it is to be mentioned that, this asymmetric theory does not have yet the complete or through experimental verification. We know merely the order of magnitudes and mutual relation between the material constants in studying most of the dynamic problem. However, the complete correspondence of the experiment and theory exists in the case of discrete media (spatial grillages) where all the material constants may be uniquely determined. When passing from such a grillage to a continuous medium we obtain exactly Cosserat’s continuum. The couple-stress theory of elasticity, which is now even considered as an Utopian theory by some sections of research workers, will enhance the speed of development in science and technology in the future. We must remember that the phenomenon ‘Utopian to reality’ occurs as a repeated feature in the history of science.