BIFURCATION OF SIR EPIDEMIC MODEL WITH IMMIGRATION USING NON-MONOTONE INCIDENCE RATE OF THE INFECTIVES

Bifurcation of SIR epidemic model with immigration using non-monotonic incidence rate of the infectives describe the psychological effect of certain serious diseases on the community when the number of infectives is getting larger. Now assume that the treated individual may be again infected by infectious individual. This model undergoes a sequence of saddle-node bifurcation, subcritical bifurcation and homo-clinic bifurcation. Since the model is globally stable in the absence of the removal rate, this suggests that a constant removal rate of the infectives induces the periodic oscillations of diseases. The global analysis of the model, the existence and non-existence of limit cycles are presented.

10.1 Introduction

Kermack and McKendrick [44] constructed a system of ordinary differential equations to study epidemiology. From then on, mathematical models have become important tools in analyzing the spread and control of epidemic diseases. In classical epidemic models, the treatment rate is assumed to be proportional to the number of the infectives. That is, it is usually assumed that the capacity for the treatment of a disease in a community is a constant treatment rate \(0 \leq r \leq 1\). Wang et. al. [56] incorporated the following piecewise linear treatment function into an SIR model:

\[
T(I) = \begin{cases} 
  rI, & \text{if } 0 \leq I \leq I_0 \\
  rI_0, & \text{if } I > I_0 
\end{cases}
\]

where \(rI_0\) is the capacity of treatment for infected population, and \(r\) is a positive constant. It is found that a backward bifurcation occurs if the capacity is small, and there exist bi-stable endemic equilibrium if the capacity is low. In addition, Zhang and Liu [71] studied an SIR model with saturated treatment function:
where $\alpha, \beta > 0$. They found sufficient conditions for global stability of endemic and disease-free equilibrium, and they discussed the equilibrium effects of the magnitudes of the parameters in the treatment function. After studying the cholera epidemic spread in Bari in 1973, Capasso and Serio [11] introduced a saturated incidence rate $g(I)S$ into epidemic models. Wang and Ruan [57] studied an epidemic model with a specific nonlinear incidence rate

$$g(I)S = \frac{kI^2S}{1 + \alpha I^2}$$

and presented a detailed qualitative and bifurcation analysis of the model. The general incidence rate

$$g(I)S = \frac{kI^pS}{1 + \alpha I^q}$$

was proposed by Liu et al. [47] and used by a number of authors, Derrick and Van Den Driessche [13], Hethcote and Van Den Driessche [37], Alexander and Moghadas [1], Arino et al. [5], Liu et al. [48], Xiao and Ruan [65] etc. Gumel [30] and Wang et al. [59] introduced the infectives rate at the late stage of the SARS outbreak, even when the number of infective individuals were getting relatively larger.

Treatment including isolation or quarantine is an important method to decrease the spread of diseases such as measles, AIDS, tuberculosis, and flu. Suppose that the capacity for the treatment of a disease in a community is a constant $r$ with immigration. In order to easily understand its effect, consider a case that the removal rate of infectives equals $r$. This means that, use the maximal treatment capacity to cure or isolate infective so that the disease is eradicated. This can occur if the disease is so dangerous that we hope to wipe out it quickly, or the disease spreads rapidly so that the treatment capacity is insufficient for treatment in a period (for example flu).

Bifurcation of SIR epidemic model with immigration using non-monotone incidence rate is considered in this chapter. This chapter is prearranged as follows: In section 10.2, the model is formulated and the description is given. Bifurcation is also analyzed by some theorems in the
same section. Numerical simulation and discussions are given in section 10.3. In the last section 10.4, conclusion is presented.

10.2 MODEL DESCRIPTION AND ANALYSIS

10.2.1 Model Description

Kar and Batabyal [41] used the treatment function in Xiao and Ruan [65] SIR model, the model changes into

\[
\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1 + \alpha I^2} + \beta R
\]

\[
\frac{dI}{dt} = \frac{\lambda IS}{1 + \alpha I^2} - (d + m)I - T(I)
\]

\[
\frac{dR}{dt} = mI - (d + \beta)R + T(I)
\]

where \( T(I) = \begin{cases} rI, & \text{if } 0 \leq I \leq I_0 \\ K, & \text{if } I > I_0 \end{cases} \)

Wendi and Wang [61] considered the above model with saturated incidence rate in the bifurcation of SIR epidemic model. The model takes the form

\[
\frac{ds}{dt} = a - dS - \lambda SI + \mu
\]

\[
\frac{dI}{dt} = \lambda SI - (d + m + \varepsilon)I - T(I)
\]

\[
\frac{dR}{dt} = mI - dR + T(I)
\]

where \( T(I) = \frac{\beta I}{1 + \alpha I} \)

Wendi et. al. [59] proposed an SIR epidemic model with the removal rate of the infectives. The model is of the form

\[
\frac{dS}{dt} = a + \mu - dS - \lambda SI
\]

\[
\frac{dI}{dt} = \lambda SI - (d + m)I - T(I)
\]
\[
\frac{dR}{dt} = mI - dR + T(I)
\]

where \( T(I) = \begin{cases} r & \text{for } I > 0 \\ 0 & \text{for } I = 0 \end{cases} \)

In this chapter, bifurcation of SIR epidemic model with immigration using non-monotonic incidence rate is discussed.

### 10.2.2 Assumptions

- Consider Wendi et. al. [59] SIR epidemic model.
- Assume that the population consists of three types of individuals. They are susceptible, infective and recovered individuals.
- Let the non-monotonic incidence rate be \( \frac{AIS}{1 + \alpha I} \).
- Assumed that the capacity for the treatment of a disease in a community is a constant treatment rate \( 0 \leq r \leq 1 \).
- The treatment rate of the infectives
  \[
  T(I) = \begin{cases} r & \text{for } I > 0 \\ 0 & \text{for } I = 0 \end{cases}
  \]
- Consider the susceptible rate with immigration.
- Assume that the immigration is constant.
- The non-monotonic incidence rate is considered with a single parametric measure \( \alpha \) and two different cases of treatment functions \( T(I) \) are considered.

### 10.2.3 Notations

- \( S \) : Number of susceptibles
- \( I \) : Number of infectives.
- \( R \) : Number of removed or recovered individuals.
- \( a \) : Recruitment rate of the population
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\[ d \]: Natural death rate of the population

\[ \lambda \]: The proportionality constant

\[ m \]: Natural recovery rate of the infective individuals

\[ \beta \]: The rate at which recovered individuals lose immunity and return to susceptible class

\[ \mu \]: Increase of susceptibles at a constant rate

\[ \alpha \]: The parameter measures of the psychological or inhibitory effect

\[ E_0 \]: Disease-free equilibrium.

\[ E^* \]: Endemic equilibrium

\[ R_0 \]: Basic Reproduction Number

\[ S^* \]: Susceptible rate at the endemic equilibrium

\[ R^* \]: Recovered or removed rate at the endemic equilibrium

\[ I^* \]: Infected rate at the endemic equilibrium

10.2.4 Mathematical Model

The model to be studied takes the following form:

\[
\begin{align*}
\frac{dS}{dt} &= a + \mu - dS - \frac{\lambda SI}{1 + \alpha I} \\
\frac{dI}{dt} &= \frac{\lambda SI}{1 + \alpha I} - (d + m)I - T(I) \\
\frac{dR}{dt} &= mI - dR + T(I)
\end{align*}
\]  \hspace{1cm} (10.1)

Now assume a bilinear incidence rate in (10.1). A good alternative for this is a modified standard incidence rate \( \frac{\lambda SI}{1 + \alpha I} \). In (10.1), \( T(I) \) is the removal rate of infective individuals due to the treatment of infectives. Suppose that the treated infectives become recovered when they are treated in treatment sites.
Suppose that
\[ T(I) = \begin{cases} 
  r & \text{for } I > 0 \\
  0 & \text{for } I = 0 
\end{cases} \quad (10.2) \]

where \( r > 0 \) is a constant and represents the capacity of treatment for infectives. This means that, a constant removal rate for the infectives until the disease disappears. Suppose that \((S(t), I(t), R(t))\) is a solution of (10.1), if \( S(t) > 0, I(t) > 0, R(t) > 0, \) for \( 0 \leq t < t_0 \) and \( I(t_0) = 0. \)

\[
\frac{dS}{dt} = a + \mu - dS \\
I(t) = 0 \quad \text{for } t \geq t_0 \\
\frac{dR}{dt} = -dR
\]

Consequently, \( R^3 \) is positively invariant for system (10.1). The purpose of this chapter is to show that this removal rate has significant effects on the dynamics of (10.1). Next to prove that (10.1) undergoes a sequence of bifurcations including saddle-node bifurcation, subcritical Hopf bifurcation, and homoclinic bifurcation. Here a global analysis of the model and discuss the existence and non-existence of limit cycles are presented.

Best possible capacity for treatment can be chosen according to our results. Now simplify the model before going into any detail. Since the first two equations are independent of the third one and its dynamic behavior is trivial when \( I(t_0) = 0 \) for some \( t_0 > 0 \), it suffices to consider the first two equations with \( I > 0 \). Thus the reduced model is

\[
\begin{align*}
  \frac{dS}{dt} &= a + \mu - dS - \frac{\lambda SI}{1 + \alpha I} \\
  \frac{dI}{dt} &= \frac{\lambda SI}{1 + \alpha I} - (d + m)I - r 
\end{align*}
\quad (10.3)
\]

It is assumed that all the parameters are positive constants.
10.2.4 Bifurcations

In this section, first consider the equilibrium of (10.3) and their local stability. Then study the Hopf bifurcation and the Bogdanov–Takens bifurcation of (10.3). In order to find the endemic equilibrium of (10.3), set

\[ a + \mu - dS - \frac{\lambda SI}{1 + \alpha I} = 0 \]

\[ a + \mu - S[d + \frac{\lambda I}{1 + \alpha I}] = 0 \]

\[ S = \frac{a + \mu}{d + \frac{\lambda I}{1 + \alpha I}} \]

Substitute \( S = \frac{a + \mu}{d + \frac{\lambda I}{1 + \alpha I}} \) into \( \frac{\lambda SI}{1 + \alpha I} - (d + m)I - r = 0 \) to obtain the quadratic equation

\[ [(\alpha d + \lambda)(d + m)]I^2 + [d(d + m) + r(\alpha d + \lambda) - (\alpha d + \lambda)(a + \mu)]I + rd = 0 \]

(10.4)

Set \( R_0 = \frac{(\alpha d + \lambda)(a + \mu)}{d(d + m)}, \quad H = \frac{(\alpha d + \lambda)r}{d(d + m)} \)

Then (10.4) can be written as

\[ [(\alpha d + \lambda)(d + m)]I^2 + [(\alpha d + \lambda)r + d(d + m) - (\alpha d + \lambda)(a + \mu)]I + rd = 0 \]

\[ \frac{(\alpha d + \lambda)}{d} I^2 - \left( \frac{(\alpha d + \lambda)(a + \mu)}{d(d + m)} - 1 - \frac{(\alpha d + \lambda)r}{d(d + m)} \right) I + \frac{r}{d + m} = 0 \]

\[ \frac{(\alpha d + \lambda)}{d} I^2 - (R_0 - 1 - H)I + \frac{r}{d + m} = 0 \]

(10.5)

\[ R_0 = \frac{(\alpha d + \lambda)(a + \mu)}{d(d + m)} \]

is the reproduction number of (10.3) in the absence of the removal rate. It is evident that (10.5) does not have a positive solution if \( R_0 \leq 1 \). If \( R_0 > 1 \), it is easy to see that (10.5) does not have a positive solution if
(\sqrt{R_0} - 1)^2 < H \quad \text{(10.6)}

Admits a positive solution if

(\sqrt{R_0} - 1)^2 = H \quad \text{(10.7)}

And has two positive solutions if

(\sqrt{R_0} - 1)^2 > H > 0 \quad \text{(10.8)}

Thus, (10.3) does not have a positive equilibrium if \( R_0 \leq 1 \) or (10.6) holds. Furthermore, (10.7) implies that (10.3) has one endemic equilibrium and (10.8) implies that (10.3) has two endemic equilibrium. If \( N = S + I \), then

\[
\frac{dN}{dt} = a + \mu - dS - \frac{\lambda SI}{1 + \alpha I} + \frac{\lambda SI}{1 + \alpha I} - (d + m)I - r
\]

\[
= a + \mu - r - mI - d(S + I)
\]

\[
= a + \mu - r - mI - dN \quad \text{where} \quad N = S + I
\]

\[
\leq a + \mu - r - dN
\]

\[
\frac{dN}{dt} \leq a + \mu - r - dN.
\]

It follows that the positive solutions of (10.3) are bounded. If \( R_0 \leq 1 \) or (10.6) holds, it follows that \( I(t) \) becomes 0 in finite time, i.e., the disease disappears in a finite time. Now, it proposes the following assumption:

\[ R_0 > 1 \quad \text{and} \quad 0 < H < (\sqrt{R_0} - 1)^2 \quad (*) \]

Let (*) hold. Then (10.3) admits two endemic equilibriums \( E_1 = (S_1, I_1) \) and \( E_2 = (S_2, I_2) \), where

\[
I_1 = \frac{d}{2(\alpha d + \lambda)} \left[ (R_0 - 1 - H) + \sqrt{(R_0 - 1 - H)^2 - 4(H + \frac{\alpha r}{d + m})} \right]
\]

\[
S_1 = \frac{(\alpha + \mu)(1 + \alpha I_1)}{d(1 + \alpha I_1) + \lambda I_1}
\]

\[
I_2 = \frac{d}{2(\alpha d + \lambda)} \left[ (R_0 - 1 - H) - \sqrt{(R_0 - 1 - H)^2 - 4(H + \frac{\alpha r}{d + m})} \right]
\]
\[ S_2 = \frac{(\alpha + \mu)(1 + \alpha I_2)}{d(1 + \alpha I_2) + \lambda I_2} \]

Although the endemic equilibrium occurs under the assumption (*), it shows that the disease can disappear in a range of the parameters. This means that it is unnecessary to increase the removal rate \( r \) to \( H > (\sqrt{R_0} - 1)^2 \) to make the disease disappear. By analyzing the stability of these two equilibria, the Jacobian matrix of (10.3) at \((S_1, I_1)\) is

\[
J_1 = \begin{bmatrix}
-d - \frac{\lambda I_1}{1 + \alpha I_1} & -\frac{\lambda S_1}{1 + \alpha I_1} \\
\frac{\lambda I_1}{1 + \alpha I_1} & \frac{\lambda S_1}{1 + \alpha I_1} - \frac{\alpha \lambda S_1 I_1}{(1 + \alpha I_1)^2} - (d + m)
\end{bmatrix}
\]

Note that \( a + \mu - d S_1 = \frac{\lambda S_1 I_1}{1 + \alpha I_1} = (d + m)I_1 - r. \)

\[ \Rightarrow S_1 = \frac{a + \mu - (d + m)I_1 + r}{d} \]

Thus

\[
\det(J_1) = d(d + m) \left[ -R_0 + \frac{2(\alpha d + \lambda)}{d} I_1 + H + 1 \right]
\]

\[ = -d(d + m) \sqrt{(R_0 - 1 - H)^2 - 4(H + \frac{\alpha r}{d + m})} < 0. \]

It follows that \((S_1, I_1)\) is a saddle point. The Jacobian matrix of (10.3) at \((S_2, I_2)\) is

\[
J_2 = \begin{bmatrix}
-d - \frac{\lambda I_2}{1 + \alpha I_2} & -\frac{\lambda S_2}{1 + \alpha I_2} \\
\frac{\lambda I_2}{1 + \alpha I_2} & \frac{\lambda S_2}{1 + \alpha I_2} - \frac{\alpha \lambda S_2 I_2}{(1 + \alpha I_2)^2} - (d + m)
\end{bmatrix}
\]

\[
\det(J_2) = d(d + m) \left[ -R_0 + \frac{2(\alpha d + \lambda)}{d} I_2 + H + 1 \right]
\]

\[ = -d(d + m) \sqrt{(R_0 - 1 - H)^2 - 4(H + \frac{\alpha r}{d + m})} < 0. \]
Similarly,
\[
\det(J_2) = d(d + m) - \sqrt{(R_0 - 1 - H)^2 - 4(H + \frac{\alpha r}{d + m})} > 0
\]
Thus, \((S_2, I_2)\) is a focus, a node, or a centre. The stability of this equilibrium is stated in the following theorem.

**Theorem: 10.2.1**

Let (*) hold. Then

i) \(E_2\) is stable if either

\[
\frac{(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3}{m} \leq (\alpha d + \lambda)r
\]

(or)

\[
(\alpha d + \lambda)r < \frac{(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3}{m}
\]

and

\[
(\alpha d + \lambda)r < \frac{1}{2} \left[ 2(\alpha d + \lambda)(a + \mu) + (2d + m)(d + m) \left( 1 - \sqrt{1 + \frac{4(\alpha d + \lambda)(a + \mu)}{(d + m)^2}} \right) \right]
\]

(10.9)

ii) \(E_2\) is unstable if

\[
(\alpha d + \lambda)r < \frac{1}{2} \left[ 2(\alpha d + \lambda)(a + \mu) + (2d + m)(d + m) \left( 1 - \sqrt{1 + \frac{4(\alpha d + \lambda)(a + \mu)}{(d + m)^2}} \right) \right]
\]

(10.10)

**Proof**

Since \(S_2 = \frac{a + \mu - (d + m)I_2 + r}{d}\)
\[
\text{Trace}(J_2) = \frac{-(2d + m)(\alpha d + \lambda)}{d} I_2 \nabla \frac{2d^2 - (\alpha d + \lambda)(a + \mu) + r(\alpha d + \lambda) + dm}{d}
\] (10.12)

Thus, the \textit{Trace} is negative if \(2d^2 - (\alpha d + \lambda)(a + \mu) + r(\alpha d + \lambda) + dm \geq 0\).

Suppose \(2d^2 - (\alpha d + \lambda)(a + \mu) + r(\alpha d + \lambda) + dm < 0\) (10.13)

Now find the condition under which \(\text{Trace}(J_2) = 0\)

Set \(D \triangleq \frac{d(d + m)}{(\alpha d + \lambda)(2d + m)} \left( \frac{d}{d + m} + 1 - R_0 + H \right)\)

The equation (10.12) implies that \(\text{Trace}(J_2) = 0\)

\[I_2 = \frac{2d^2 - (\alpha d + \lambda)(a + \mu) + r(\alpha d + \lambda) + dm}{(2d + m)(\alpha d + \lambda)} = D_1.\] (10.14)

If \(D_2 = -\frac{2d}{2d + m} + \frac{m}{2d + m}(R_0 - 1 - H)\)

From the definition of \(I_2\), \(\text{Trace}(J_2) = 0\) is equivalent to

\[D_2 = \sqrt{R_0 - 1 - H)^2 - 4(H + \frac{\alpha r}{d + m})}\] (10.15)

Thus, the set of \(\text{Trace}(J_2) = 0\) is empty if

\(H \geq R_0 - 1 - \frac{2d}{m}\) (10.16)

Suppose \(H < R_0 - 1 - \frac{2d}{m}\) (10.17)

Taking square on both sides of (10.15) and simplifying the resulting equation, it implies

\[D_3 \triangleq r^2(\alpha d + \lambda) + (-3dm - 2(\alpha d + \lambda)(a + \mu) - m^2 - 2d^2)r\]
\[+ (\alpha d + \lambda)(a + \mu)^2 - d(a + \mu)\gamma - 2(a + \mu)d^2 = 0\] (10.18)

Hence
\[ r = \frac{1}{2(\alpha d + \lambda)} \left[ 2(\alpha d + \lambda)(a + \mu) + (2d + m)(d + m) \left( 1 \pm \sqrt{1 + \frac{4(\alpha d + \lambda)(a + \mu)}{(d + m)^2}} \right) \right]. \]

From (10.13)
\[ \frac{1}{2(\alpha d + \lambda)} \left[ 2(\alpha d + \lambda)(a + \mu) + (2d + m)(d + m) \left( 1 - \sqrt{1 + \frac{4(\alpha d + \lambda)(a + \mu)}{(d + m)^2}} \right) \right]. \]

(10.19)

In (H_1), (10.17) and (10.19) are the necessary and sufficient conditions for \( \text{Trace}(J_2) = 0 \). Now \( E_2 \) is stable if (10.9) is valid. The previous discussion shows that the stability of \( E_2 \) does not change if (10.16) hold. Note that (10.16) is equivalent to
\[
\frac{(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3}{m} < (\alpha d + \lambda)r
\]

By the definitions of \( D_1 \) and \( D_2 \),
\[
\text{Trace} (J_2) = -\frac{(2d + m)(\alpha d + \lambda)}{d} (I_2 - D_1)
\]
\[
= -\frac{2d + m}{2} \left( (R_0 - 1 - H)^2 - 4(H + \frac{ar}{d + m} - D_2) \right)
\]

(10.20)

Thus, (10.16) implies that \( \text{Trace}(J_2) < 0 \). Therefore, \( E_2 \) is stable if (10.9) holds. It follows that \( \text{Trace}(J_2) < 0 \) if (10.10) is valid and that \( \text{Trace}(J_2) > 0 \) if (10.11) holds.

\[
\frac{(R_0 - 1 - H)^2 - 4(H + \frac{ar}{d + m} - D_2)}{(2d + m)^2(d + m)} - D_2^2 = \frac{4(\alpha d + \lambda)}{d(2d + m)^2(d + m)} D_3.
\]

Now verify the existence of a Hopf bifurcation in (10.3) and determine its direction. Set
\[
h_0 = \frac{1}{2(\alpha d + \lambda)} \left[ 2(\alpha d + \lambda)(a + \mu) + (2d + m)(d + m) \left( 1 - \sqrt{1 + \frac{4(\alpha d + \lambda)(a + \mu)}{(d + m)^2}} \right) \right]
\]
Theorem: 10.2.2

Let (*) hold. Assume that

\[ r < \frac{(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3}{m} \]  \hspace{1cm} (10.21)

Then there is a family of unstable limit cycles if \( r \) is less than and near \( h_0 \), i.e., a subcritical Hopf bifurcation occurs when \( r \) passes through \( h_0 \).

Proof

Suppose \( r = h_0 \). Then \( \text{Trace}(J_2) = 0 \). It follows from (10.14) that

\[ I_2 = \frac{d + m}{\alpha d + \lambda} \left( -\frac{d}{d + m} - \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4(\alpha d + \lambda)(a + \mu)}{(d + m)^2}} \right) \]

\[ S_2 = \frac{d + m}{2(\alpha d + \lambda)} \left( 1 + \sqrt{1 + \frac{4(\alpha d + \lambda)(a + \mu)}{(d + m)^2}} \right) \]

Set \( \omega = \sqrt{\text{det}(J_2)} \)

\[ = \sqrt{d(d + m)\sqrt{(R_0 - 1 - H)^2 - 4(H + \frac{\alpha r}{d + m})}} \]

Then the eigen values of \( J_2 \) are \( \lambda_1 = \omega \) and perform coordinate transformation by \( x = S - S_2, y = I - I_2 \).

Then system (10.3) becomes

\[ \frac{dx}{dt} = -[d + (\alpha d + \lambda)I_2]x - (\alpha d + \lambda)S_2y - (\alpha d + \lambda)xy, \]

\[ \frac{dy}{dt} = (\alpha d + \lambda)xi_2 + [(\alpha d + \lambda)S_2 - d - m]y + (\alpha d + \lambda)xy \]  \hspace{1cm} (10.22)

Setting \( x = -(\alpha d + \lambda)S_2y \),

\[ y = wu + [d + (\alpha d + \lambda)I_2]v \] and using

\[ \text{Trace}(J_2) = (\alpha d + \lambda)S_2 - 2d - m - (\alpha d + \lambda)I_2 = 0 \]

\[ w^2 = \text{det}(J_2) = -d(\alpha d + \lambda)S_2 + d^2 + dm + (\alpha d + \lambda)I_2 + (\alpha d + \lambda)mi_2, \]
We obtain
\[
\frac{du}{dt} = -wv + \frac{(\alpha d + \lambda)v[-(\alpha d + \lambda)S_d + (\alpha d + \lambda)S_2 + (\alpha d + \lambda)I_2 + d][wu + dv + (\alpha d + \lambda)I_d v]}{w}
\]
\[
\frac{dv}{dt} = -wv + f(u,v)
\]
\[
\frac{dv}{dt} = wu - (\alpha d + \lambda)v[wu + dv + (\alpha d + \lambda)I_d v]
\]
\[
\frac{dv}{dt} = wu - g(u,v)
\]
where \(f = \frac{(\alpha d + \lambda)v[-(\alpha d + \lambda)S_d + (\alpha d + \lambda)S_2 + (\alpha d + \lambda)I_2 + d][wu + dv + (\alpha d + \lambda)I_d v]}{w}\)

\(g = g = -(\alpha d + \lambda)v[wu + dv + (\alpha d + \lambda)I_d v]\)

Using the fact that \((\alpha d + \lambda)S_2 - 2d - m - (\alpha d + \lambda)I_2 = 0\), then it gives
\(g = wf / (d + m)\). If
\[
\eta = \frac{1}{16}[f_{uuu} + f_{uvv} + g_{uuu} + g_{vvv}] + \frac{1}{16}\omega[f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{ vv}]
\]
By some tedious calculations, it implies
\[
-(\alpha d + \lambda)^2[-(\alpha d + \lambda)S_d + (\alpha d + \lambda)S_2 + (\alpha d + \lambda)I_2 + d][d + (\alpha d + \lambda)I_2]
\]
\[
\eta = \frac{-3d^2 - 4dm - m^2 - \omega^2 - 2d(\alpha d + \lambda)I_2 - 2m(\alpha d + \lambda)I_2}{8\omega^2(d + m)^2}
\]

Note that \(-(\alpha d + \lambda)S_2 + (\alpha d + \lambda)I_2 + d = -d - m\).
\[
(\alpha d + \lambda)^2[d + (\alpha d + \lambda)I_2][2d^2 + 3dm + m^2]
\]
\[
\eta = \frac{+(\alpha d + \lambda)S_2 d + dm S_2 + m (\alpha d + \lambda)I_2}{8\omega^2} > 0
\]

The conclusion of the theorem follows.

As an example, let \(a = 4, \mu = 4, d = 0.1, \lambda = 0.1, m = 1, \alpha = 0.2\) then
\[
(\sqrt{R_0} - 1)^2 d(d + m) = 0.4183, \quad \frac{(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3}{m} = 0.828 \quad \text{and} \quad h_0 = 2.2917.
\]
Then Theorem 10.3.2 shows that there is an unstable limit cycle when \( r \) decreases from 2.2917.

At this time, the local stability of the equilibrium of model (10.3) is clear. In order to determine the global dynamics of the model, it is to investigate its global bifurcation. Suppose

\[
R_0 > 1 \text{ and } H = (\sqrt{R_0} - 1)^2
\]

Then (10.3) has one unique positive equilibrium \((S^*, I^*)\) where

\[
I^* = \frac{d}{(\alpha d + \lambda)}(\sqrt{R_0} - 1), \quad S^* = \frac{\alpha + \mu}{d\sqrt{R_0}}.
\]

The Jacobian matrix of (10.3) at this point is

\[
J_0 = \begin{bmatrix}
-\frac{\lambda I^*}{1 + \alpha I^*} & \frac{\lambda S^*}{1 + \alpha I^*} \\
\frac{\lambda I^*}{1 + \alpha I^*} & \frac{\lambda S^*}{1 + \alpha I^*}
\end{bmatrix}
\]

(10.24)

Suppose

\[
\sqrt{R_0} = 1 + \frac{d}{\alpha d + \lambda}
\]

(***)

By (10.24)

\[
\det(J_0) = -d(\alpha d + \lambda)S^* + d^2 + md + (\alpha d + \lambda)dI^* + \alpha d + (\alpha d + \lambda)\lambda m I^*
\]

\[
= -d(\alpha d + \lambda)\left(\frac{\alpha + \mu}{d\sqrt{R_0}}\right) + d^2 + md + (\alpha d + \lambda)d\left(\frac{d}{(\alpha d + \lambda)}(\sqrt{R_0} - 1)\right)
\]

\[
+ (\alpha d + \lambda)m\left(\frac{d}{(\alpha d + \lambda)}(\sqrt{R_0} - 1)\right)
\]

\[
= -(\alpha d + \lambda)(\alpha + \mu) + d^2 \sqrt{R_0} + md\sqrt{R_0} + d^2(\sqrt{R_0} - 1) + md(\sqrt{R_0} - 1)
\]

\[
= \frac{-(\alpha d + \lambda)(\alpha + \mu) + d^2 R_0 + mdR_0}{\sqrt{R_0}} = 0
\]

Furthermore, (***i) implies that

\[
\text{Trace}(J_0) = -2d - (\alpha d + \lambda)I^* + (\alpha d + \lambda)S^* - m
\]
\[
-2d - (\alpha d + \lambda) \left( \frac{d}{(\alpha d + \lambda) \sqrt{R_0} - 1} \right) + (\alpha d + \lambda) \left( \frac{a}{d \sqrt{R_0}} \right) - m \\
= -2d - d \sqrt{R_0} - 1 + (\alpha d + \lambda) a - m \\
= - \frac{d^2 \sqrt{R_0} + d^2 R_0 + m d R_0 + m d R_0}{\sqrt{R_0}} = 0
\]

Thus, (***) imply that the Jacobian matrix has a zero eigenvalue with multiplicity 2. This suggests that (10.3) may admit a Bogdanov–Takens bifurcation.

### 10.3 Global Analysis

The objective of this section is to study the global structure of (10.3). Suppose that (*) holds in this section, the system (10.3) does not have a limit cycle and it is easy to classify its dynamical behavior. If \(E_2\) is unstable, any positive semi-orbit except the two equilibriums and the stable manifolds of \(E_1\) intersects the positive \(S\)-axis in finite time. If \(E_2\) is stable, there is a region whose boundary includes the two stable manifolds of \(E_1\) such that any positive semi-orbit inside this region tends to \(E_2\) as \(t\) tends to infinity and any positive semi-orbit outside this region meets the positive \(S\)-axis in a finite time. A typical phase portrait is shown in Fig. 1. If \(E_2\) is stable, there is a region whose boundary includes the two stable manifolds of \(E_1\) such that any positive semi-orbit inside this region

![Figure 10.1](image-url)
Extinction of the disease, where

\[ a = 2.5, \mu = 1.5, \alpha = 1, d = 0.1, m = 0.8, \lambda = 0.2, r = 0.8 \]

tends to \( E_2 \) as \( t \) tends to infinity and any positive semi-orbit outside this region meets the positive \( S \)-axis in a finite time. A typical phase portrait is shown in Figure 10.2.

![Figure 10.2](image)

Persistence of the disease where

\[ a = 2.5, \mu = 1.5, \alpha = 1, d = 0.1, m = 0.8, \lambda = 0.2, r = 0.6 \]

When (10.3) admits a limit cycle, more complicated dynamical behavior will occur, as it is suggested by the Bogdanov–Takens bifurcation in Section 10.2.4.

Let \( x = S - S_2, \ y = I - I_2. \)

Then (10.3) becomes

\[
\begin{align*}
\frac{dx}{dt} &= -(d + \alpha d + \lambda)I_2 x - (\alpha d + \lambda)S_2 y - (\alpha d + \lambda)xy, \\
\frac{dy}{dt} &= (\alpha d + \lambda)xy + (\alpha d + \lambda)I_2 x + [(\alpha d + \lambda)S_2 - d - m]y.
\end{align*}
\]

Set \( X = -dx - (d + m)y, \ Y = x + y \) and rewrite \( X, Y \) as \( x, y \) respectively (10.25) becomes
\[
\frac{dx}{dt} = [-2d - m + (\alpha d + \lambda)(S_2 - I_2)]x + [-dm - d^2 + (\alpha d + \lambda)(dS_2 - I_2m - dI_2y)]
\]
\[
+ \frac{(\alpha d + \lambda)}{m} x^2 + \frac{(\alpha d + \lambda)(2d + m)}{m} xy + \frac{(\alpha d + \lambda)d(d + m)}{m} y^2
\]
\[
\frac{dy}{dt} = x
\]

(10.26)

If \( \Delta = (R_0 - 1 - H)^2 - 4H \) it is easy to see that

\[
[-dm - d^2 + (dS_2 - I_2m - dI_2)(\alpha d + \lambda)] = -\det(J_2)
\]
\[
= -d(d + m)\sqrt{\Delta} < 0
\]

\[-2d - m + (S_2 - I_2)(\alpha d + \lambda)] = \text{Trace}(J_2).
\]

Make the change of the variables

\[
X = x, \quad Y = d(d + y)^{1/2} \Delta^{1/4} y, \quad \theta = d(d + y)^{1/2} \Delta^{1/4} t
\]

and rewrite \( X, Y \) and \( \theta \) as \( x, y \) and \( t \), respectively. System (10.26) becomes

\[
\frac{dx}{dt} = -y + [d(d + m)]^{-1/2} x + \frac{(\alpha d + \lambda)}{m} [d(d + m)]^{-1/2} \Delta^{-1/4}
\]
\[
+ \frac{(\alpha d + \lambda)(2d + m)}{dm(d + m)} \Delta^{-1/2} + (\alpha d + \lambda)[d(d + m)]^{-1/2} \Delta^{-3/4} / m
\]
\[
\frac{dy}{dt} = x
\]
\[
\frac{dx}{dt} = -y + \delta x + lx^2 + mxy + ny^2
\]
\[
\frac{dy}{dt} = x
\]

(10.27)

where

\[
\delta = [d(d + m)]^{1/2} \Delta^{-1/4} \text{Trace}(J_2), \quad I = \frac{\alpha d + \lambda}{m} [d(d + m)] \Delta^{-1/4}
\]
\[
m = \frac{(\alpha d + \lambda)(2d + m)}{dm(d + m)} \Delta^{-1/2}, \quad n = \frac{(\alpha d + \lambda)[d(d + m)]^{-1/2} \Delta^{-3/4}}{m}
\]

Now the following theorem is required.
Theorem: 10.3.1

Let (*) hold. Then there is no limit cycle in (10.3) if one of the following holds:

i) \( E_2 \) is unstable if

\[
(\alpha d + \lambda)r < \frac{(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3}{m}
\]

holds.

ii) \( E_2 \) is stable if either

\[
(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3 \leq (\alpha d + \lambda)r
\]

(or)

\[
(\alpha d + \lambda)r < \frac{(\alpha d + \lambda)(a + \mu) - 3d^2 - dm - 2d^3}{m}
\]

is valid

Proof

It suffices to prove that the system (10.27) does not have a limit cycle. Suppose that the assumption (i) is valid.

By the discussions in the proof of Theorem 10.3.1, then \( \text{Trace}(J_2) > 0 \). Hence \( \delta > 0 \), and therefore \( \delta m(1 + n) > 0 \). It follows from that there is no limit cycle in (10.27).

Suppose (ii) holds, now transform (10.27) by \( X = mx, \ Y = my \) to obtain

\[
\begin{align*}
\frac{dX}{dt} &= -Y + \delta X + \frac{1}{m} X^2 + XY + \frac{n}{m} Y^2 \\
\frac{dY}{dt} &= X
\end{align*}
\]  

(10.29)

Following the proof of Theorem 10.3.1, then the assumption (ii) implies that \( \text{Trace}(J_2) < 0 \). As a consequence that \( \delta < 0 \). Since \( \frac{1}{m} > 0 \) and \( \frac{n}{m} > 0 \), it follows from that there is no limit cycle in (10.2) if

\[
\delta + \frac{m}{2n} \leq 0.
\]

(10.30)

By (10.7), then (10.30) is equivalent to
Trace \( (J_2) \leq -\frac{2d + m}{2} \lambda^{1/2} \) \hspace{1cm} (10.31)

By (10.12) and the definition of \( I_2 \), then (10.31) is equivalent to

\[
m(R_0 - 1 - H) - 2d \leq 0 \] \hspace{1cm} (10.32)

It is easy to verify that (10.32) is equivalent to (10.6) consequently, there is no limit cycle in (10.3). If (10.11) is valid, since \( E_2 \) is unstable and there is no limit cycle, any orbit except the two endemic equilibria and the stable manifolds of \( E_1 \) meets the positive \( S \)-axis in finite time, i.e., the disease becomes extinct in finite time. If the assumption (ii) of Theorem 10.3.1 holds, then there is no limit cycle in (10.3) and \( E_2 \) is stable. By Theorem 10.3.1, the significant change of dynamical behavior of (10.3) can only occur in the case where (10.10) holds. Since a homo-clinic orbit is important in determining the asymptotic behavior of (10.3), we now present a different way to show the existence of a homo-clinic orbit in (10.3) where the homo-clinic orbit may not be in a small neighborhood of a degenerate equilibrium.

Choose \((a_0, m_0, \mu_0, d_0, \lambda_0, r_0)\) such that (10.21) holds

when \( a = a_0, \ d = d_0, \ \lambda = \lambda_0, \ m = m_0, \ r = r_0, \ \mu = \mu_0, \ \alpha = \alpha_0 \) and

\[
h_0 = \frac{1}{2} \left[ 2(a_0d_0 + \lambda_0)(a_0 + \mu_0) + (2d_0 + m_0)(d_0 + m_0) \left( 1 - \frac{4(a_0d_0 + \lambda_0)(a_0 + \mu_0)}{(m_0 + d_0)^2} \right) \right] \] \hspace{1cm} (10.33)

Fix \( d = d_0, \ \lambda = \lambda_0, \ m = m_0, \ \alpha = \alpha_0 \) and set

\[
\Delta_0 = \left( \frac{(a_0d_0 + \lambda_0)(a_0 + \mu_0)}{d_0(d_0 + m_0)} - 1 - \frac{(a_0d_0 + \lambda_0)r_0}{d_0(d_0 + m_0)} \right)^2 - 4 \frac{(a_0d_0 + \lambda_0)r_0}{d_0(d_0 + m_0)}.\]

Vary \( r \) and \( a + \mu \) by

\[
r = r_0 - \theta,\]

\[
\left( \frac{(a_0d_0 + \lambda_0)(a + \mu)}{d_0(d_0 + m_0)} \right)^2 = \sqrt{\Delta_0 + \left( \frac{1 + (a_0d_0 + \lambda_0)}{d_0(d_0 + m_0)} \right)^2} + \sqrt{4 \frac{(a_0d_0 + \lambda_0)m_0}{d_0(d_0 + m_0)}} \]

\[
\frac{(a_0d_0 + \lambda_0)(a + \mu)}{d_0(d_0 + m_0)} = 1 + \frac{(a_0d_0 + \lambda_0)r}{d_0(d_0 + m_0)} + \sqrt{\Delta_0 + 4 \frac{(a_0d_0 + \lambda_0)r}{d_0(d_0 + m_0)}} \] \hspace{1cm} (10.34)
We can see that $\Delta$ is invariant as $\theta$ varies. As a consequence, by (10.28) we can see that $l, m, n$ are invariant as $\theta$ varies. Furthermore, by (10.12) and the definition of $I_2$, we have

\[
\text{Trace}(J_2) = \frac{m_0}{2} (R_0 - 1 - H) - d_0 - \frac{2d_0 + m_0}{2} \sqrt{\Delta} \\
= \frac{m_0}{2} \sqrt{\Delta_0 + 4 \left( \frac{\alpha_0 d_0 + \lambda_0}{d_0 (d_0 + m_0)} \right) r} - d_0 - \frac{2d_0 + m_0}{2} \sqrt{\Delta_0}.
\]

It follows that Trace $(J_2)$ is decreasing, and therefore, $\delta$ is decreasing, as $\theta$ increases. Now, it is easy to check that (10.27) is a rotated vector field with respect to parameter $\theta$. If we increase $\theta$ from 0, it follows from Theorem 10.3.2 that an unstable limit cycle is produced due to Hopf bifurcation and this limit cycle expands as $\theta$ increases. Moreover, (H1) holds as $\theta$ varies because we have (10.34). When $\theta$ is increased to $r_0$, the equilibrium $E_1$ of (10.3) becomes $(a + \mu/d_0, 0)$ (a disease-free equilibrium) and the equilibrium $E_2$ of (10.3) becomes

\[
(I_2, S_2) = \left( \frac{d_0}{(\alpha_0 d_0 + \lambda_0) \sqrt{\Delta_0}}, \frac{a + \mu}{d_0 (1 + \sqrt{\Delta_0})} \right)
\]

Since $E_2$ is globally stable in model (10.3) when $r = 0$, it follows that the trivial equilibrium $(0, 0)$ of (10.27) is globally stable at this time. Hence, the unstable limit cycle must meet a homo-clinic orbit before $\theta = r_0$.

This means that there exists a homo-clinic orbit in the following system

\[
\frac{dS}{dt} = a + \mu - dS - \frac{\lambda SI}{1 + \alpha I} \\
\frac{dI}{dt} = \frac{\lambda SI}{1 + \alpha I} - (d + m)I - r
\]

which is considered in the $R_2$ plane. By the form of the above system, a homo-clinic orbit starting from the interior of $R^+_2$ cannot meet the nonnegative $S$-axis and the positive $I$-axis. This shows that the homo-clinic orbit starting from the interior of $R^+_2$ must lie in the interior of $R^+_2$. Therefore, we can state the following result.
10.4 NUMERICAL SIMULATION AND DISCUSSIONS

In this section, to understand our results more intuitively some numerical simulations are carried out. To exhibited the bifurcation of a system (10.3) such as the stability of the endemic equilibrium, the existence of the periodic orbit and the bi-stable equilibrium respectively.

Case (i)

\[
R_0 = \frac{(\alpha d + \lambda)(a+ \mu)}{d(d + m)}
\]

\[\lambda = 0.1, a = 2, d = 1, m = 0.02, \mu = 0.2, \alpha = 0.3\]

\[R_0 = \frac{(0.4)(2 + 0.2)}{(1 + 0.02)} = 0.86 < 1\]

This holds (10.5).

Case (ii)

\[\lambda = 2, a = 4, d = 2, m = 0.1, \mu = 0.12, \alpha = 2\]

\[R_0 = \frac{6(4 + 0.12)}{2(2 + 0.1)} = 6 > 1\]

\[R_0 < 1\] does not have a positive equilibrium. \[R_0 < 1\] this holds (10.6).

In this chapter, an epidemic model with a constant removal rate of the infective individuals is to understand the effect of the treatment capacity on the disease transmission. If the parameters satisfy (10.11), Theorems 10.2.1 and 10.3.1 imply that the disease becomes extinct in a finite time because the endemic equilibrium \(E_2\) is unstable and there is no limit cycle in (10.3). It is carried out a global qualitative analysis of the model. The result on the nonexistence of a limit cycle in (10.3) gives us the global structure of the model and indicates that complicated behavior of the model can only occur when (10.10) holds. Theorem 10.4.2 presents the existence of a homo-clinic orbit in (10.3) in a large range of parameters.
10.5 CONCLUSION

An epidemic model with a constant removal rate of infective individuals is proposed to understand the effect of limited resources for treatment of infectives on the disease spread. And also proceed that an endemic model with a constant removal rate of the infective of individuals to understand the effect of the treatment capacity on the disease transmission with immigration. It is found that it is unnecessary to take such a large treatment capacity that endemic equilibrium disappear to eradicate the disease.

It is shown that the outcome of disease spread may depend on the position of the initial states for certain range of parameters. If the parameters satisfy (10.9), there is a region such that the number of infectives tends to $I_2$ if the initial position lies in the region and the disease dies out if the initial position lies outside the region. If the parameters satisfy (10.10), the disease is persistent if the initial position lies in the region and the disease becomes extinct if the initial position lies outside this region. Since the eventual behavior is related to the initial positions, this model may be more realistic and useful.