CHAPTER 1  
INTRODUCTION

The spread of infectious diseases have always been of concern and a threat to public health. It has caused serious problems for the survival of human beings and other species, and for the economic and social development of the human society. In our life there are different infectious diseases. To prevent and to control infectious diseases more effectively, it is important to understand the mechanism of the spread and then provide useful predictions and guidance so that better strategies can be established. An epidemic model, describe the situation where, there is infection as well as removal is considered. Usually an infectious disease spreads in a population when one or more infectives enter into the population from outside. Generally, there is a time gap between the receipt of infectious and the appearance of symptoms in the case of most of the infectious diseases. After an individual gets infected by a disease like chicken pox, measles, etc., the symptoms of the disease are manifested on the body of the person after a gap of about two weeks. Infectious diseases are caused by pathogenic micro organisms, such as bacteria, viruses, and parasite or fungal, the diseases can be spread directly or indirectly from one person to another person. The research in infectious diseases can be basically classified as descriptive, analytical, experimental, and theoretical. Mathematical models can explain good knowledge of how infectious diseases spread, to make reliable predictions, to provide useful prevention and to control strategies and instructions through analysis.

Mathematical modeling can be used for comparing different diseases in the same population, the same diseases at different times or the same diseases in different population. Epidemiological models are helpful when comparing the effects of prevention or control procedures. Infectious diseases are illnesses caused by a disease agent that can be transmitted from organism to organism. Diseases can take on many forms but are generally characterized by the weakening of the infected organism, the degree of which depends on the organisms resistance to the infection and the virulence of the disease
agent causing it. Mathematical modeling is an important tool to understand and predict the spread of infectious diseases. A tremendous number of models have been formulated, analyzed and applied to a variety of infectious diseases qualitatively and quantitatively. Various epidemic models have been formulated and analyzed by many researchers. Bernoulli used mathematical models for smallpox in 1760. Since that time, theoretical epidemiology has witnessed numerous developments. Hamer formulated a discrete-time model for the spread of measles in 1906. A physician, Dr. Ross, used a differential equation model to describe the transmissions of Malaria between human beings and mosquitoes in 1911. Kermack and McKendrick [44] formulated a well-known and well-recognized SIR (Susceptible–Infective–Recovered) compartmental model, in 1927. The analysis of deterministic mathematical models has been focused on the wellposedness of the models and their solutions, persistence of diseases, existence and stability of steady states - which characterize the diseases spreading or being endemic, existence and stability of periodic solutions - which describe the oscillations of disease transmissions, and occurrence of bifurcations. In earlier deterministic mathematical models for epidemic transmissions, a constant population size was usually assumed. Now it is assumed to be finite and the total population at any time may be divided into three disjoint classes susceptible, infective and removals. It took the advantage of the tractability of the mathematical analysis. As progresses are being made in epidemic modeling, more advanced mathematical tools have been developed and are ready to be applied. More realistic mathematical models for infectious diseases have been dramatically developed lately. Hence, more advanced mathematical techniques, such as the theories of bifurcation and saturation have been broadly employed and utilized in the model analysis, and high-speed computers have also been used for more complicated simulations.

All the above mathematical models are discussed and analyzed with constant rate of susceptible. Sometimes the susceptible rate may increase unexpectedly. In this case, the hospitals will have to meet some difficult problems. To solve this kind of problem, in the present study, the SIR model with immigration (i.e., increase of susceptible) is introduced. There are
three types of incidence rates viz., non-monotonic, nonlinear and saturated incidence rates under treatment is considered to offer constant treatment to all the infective.

1.1 PRELIMINARIES

1.1.1 Mathematical Modeling

A mathematical model is a model by using mathematical concepts such as functions and equations. When mathematical models are created, they have moved from the real world into the abstract world of mathematical concepts, which is where the model is built. Then it is manipulated to solve the model using mathematical or statistical techniques. Mathematical modeling is the art of translating problems from an application area into tractable mathematical formulations whose theoretical and numerical analysis provides in-sight, answers, and guidance useful for the originating application.

- A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is mathematical modeling.
- A mathematical model is an abstract, simplified, mathematical construct related to a part of reality and created for a particular purpose.

1.1.2 Simulation

Simulation is the imitation of the operation of a real-world process or system over time. Simulation modeling involves the use of a computer program or some technological tool to generate a scenario based on a set of rules. These rules arise from an interpretation of how a certain process is supported to evolve or progress. A simulation is an applied methodology that can describe the behavior of that system using either a mathematical model or a symbolic model. Simply, simulation is the imitation of the operation of a real-world process or system over a period of time. Definition of simulation range from:
• A method for implementing a model over time.
• A technique for testing, analysis, or training in which real-world systems are used or where real-world and conceptual systems are reproduced by a model.
• A methodology for extracting information from a model by observing the behavior of the model as it is executed.
• A non-technical term meaning is not real i.e., imitation.

1.1.3 Epidemics

An epidemic is a widespread occurrence of an infectious disease within a particular time. Epidemics have ever been a great concern of human kind, because the impact of infectious diseases on human and animal is enormous, both in terms of suffering from social and economic consequences.

An epidemic may be restricted to one location; however, if it spreads to other countries or continents and affects a substantial number of people, it may be termed as pandemic. Epidemics are noted as follows:

• Any disease that becomes widespread over a large population about the same time is considered an epidemic.
• Disease that spreads rapidly and affects an inordinately large number of people within a very short period.
• Unexpected and sudden increase in the number of people affected by a particular disease within a geographical region.
• Affecting or tending to affect the disproportionately the large number of individuals within a population, or community or region at the same (typhoid was epidemic).
• Spreading rapidly and extensively by infection and affecting many individuals in an area or a population at the same time an epidemic outbreak of influenza.
• Affecting many individuals at the same time and spreading from person to person in a locality where the disease is not permanently prevalent.
Mathematical modeling is an essential tool in studying a diverse range of such epidemic diseases to gain a better understanding of transmission mechanisms and make predictions; determine and evaluate control strategies.

**Compartment Model**

Dynamic models for infectious diseases are mostly based on compartment structure that were initially proposed by Kermack and McKendrick [44], and developed later by many other bio-mathematicians.

To formulate a dynamic model for the transmission of an epidemic disease, the population in a given region is often divided into several different groups of compartments. Such a model describing the dynamic relations among these compartments is called a compartment model.

**Kermack-McKendrick SIR Epidemic Model**

In the compartment model studied by Kermack and McKendrick [44] in 1927, the population is divided into three compartments: a susceptible compartment, labeled $S$, in which all the individuals are susceptible if they contact with a disease; an infected compartment, labeled $I$, in which all the individuals are infected by the disease and infectious; and a removed compartment, labeled $R$, in which all the individuals are removed or recovered from the infection. Denote the numbers of individuals in the compartments $S$, $I$, and $R$, at time $t$, as $S(t)$, $I(t)$, and $R(t)$, respectively.

- $S$ is the number of susceptible, who are not infected but could become infected.
- $I$ is the number of infective. These individuals have the disease and can transmit it to the susceptible.
- $R$ is the number of removed individuals. These may or may not have the disease, but they can't become infected and they can't transmit the disease to others.

Thus the following three assumptions were

1. The disease spreads in a closed environment; that is, there is no emigration nor immigration, and neither birth nor death in the
population, so that the total population remains a constant $N$ for all $t$, that is, $S(t) + I(t) + R(t) \equiv N$.

2. The number of susceptible who are infected by an infected individual per unit of time, at time $t$, is proportional to the total number of susceptible with the proportional coefficient (transmission coefficient) $\lambda$, so that the total number of newly infective, at time $t$, is $\lambda S(t)I(t)$.

3. The number removed (recovered) individuals from the infected compartment per unit time is $mI(t)$ at time $t$, where $m$ is the recovery rate coefficient, and the recovered individuals gain permanent immunity.

The rate of change in the number of susceptible in the time interval is given by $\frac{dS}{dt}$ and rate of change in the number of infective is given by $\frac{dI}{dt}$.

Based on these assumptions, the corresponding SIR model is

$$\frac{dS}{dt} = -\lambda SI$$
$$\frac{dI}{dt} = \lambda SI - mI$$
$$\frac{dR}{dt} = mI$$

Figure 1.1: The three compartments versus time
Immigration

The process in which some individuals are added from outside to the population is known as immigration. Persons entering into the infected community from outside are called immigration. In epidemiology, immigration is the increase of susceptible at a constant rate. It is denoted by $\mu$. Immigration is a result of a number of factors, including economic and or political reasons, family re-unification, natural disasters or the wish to change one surrounding voluntarily.

Epidemic Model with Immigration

Communicable diseases may be introduced into a population by the arrival (input) of infective from outside the population. It incorporates the immigration of individuals including the susceptible, infective and removed into epidemic models. Now the model is of the form

$$
\begin{align*}
\frac{dS}{dt} &= -\lambda SI + \mu \\
\frac{dI}{dt} &= \lambda SI - mI \\
\frac{dR}{dt} &= mI
\end{align*}
$$

Using the case of measles, for example, there is an arrival of new susceptible individuals into the population. For this type of situation births and deaths must be included in the model. The following differential equations represent the SIR model with births and deaths and with general incidence rate.

$$
\begin{align*}
\frac{dS}{dt} &= a - dS - \lambda SI + \beta R + \mu \\
\frac{dI}{dt} &= \lambda SI - (d + m)I \\
\frac{dR}{dt} &= mI - (d + \beta)R
\end{align*}
$$

where $a$ is the recruitment rate of the population, $d$ the natural death rate of the population, $\lambda$ the proportionality constant, $m$ is the natural recovery rate of the infective individuals, $\beta$ the rate at which the recovered individuals lose
immunity and return to susceptible class and $\mu$ the increase of susceptible at a constant rate.

**Endemic**

In epidemiology, an infection is said to be endemic in population when that infection is maintained in the population without the need for external inputs.

- The term endemic refers to the state of being peculiar to or prevalent in a particular region, locality or people.
- It may also refer to the state of relating to a pathogen or disease that is found in or confined to a particular region or people.
- A disease that is constantly present to a greater or lesser degree in people of a certain class or in people living in a particular location.

Epidemiology is literally the study of epidemics. There are two basic types of epidemiology namely descriptive and analytical. The descriptive type deals with specific population groups, whereas the analytical type focuses on health risk factors and preventive measures.

**Epidemic Model with Treatment**

An epidemic model with treatment is constructed by Wang [61]. The model is of the form

$$\frac{dS}{dt} = -dS - \lambda SI + \beta R + \mu$$

$$\frac{dI}{dt} = \lambda SI - (d + m)I - T(I)$$

$$\frac{dR}{dt} = mI - (d + \beta)R + T(I)$$

\[
T(I) = \begin{cases} 
  rI, & \text{if } 0 \leq I \leq I_c \\
  rI_c, & \text{if } I \geq I_c 
\end{cases}
\]

**Incidence Rate**

Incidence in an epidemiological model is the rate at which susceptible become infectious. The incidence rate is the number of new cases per population at risk in a given time period.
The probability of developing a particular disease during a given period of time, the numerator is the number of new cases during the specified time period and the denominator is the population at risk during period.

1.2 SOME BASIC CONCEPTS

Monotonic

Monotonic is a property of many logical systems that states that the hypothesis of any derived fact may be freely extended with additional assumptions.

Monotonic is a sequence or function consistently increasing and never decreasing or consistently decreasing and never increasing in value.

If the sequence \((a_n)\) is such that

\[ a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots \]

then \((a_n)\) is said to be a monotonic decreasing sequence.

If the sequence \((a_n)\) is such that

\[ a_1 \leq a_2 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots \]

then \((a_n)\) is said to be a monotonic increasing sequence.

Non-monotonic

Non-Monotonic is a formal logic whose consequence relation is not monotonic. Most studied formal logics have monotonic consequence relation, meaning that adding a formula to a theory never produce a reduction of its set of consequences. A sequence which is not monotonic is called a non-monotonic sequence.

Basic Reproduction Number

The reproduction number \(R_0\) is the products of infection rates and duration of infection.

i.e., Basic Reproduction Number \((R_0) = \) (Rate of secondary infection) \(\times\)

\[(\text{Duration of infection})\]

If \(R_0<1\) the infection cannot grow and if \(R_0>1\) the disease can invade the population.
The basic reproduction number $R_0$ is a dimensionless number and not a rate, which would have units of time. Some authors incorrectly, call $R_0$ the basic reproductive rate.

There are two methods often used to determine the basic reproductive number of epidemic models. One method is to find conditions for the local stability of the disease-free equilibrium of the model. The other method is to use a next generation operators Diekmann et. al. [16], Diekmann and Heesterbek [15], Vanden Driessche and Watmough [54]. Moreover, the basic reproductive number can also be determined either by finding the conditions on the existence of an endemic equilibrium of epidemiologic meaning of parameters in the model.

**Periodic Orbits**

An orbit that repeats is called a periodic orbit. A periodic orbit corresponds to a special type of solution for a dynamical system, namely one which repeats itself in time.

A cycle or periodic orbit of a system of equation $\dot{x} = f(x)$ is any closed solution curve of the system which is not an equilibrium point of the system. A periodic orbit $\Gamma$ is called unstable if it is not stable; and $\Gamma$ is called asymptotically stable if it is stable.

Periodic orbits play a very important role in understanding the behavior of non-linear dynamical system. In particular they determine a large part of structure of strange attractors.

**First Octant**

Octant is a term used in mathematics to mean any of the eight divisions into which the three planes consist the Cartesian co-ordinate axis divide space or simply a $45^\circ$ arc. It is invented by Clayden, Greevers, Warren and others. In astronomy, it means that the location of a celestial body when it is at an angular distance of $45^\circ$ from a reference body. The first octant is the region of 3D co-ordinate axis system where $x > 0, y > 0, z > 0$. 
Octal is a word of octant. Octant refers to one eighth of a circle.

Figure 1.2: First Octant in 3D

1.2.1 Equilibrium

Equilibrium is a solution that does not change with time. This means, if the system starts at equilibrium, the state will remain at the equilibrium forever.

- Equilibrium of a dynamical system is a value of the state variables where the state variables do not change.

- Given an equation \( f(x) \), a point \( x^* \) is an equilibrium point is \( f(x^*) = 0 \).

Equilibrium Solution

A constant solution of a differential equation is called an equilibrium solution.

Disease-free Equilibrium

The equilibrium points of given system at the origin \((0,0)\) are called disease-free equilibrium \(0\).

Endemic Equilibrium

The positive solution of the given system is called endemic equilibrium \(E^*(S^*, I^*, R^*)\).
1.2.2 Global Analysis

In Mathematics, global analysis, also called analysis or manifolds, is the study of the global and topological properties of differential equations on manifolds and vector space bundles.

Global analysis uses techniques in infinite-dimensional manifold theory and topological spaces of mappings to classify behaviors of differential equations, particularly nonlinear differential equations.

- Global analysis is simply the study of differential equations, both ordinary and partial. Thus one can consider global analysis as differential equations from a global or topological point of view.

- Global analysis, or analysis on manifolds, studies the global nature of differential equations on manifolds.

1.2.3 Bifurcation

For classical epidemic models, it is common that a basic reproduction number is a threshold in a sense that a disease is persistent if the basic reproduction number is greater than 1, and dies out if it is below 1. In this case, the bifurcation leading from a disease free equilibrium to an endemic equilibrium is forward.

In recent years, papers found backward bifurcations due to social groups with different susceptibilities, pair formation, macro parasite infection, nonlinear incidences, and age structures in epidemic models. In this case, the basic reproduction number does not describe the necessary elimination effort; rather the effort is described by the value of the critical parameter at the turning point. Thus, it is important to identify backward bifurcations to obtain thresholds for the control of diseases.

Bifurcation means to "divide into two branches."

Hopf Bifurcation

The appearance or the disappearance of a periodic orbit through a local change in the stability properties of a steady point is known as the Hopf bifurcation.
Saddle Node Bifurcation

A saddle node bifurcation is a collision and disappearance of two equilibria in dynamical systems.

In systems generated by autonomous ODEs, this occurs when the critical equilibrium has one zero eigen value. This phenomenon is also called fold or limit point bifurcation. A discrete version of this bifurcation is considered in the article Saddle-node bifurcation for maps.

Homo-clinic Bifurcation

In Mathematics, a homo-clinic bifurcation is a global bifurcation which often occurs when a periodic orbit collides with a saddle point.

1.3 THE STABILITY ANALYSIS

Critical Point

A critical point of the system of equations \( \frac{dx}{dt} = F(x, y), \frac{dy}{dt} = G(x, y) \) is a point \((x^*, y^*)\) such that \(F(x^*, y^*) = 0 = G(x^*, y^*)\)

Stable Point

A critical point \((x^*, y^*)\) is said to be stable provided that if the initial point \((x_0, y_0)\) is sufficiently close to \((x^*, y^*)\) then the point \((x(t), y(t))\) remain close to \((x^*, y^*)\) ∀ \(t > 0\).

Unstable Point

A critical point which is not stable is called unstable critical point.

Asymptotically Stable Point

A critical point \((x^*, y^*)\) is said to be asymptotically stable if it is stable and every trajectory that begins sufficiently close to \((x^*, y^*)\), also approaches as \(t \to \infty\).

- Critical point is a saddle point and is unstable.
- The critical point is a nodal source and is unstable.
Bi-stable

A system that exhibits two distinct steady states is called bi-stable. The defining characteristic of bi-stability is simply that two stable states (minima) are separated by a peak (maximum).

1.3.1 Linear Stability Analysis for Systems of Ordinary Differential Equations

The procedure to determine stability of \((X^*, Y^*)\) is as follows:

- Compute all partial derivatives of the right hand-side of the original system of differential equations and construct the Jacobian matrix \(J\).
- Evaluate the Jacobian matrix \(J\), at the steady state \((X^*, Y^*)\).
- Compute eigenvalues.
- Conclude stability or instability based on the real parts of the eigenvalues.

Conditions of Stability

- If the eigen values of \(J\) all have real parts less than zero, then the steady state is stable.
- If at least one of the eigen values of \(J\) has real part greater than zero, then the steady state is unstable.
- If at least one of the eigen values of \(J\) has real part equal to zero then no conclusion can be made from the linear analysis. This is a
... borderline case between stability and instability. In these cases, nonlinear terms need to be considered.

1.3.2 Classification of Steady States

In addition to determining the linear stability of steady states, to classify them further by the eigenvalues of the Jacobian $J(x^*, y^*)$.

- $\lambda_i \in R, \lambda_{-i} < 0 \Rightarrow (x^*, y^*)$ is a stable node (or sink).
- $\lambda_i \in R, \lambda_{-i} > 0 \Rightarrow (x^*, y^*)$ is an unstable node (or source).
- $\lambda_i \in R, \lambda_1 < 0 < \lambda_2 \Rightarrow (x^*, y^*)$ is a saddle point (unstable).
- $\lambda_i \in C, \lambda_i = \alpha \pm i\beta, \alpha < 0 \Rightarrow (x^*, y^*)$ is a stable spiral (or stable focus).
- $\lambda_i \in C, \lambda_i = \alpha \pm i\beta, \alpha > 0 \Rightarrow (x^*, y^*)$ is an unstable spiral (or unstable focus).
- $\lambda_i \in C, \lambda_i = \alpha \pm i\beta, \alpha = 0 \Rightarrow (x^*, y^*)$ is a centre (neutrally stable).

1.3.3 Steady States with $\lambda_{1,2} \in R$

![Figure 1.4: Steady States with Eigen Values $\lambda_{1,2} \in R$](image-url)
1.3.4 Steady States with $\lambda_{i,j} \in C, \lambda_i = \alpha + i \beta$

![Steady States with Eigen values](image)

Figure 1.5: Steady States with Eigen values $\lambda_{i,j} \in C$

1.4 REVIEW OF LITERATURE

Early studies related to epidemic models were reported by a number of researchers namely Kermack and Mckendrick [44], Hoppensteadt and Waltman [38], Frank Hoppensteadt [26], Diekmann [14], Capasso and Serio [11], Anderson and May [2], Liu, Levin and Iwasa [47], Liu, Hethcote and Levin [48], Hethcote and Levin [34], Hethcote and Van Den Driessch [37], Mena Lorca and Hethcote [50], Derrick and Van den Driessche [13], Anderson and May [3], Wu and Ferg [64], Diekmann and Heesterbeek [15], Edward Bender [20], Hethcote [33], Brauer and Van den Driessche [8], Esteva and Matias [22], Van den Driessche and Watmough [54], Ruan and Wang [52], Wiggins [63], Alexander and Moghadas [1], Gumel [30], Li Jian-Quan el. al. [46], Daley and Gani [12], Wendi and Wang [61], Pundir and Pundir [53], Lawrence Perko [45], Xiao and Ruan [65], D’Onofrio et. al. [17], Vincenzo Capasso [55], Zhian Ma and Jia Li [75], Cai et. al. [10], Yuan and
Bo Li [69], Hai-Feng Huo and Zhan-Ping Ma [32], Keddar [40], Kar and Ashim Batabyal [41], Pathak, Maiti and Samanta [51], Zhang Zhonghua and Suo Yaohong [72], Yakui Xue and Xiafeng Duan [67], Fathalla, Rihan and Naim Anwar [24], Gajendra Ujjainkar, Gupta, Singh and Khandelud [28], Wang et. al. [60], Ankit Agrawal and Saxena [4] and so on.

In these articles, discussion is given on the description of the epidemic models, development of the system, capacity of the treatment and their measures.

In this section, a brief survey of some selected research articles with the above classification is provided.

**Models Relating to SIR Epidemic**

Kermack and Mckendrick [44] constructed a system of ordinary differential equations to study epidemiology. Simple mass action was introduced in classical model $\beta SI$, where $\beta$ is transmission rate, $S$ is susceptible population and $I$ is infectious population. Hoppensteadt and Waltman [38] have solved some problems in epidemics. Diekmann [14] analyzed the limiting behavior of an epidemic model.

Capasso and Serio [11] introduced a saturated incidence rate into epidemic models. The general incidence rate $\left( \frac{\kappa PS}{1 + \kappa I} \right)$ was proposed by Liu, Levin and Iwasa [47]. If the function $\left( \frac{\kappa PS}{1 + \kappa I} \right)$ is non-monotone, that is, $\left( \frac{\kappa PS}{1 + \kappa I} \right)$ is increasing when $I$ is small and decreasing when $I$ is large.

Hethcote et. al. [35] proposed non-linear oscillations in epidemic models. A very general form of non-linear incidence rate was considered by Derrick and Driessche [13].

Diekmann et al. [15] introducing the infective rate at the late stage of the SARS outbreak, even when the number of infective individuals were getting relatively larger. Brauer and Van den Driessche [8], considered the models for the transmission of disease with immigration of infective.
Ruan and Wang [52] studied an epidemic model with a specific nonlinear incidence rate $\frac{\lambda I}{1 + \alpha I^2}$ and presented a detailed qualitative and bifurcation analysis of the model. The capacity for the treatment of a disease in a community is a constant treatment rate $0 \leq r \leq 1$. Wang and Ruan [57] incorporated the following piecewise linear treatment function

$$
( ) = \begin{cases} 
  rI, & \text{if } 0 \leq I \leq I_c, \\
  rI_i, & \text{if } I > I_c,
\end{cases}
$$

In practical view, the above form of treatment function is justified where patients have to be hospitalized and the number of beds is limited or the medicines are not sufficient.

Li Jian-quan, Zhang Juan, Ma Zhi-en [46] discussed an epidemic models of SIR type and SIRS type with general contact rate and constant immigration of each class by means of theory of limit system and suitable Liapunov functions. Alexander and Moghadas [1] presented the periodicity in an epidemic model with a generalized non-linear incidence.

Wendi and Wang [61] proposed and analyzed the backward bifurcation of SIR epidemic model with treatment rate. Also they studied an epidemic model with a specific nonlinear incidence rate and presented a detailed qualitative and bifurcation analysis of the model. It is found that a backward bifurcation occurs if the capacity is small, and there exist bi-stable endemic equilibrium if the capacity is low.

Xiao and Ruan [65] considered a global analysis of an epidemic model with non-monotone incidence rate. A more general incidence $\frac{\lambda I^q S}{1 + \alpha I^p}$ was proposed by many other researchers [18, 46]. Yuan and Bo li [69], analyzed a Global Dynamics of an Epidemic Model with a Ratio-Dependent Nonlinear Incidence Rate. They used the nonlinear incidence rate $\frac{\lambda I}{1 + \alpha I^2}$ in the SIR epidemic model.

Kar and Batabyal [41] proposed an SIR model with non-monotonic incidence rate suggested by Xiao and Ruan incorporating the above
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treatment function and non-monotonic incidence rate under a treatment function. Hai-Feng Huo and Zhan-Ping Ma [32] is studied a delayed epidemic model with non-monotonic incidence rate which describes the psychological effect of certain serious on the community when the number of infective is getting larger.

Zhang Zhonghua and Suo Yaohong [73] discussed the existence and asymptotical stability of the equilibria, direction of Hopf bifurcation and also the disease eradication not only depends on $R_0$ but also on the initial sizes of all sub-populations.

Yakui Xue and Xiafeng Duan [67] is investigated an SIR epidemic model with nonlinear incidence rate and double delays due to the force of infection and temporary immunity period.

Gajendra et. al. [28] explained a modified SIR epidemic model with non-monotonic incidence rate under treatment with the new incidence rate $\frac{\lambda SI}{1 + \alpha I + \alpha_1 I^2}$. Fathalla Rihan and Naim Anwar [24] considered a delayed SIR epidemic model in which the susceptible are assumed to satisfy the logistic equation and the incidence term is of saturated form with the susceptible. They investigated the qualitative behavior of the model and find the conditions that guarantee the asymptotic stability of corresponding steady states.

Ankit Agrawal [4] has proposed an incidence rate as $\frac{\lambda I}{\rho + \beta I}$. It has been made to develop realistic mathematical models for the transmission dynamics of infectious diseases.

Motivation for Present Work

In all the above cited works, the SIR epidemic models were discussed and analyzed with constant rate of susceptible. If the number of susceptibility will increase, then the hospitals will meet some difficult problems to offer constant treatment to all the infective. As the improving of the hospital’s treatment conditions, such as effective medicines, skillful techniques, the treatment rate will be increased. At last, the treatment rate
of the disease won’t increase until a plateau is reached. Therefore, constant recovery is a poor description of real-world infections. Aiming at this kind of improvement of circumstance, in the present work, the SIR model with immigration (i.e., increase of susceptible) is considered. In SIR epidemic model with immigration using non-monotonic incidence rate under treatment, nonlinear incidence rate under treatment and generalized saturated incidence rate under treatment are used to study the stability of the disease free equilibrium and endemic equilibrium. These models are also used to determine the local and global stability conditions and to verify the existence of bi-stability. It is found that a backward bifurcation occurs if the capacity is small, and there exist bi-stable endemic equilibrium if the capacity is low.

It explains, either the number of infective individuals tends to zero as time evolves or the disease persists. It is shown that this kind of treatment rate leads to the existence of multiple endemic equilibria.

1.5 PROFILE OF THE RESEARCH WORK

The research work carried out by the author is presented in chapters 2 to 10. In all these chapters, SIR epidemic model with immigration and modified SIR epidemic model with immigration is considered. The models are designed by the systems of ordinary differential equations using simple mass action. Some numerical simulations and analysis are given to illustrate the analytical results. Three types of incidence rates are used in the present work. They are i) Non-monotonic incidence rate under treatment, ii) Non-linear incidence rate under treatment and iii) Generalized saturated incidence rate under treatment. Also bifurcation of SIR epidemic model is discussed with non-monotonic incidence rate.

Chapter 2: Modeling and Simulation Analysis of SIR Epidemic Models with Immigration using Non-monotonic Incidence Rate under Treatment

In this chapter, two mathematical models are constructed with the system of ordinary differential equations and assuming that the incidence rate is \( \frac{\lambda IS}{1 + \alpha T} \). In this work the treatment function \( T(I) \) is defined by
This means that the treatment rate is proportional to the infective when the number of infectious is less or equal to some fixed value \( I_0 \) and the treatment is constant when the number of infective crosses the fixed value \( I_0 \). The stability analysis of the model is discussed with the above two cases of treatment function \( T(I) \).

**Chapter 3: Global Analysis of Modified SIR Epidemic Models with Immigration using Non-Monotone Incidence Rate under Treatment**

This chapter is constructed with two kinds of epidemic models. Those are i) Modified SIR epidemic model with immigration using non-monotonic incidence rate and ii) Modified SIR epidemic model with immigration using non-monotonic incidence rate under treatment. The modified SIR epidemic model with immigration using non-monotonic incidence rate under a limited resource for treatment is proposed to recognize the effect of the capacity of the treatment. The incidence rate \( \frac{\lambda SI}{1 + \alpha I + \alpha I^2} \) is considered in this chapter. Two different cases of treatment functions \( T(I) \) are discussed and the unique positive equilibrium point \( (\ast, \ast) \) is used to understand the stability condition of the treatment. Also it explains the stability of the disease-free equilibrium and the endemic equilibrium.

**Chapter 4: Modeling and Simulation Analysis of SIR Epidemic Models with Immigration using Nonlinear Incidence Rate**

In this chapter, two mathematical models are constructed. They are i) SIR epidemic model with immigration using non-linear incidence rate and ii) Modified SIR epidemic model with immigration using non-linear incidence rate. The infectious force takes the form \( \left( I, S \right) = g \left( \frac{I}{S} \right) = \frac{\lambda IS}{S^2 + \alpha I} \). It will show the properties of disease-free equilibrium and endemic equilibrium and perform a global qualitative analysis of model.
Chapter 5: Modeling and Simulation Analysis of SIR Epidemic Models with Immigration using Nonlinear Incidence Rate under Treatment

To incorporate the effect of the behavioral changes of the susceptible individuals, a nonlinear incidence rate given by \( \frac{\lambda (I / S)}{1 + \alpha (I / S)} \) is considered in this chapter. In this work, the treatment function \( T(I) \) is defined by \[
T(I) = \begin{cases} 
0, & \text{if } I > I_0 \\
\frac{r I}{I}, & \text{if } 0 \leq I \leq I_0 
\end{cases}
\]

This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value \( I_0 \) and the treatment is constant when the number of infective crosses the fixed value \( I_0 \). From these equilibrium points and stability analysis, the condition of the treatment can be evaluated.

Chapter 6: SIR Epidemic Models with Immigration using Generalized Saturated Incidence Rate

On account of the effect of limited treatment resources on the control of epidemic disease, SIR epidemic models with generalized saturated incidence rate is incorporated. In this chapter, two SIR epidemiological models with asymptotically homogeneous incidence rate function is introduced with the transmission rates i) \( \phi = \frac{\lambda SI}{\rho + \alpha I} \) and ii) \( \phi = \frac{\lambda SI}{\rho + \alpha_1 I + \alpha_2 I^2} \), where \( \lambda \) is the proportionality constant \( \rho \) is the positive constant \( \geq 1 \), \( \alpha \) is a positive parameter and \( \lambda SI \) is the infection force of the disease. This transmission rates displays a saturation effect accounting for fact that the number of contacts an individual reaches some maximal value due to spatial distribution of the population. The stability analysis of the disease-free and the endemic equilibrium are discussed with a nonlinear incidence rate.

Chapter 7: SIR Epidemic Models with Immigration using Generalized Saturated Incidence Rate under Treatment

In this chapter, two mathematical models are designed with two different transmission rates. The treatment function \( T(I) \) which is useful to
find the capacity of the treatment. These SIR models are helpful to obtain the properties of the equilibrium points and analyze sufficient conditions under which the equilibrium points are unique or global. It also provides learning about the transmission rate effects and these modeling results are helpful to predict the developing tendency of disease.

Chapter 8: Qualitative Analysis of SIR Epidemic Model with Immigration using Saturated Incidence Rate under Treatment

In this chapter, the SIR epidemic model on immigration under saturated treatment rate is formulated and analyzed qualitatively by the numerical illustrations and diagrams. Here $\lambda SI$ is the bilinear incidence rate and the removal rate of the infected individuals to be

\[
(\cdot) = \frac{\beta I}{1 + \alpha I}, \text{I > 0, and } \alpha, \beta > 0
\]

Chapter 9: Backward Bifurcation of SIR Epidemic Model with Immigration using Non-monotone Incidence Rate under Treatment

In this chapter, the backward bifurcation of SIR epidemic model is occurred because of the insufficient capacity of treatment. An insufficient capacity of the treatment is a source of the backward bifurcation. Here the incidence rate $\frac{\lambda SI}{1 + \alpha I}$ has introduced to control the disease. Because of the limited resources, this model has bi-stable endemic equilibrium. The existence of endemic equilibrium and the existence of backward bifurcation investigate the effect of the limited medical resources and their supply efficiency of the backward bifurcation. It is analyzed locally and globally the asymptotic stability of the disease-free equilibrium and the endemic equilibrium. Also it presents a global analysis of the model.

Chapter 10: Bifurcation of SIR Epidemic Model with Immigration using Non-monotone Incidence Rate of the Infective

This model exhibits two bifurcations viz. trans-critical bifurcation when the basic reproductive number $R_0 = 1$ and backward bifurcation where the
disease transmission rate $\lambda$ plays as control parameter. This model undergoes a sequence of saddle-node bifurcation, subcritical bifurcation and homo-clinic bifurcation. Since the model is globally stable in the absence of the removal rate, this suggests that a constant removal rate of the infectives induces the periodic oscillations of diseases. The global analysis of the model and discuss the existence and non-existence of limit cycles are presented.

**Findings and Suggestions**

From chapters 2 to 10, SIR epidemic models and modified SIR epidemic models with immigration are considered with various incidence rates under treatment. Simulation analysis, Global analysis and Qualitative analysis have been discussed in some of the models.

In future, SIR epidemic models with immigration using non-monotonic, nonlinear and saturated incidence rates under treatment are planned to work by using qualitative analysis and dynamic analysis. The SEIR model will divide the total population into four epidemiological classes which are Susceptible ($S$), Exposed ($E$), Infectious ($I$) and Recovered ($R$). This SEIR epidemic model can also discuss with immigration using non-monotonic incidence rate. Further SIR models with immigration are under investigations.