CHAPTER 8
QUALITATIVE ANALYSIS OF SIR EPIDEMIC MODEL WITH IMMIGRATION UNDER SATURATED TREATMENT RATE

In classical epidemic models, the treatment rate of the infectives is assumed to be proportional to the number of the infectives. This is based on the assumption of sufficient resource available for treatment when the number of the infectives is large. Nevertheless, this involves the best strategy or optimal available resource for treatment for every community. A community wastes resource for treatment if the resource is prepared for too large, but may have a risk of an outbreak of a disease if the available resource is too small. Thus, it is important to determine optimal resource supplies, or capacity, for the treatment of a disease. Suppose that the capacity for the treatment of a disease in a community is a constant and the recovery rate coefficient due to the treatment is \( r \). Further assume that the treatment rate is proportional to the number of the infectives when the capacity of the treatment has not been reached, and saturates to a constant when the number of the infectives is so large that the capacity of the treatment is exceeded. With these assumptions, the SIR model with immigration is constructed in this chapter. Qualitative analysis of this model is carried out under saturated treatment rate.

8.1 Introduction

Incidence rate plays an important role in the modeling of epidemic dynamics. It has been suggested by several authors [32, 48, 50, 52] that the disease transmission process may have a nonlinear incidence rate. In many epidemic models, the bilinear incidence rate \( \beta \) and the standard incidence rate \( \beta^* \) are frequently used. The bilinear incidence rate is based on the law of mass action. It has been pointed out that for standard incidence rate, it may be a good approximation if the number of available partners is large enough and everybody could not make more contacts than is practically feasible. In fact, the infection probability per contact is likely influenced by the number of infective individuals, because more infective
individuals can increase the infection risk. Wang and Ruan [57] supposed a removal rate has the form

\[ h(I) = \begin{cases} r, & I > 0, \\ 0, & I = 0. \end{cases} \]

Then they discuss the stability of equilibria and prove the model exhibits Bogdanov-Takens bifurcation, Hopf bifurcation and Homo-clinic bifurcation. However, recent studies [26, 51, 53, 56, 62] have showed that some epidemiological models have multiple endemic equilibria, and which can exist simultaneously even \( R_0 \) is less than the unity. Furthermore, the disease free equilibrium and one of the endemic equilibria can be stable simultaneity, i.e. there exist bi-stable equilibria. Therefore, it is not always effective to eliminate the disease by reducing \( R_0 \) to the values less than 1. In fact, to eradicate this kind of disease, one not only needs to reduce \( R_0 \) but also needs to restrict the initial value of each subpopulation to the domain of attraction of the disease free equilibrium. Liu et al. [47, 48] proposed the incidence rate \( \beta \frac{p}{q} \) and discovered that an SIR model can yield rich dynamics such as bi-stable equilibria, saddle-node bifurcation and Hopf bifurcation, etc. Capasso and Serio [11] introduced a saturated incidence rate \( (\cdot) \) into epidemic models, where \( (\cdot) \) tends to a saturation level when \( I \) gets large, i.e., \( (\cdot) = \frac{\beta}{1 + \alpha} \), where \( \beta \) measures the infection force of the disease and \( \frac{1}{1 + \alpha} \) measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. This incidence rate seems more reasonable than the bilinear incidence rate \( \beta \cdot \cdot \), because it includes the behavioral change and crowding the effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters. Ankit Agrawal and Saxena [4] introduced a generalized saturated incidence rate \( \frac{\lambda I}{\rho + \beta I} \). Zhan and Suo [73] considered the qualitative analysis of SIR epidemic model with saturated treatment rate. He
considered the removal rate of the infected individuals as \( h(\cdot) = \frac{\beta}{1 + \alpha} \). The recovery from the infected sub-population per unit time, have been adopted to characterize different treatment rates of epidemic disease. When the susceptible rate increases due to immigration, the hospitals have to increase the treatment rate. Due to lack of treatment condition, this chapter designed an SIR epidemic model with immigration.

The rest of this chapter is organized as follows. In the next section 8.2, the threshold conditions of disease-free equilibrium and its stability is discussed. In the same section, description of the model and existence of equilibria is given. In section 8.3, the stability of equilibrium is analyzed by qualitatively. Numerical simulation and discussions are explained with suitable diagrams in section 8.4. Section 8.5 gives the conclusion.

**8.2 MODEL DESCRIPTION AND ANALYSIS**

**8.2.1 Model Description**

In order to write down a mathematical formulation for the dynamics of the epidemic process, it is to introduce differential equations for the rates of transfer from one compartment to another. The following model is framed with general incidence rate \( \lambda \).

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \lambda SI + \beta R \\
\frac{dI}{dt} &= \lambda SI - (d + m)I \\
\frac{dR}{dt} &= mI - (d + \beta)R
\end{align*}
\]

Xiao and Ruan [65] used a non-monotonic incidence rate \( \frac{\lambda SI}{1 + \alpha I^2} \) instead of \( \lambda \). Then the above model becomes

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \frac{\lambda SI}{1 + \alpha I^2} + \beta R \\
\frac{dI}{dt} &= \frac{\lambda SI}{1 + \alpha I^2} - (d + m)I
\end{align*}
\]
Wendi and Wang [61] incorporated the linear treatment function into an SIR model:
\[
\frac{dR}{dt} = mI - (d + \beta)R
\]

Kar and Batabyal [41] used the above treatment function in Xiao and Ruan [65] SIR model, then the model convert into
\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \frac{\lambda SI}{1 + \alpha I^2} + \beta R \\
\frac{dI}{dt} &= \frac{\lambda SI}{1 + \alpha I^2} - (d + m)I - T(I) \\
\frac{dR}{dt} &= mI - (d + \beta)R + T(I)
\end{align*}
\]

Zhan and Suo [73] proposed the qualitative analysis of SIR epidemic model under saturated treatment rate. The model becomes
\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \mu \\
\frac{dI}{dt} &= \lambda SI -(d + m)I - h(I) \\
\frac{dR}{dt} &= mI - dR + h(I)
\end{align*}
\]

It is the removal or recovered rate of the infectives. In this chapter, the qualitative analysis of SIR epidemic model with immigration is considered by using saturated treatment rate. This new system is discussed with subsequent assumptions and notations.

8.2.2 Assumptions

- Consider Zhan and Suo [73] SIR epidemic model.
- Assume that the population consists of three types of individuals. They are susceptible, infective and recovered individuals.
- Let \( \lambda \) the bilinear incidence rate
• Consider the removal rate of the infected individual is
\[ \lambda = \frac{\beta}{1 + \alpha} \]

• Consider the susceptible rate with immigration

• Let immigration be constant.

### 8.2.3 Notations

- **S**: Number of susceptibles
- **I**: Number of infectives
- **R**: Number of removed or recovered individuals.
- **a**: Recruitment rate of the population
- **d**: Natural death rate of the population
- **\( \lambda \)**: The proportionality constant
- **m**: Natural recovery rate of the infective individuals
- **\( \beta \)**: The rate at which recovered individuals lose immunity and return to susceptible class
- **\( \mu \)**: Increase of susceptible at a constant rate
- **\( \alpha \)**: The parameter measures of the psychological or inhibitory effect
- **\( \mathcal{R}_0 \)**: Disease-free equilibrium
- **\( \mathcal{R}_{12} \)**: Endemic equilibria
- **\( \mathcal{R}_0 \)**: Basic reproduction number
- **\( \mathcal{R}_1 \)**: Modified reproduction number
- **\( \mathcal{R}^* \)**: Susceptible rate at the endemic equilibrium
- **\( \mathcal{I}^* \)**: Infected rate at the endemic equilibrium
8.2.4 The Mathematical Model

SIR model takes the form

\[
\begin{align*}
\frac{dS}{dt} &= a + \mu - \lambda SI \\
\frac{dI}{dt} &= \lambda SI - (d + m)I - \frac{\beta I}{1 + \alpha I} \\
\frac{dR}{dt} &= mI + \frac{\beta I}{1 + \alpha I} - dR
\end{align*}
\]  

(8.1)

For the new outbreak disease such as SARS, Tuberculosis etc., maybe saturated treatment rate is a better alternative. In fact, the treatment rate is small for there is short of effective treatment techniques at the beginning of the outbreak. Then, the treatment rate will be increased with the improving of the hospital’s treatment conditions including effective medicines, skillful techniques etc., At last, for the treatment capacity of any community is limited, the treatment will reach to its maximum if the number of infectives individuals is large enough. In view of the description the removal rate to be

\[
T(t) = \beta \frac{1}{1 + \alpha} \geq 0, \quad \alpha, \beta > 0
\]  

(8.2)

The epidemicity of disease is closely related to the stability of the equilibria of mathematical models. In classical models there usually exist a disease-free equilibrium and a unique endemic equilibrium. The stability of the disease free equilibrium is determined by a threshold parameter \( \mathcal{R}_0 \), known as the basic reproductive number, i.e. the disease free equilibrium is asymptotically stable if \( \mathcal{R}_0 < 1 \), while it is unstable and the unique endemic equilibrium is asymptotically stable if \( \mathcal{R}_0 > 1 \). Now the proposed model is

\[
\begin{align*}
\frac{dS}{dt} &= a + \mu - dS - \lambda SI \\
\frac{dI}{dt} &= \lambda SI - (d + m)I - \frac{\beta I}{1 + \alpha I} \\
\frac{dR}{dt} &= mI + \frac{\beta I}{1 + \alpha I} - dR
\end{align*}
\]  

(8.3)
It is easy to see that the treatment rate \( T(\ ) = \frac{\beta}{1 + \alpha} \) is a continuously differentiable function of \( I \) and \( T(0) = 0, \lim_{I \to \infty} T(\ ) = \frac{\beta}{\alpha} \), where \( \frac{\beta}{\alpha} \) is the maximal treatment capacity of some community.

Noticing the system (8.3) is independent of the others. So mainly investigate the dynamic properties of \( S \) and \( I \) in the present section. Therefore, only the following subsystem is focused on.

\[
\begin{align*}
\frac{dS}{dt} &= a + \mu - dS - \lambda SI \\
\frac{dI}{dt} &= \lambda SI - (d + m)I - \frac{\beta I}{1 + \alpha I}
\end{align*}
\]

(8.4)

From the system (8.4), the lemma follows.

**Lemma: 8.2.1**

If \( \frac{3}{\alpha(\ + \mu)} \), system (8.4) does not have closed orbit in the first quadrant of \((S, I)\) plane.

**Proof**

Let \( a = \frac{1 + \alpha}{\alpha} \),

\[
P = a + \mu - dS - \lambda SI
\]

and \( Q = \lambda SI - (d + m)I - \frac{\beta I}{1 + \alpha I} \)

This implies that

\[
\frac{\partial}{\partial \alpha} = \frac{(1 + \alpha)(-\mu)}{(-\alpha)^2}
\]

\[
\frac{\partial}{\partial \alpha} = \frac{\lambda \alpha 2 - \alpha 2 - \alpha 2}{(\alpha)^2}
\]

\[
\frac{\partial DP}{\partial S} + \frac{\partial DQ}{\partial I} = \frac{-a - \mu - 3\alpha I - \mu a I + \lambda a S I - n a S I}{S^2 I}
\]

where
After computation,
\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial x} = \frac{\lambda \alpha}{\beta + \mu} - \frac{\alpha - \mu}{2}
\]

Then, by Bendixson-Dulac criterion, system (8.4) does not have closed orbit in the first quadrant of \((S, I)\) plane if \(\lambda < \frac{3}{(\beta + \mu)^2}\).

In the following, the existence of equilibria is discussed. For convenience, now introduce two notations
\[
0 = \frac{(\beta + \mu) \lambda}{(\beta + \mu)}
\]
\[
1 = \frac{\lambda + \lambda \mu - \alpha}{\alpha + \lambda(\beta + \mu)}
\]
where \(0 = +\) and \(0\) is called the basic reproduction number.

Let the right hand of system (8.4) be zero. It isn’t difficult to obtain that system (8.4) always has a disease free equilibrium \(E_0 = \left(\frac{a + \mu}{d}, 0\right)\).

Next to find \(I\), take
\[
a + \mu - dS - \lambda SI = 0 \quad \text{(a)}
\]
\[
\lambda SI - (d + m)I - \frac{\beta I}{1 + \alpha I} = 0 \quad \text{(b)}
\]

From equation (a)
\[
S = \frac{a + \mu}{(d + m)} I \quad \text{and} \quad I = \frac{a + \mu - dS}{\lambda S}
\]

Using this result in (b)
\[
\lambda - \frac{(\beta I)}{1 + \alpha I} = 0
\]
Chapter 8
Qualitative Analysis of SIR Epidemic Model with Immigration

Under the condition $I \neq 0$, then the theorem follows.

**Theorem: 8.2.2**

i) If $0 > 1$, there exists a unique endemic equilibrium $E_1 = (x_1^*, y_1^*)$.

ii) If $0 = 1$ and $1 > 1$, there exists a unique endemic equilibrium $E_2 = (S_2^*, I_2^*)$.

iii) If $0 = 1$ and $1 \leq 1$, no endemic equilibrium exists.

iv) If $R_0 < 1$, $R_1 > 1$ and $\Delta > 0$, there exists two endemic equilibria: $E_3 = (x_3^*, y_3^*)$.

v) If $R_0 < 1$, $R_1 > 1$ and $\Delta = 0$, there exists a unique endemic equilibrium $E_4 = (x_4^*, y_4^*)$.

vi) If $0 < 1$, $1 > 1$ and $\Delta < 0$, no endemic equilibrium exists.

vii) If $0 < 1$ and $1 \leq 1$, no endemic equilibrium exists, $I_1^* = \frac{\lambda \alpha - nd \alpha - (\beta + n)\lambda + \lambda \mu \alpha + \sqrt{\Delta}}{2n \lambda \alpha}$, $I_2^* = \frac{\lambda \alpha - nd \alpha - (\beta + n)\lambda + \lambda \mu \alpha}{n \lambda \alpha}$, $I_3^* = \frac{\lambda \alpha - nd \alpha - (\beta + n)\lambda + \lambda \mu \alpha - \sqrt{\Delta}}{2n \lambda \alpha}$. 

\[
\begin{align*}
\lambda \left( \frac{+ \mu}{+ \lambda} \right) - (\gamma + \mu)(1 + \alpha) & = 0 \\
-\lambda \alpha^3 & + (\lambda + \lambda \mu \alpha - \alpha) \alpha + 1 = 0 \\
-\lambda \alpha^2 & + (\lambda + \lambda \mu \alpha - \alpha) \alpha + 1 = 0 \\
& + 4(\lambda + \lambda \mu \alpha - \alpha) \alpha \\
= & \frac{-(\lambda + \lambda \mu \alpha - \alpha) \alpha + \sqrt{\Delta}}{-2\lambda \alpha}
\end{align*}
\]
8.3 The Qualitative Analysis

To study the stability of the equilibria of mathematical models, first investigate the stability of the disease free equilibrium $0$. The Jacobian matrix of system (8.4) at $0$ is

$$
M_{E_0} = \begin{pmatrix}
-d & -\frac{\lambda(a + \mu)}{d} \\
0 & (\beta + n)(R_0 - 1)
\end{pmatrix}
$$

Remark:

The center manifold theorem is an extension of Lyapunov's first method to evaluate stability of systems with eigen values that have zero real part.

**Theorem: 8.3.1**

*If $0 > 1$, the disease free equilibrium $0$ is unstable, while if $0 < 1$ the disease free equilibrium $0$ is asymptotically stable.*

The matrix $M_{E_0}$ has two real eigen values: $0$, $-\lambda$ when $0 = 1$, i.e., $0$ a non-hyperbolic equilibrium. The center manifold theorem is used to study the stability of the equilibrium.

Let $y_1 = S - \frac{a + \mu}{d}$, $2 = \lambda$, then

$$
-1 = -\frac{\lambda(a + \mu)}{d} - \lambda (1 + \alpha)\frac{2}{2 + \frac{\alpha\beta}{1 + \alpha}}
$$

$$
-2 = \lambda - \frac{\alpha\beta}{1 + \alpha} (\frac{2}{2 + \frac{\alpha\beta}{1 + \alpha}})
$$

(8.5)
By the following transformation:

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} =
\begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

System (8.5) can be transformed into the standard form

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

By the center manifold theory, there exists a center manifold for system (8.6), which can be expressed locally by

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

where \( \delta \) is sufficiently small and \( DH \) is derivative of \( H \) with respect to \( \begin{pmatrix}
  1 \\
  2
\end{pmatrix} \).

To compute the center manifold \( c(0) \), assume \( \begin{pmatrix}
  1 \\
  2
\end{pmatrix} \) has the form

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

By the invariance of \( W^c(0) \) under the dynamics of (8.6) yields

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

Substituting (8.6) into (8.8), then equating coefficients on each power of \( \begin{pmatrix}
  1 \\
  2
\end{pmatrix} \) to zero yields

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 \\
  2
\end{pmatrix} = \begin{pmatrix}
  1 & -1 \\
  0 & \beta +
\end{pmatrix}
\begin{pmatrix}
  1 \\
  2
\end{pmatrix}
\]
\[3 = \frac{\lambda^3}{2}(-2\alpha\beta + 2\lambda + \lambda^2)\]

\[4 = \frac{-4\lambda}{3}((7 + 1 + 6^2)\lambda^2 - \alpha\beta(5 + 12) + 2\lambda^2\rho(3\rho - \rho + 3\rho - \rho))\]

\[5 = \frac{\lambda^5}{4}((24^3 + 44^2 + 16^2 + 1)\lambda^2 - \alpha\beta(720 + 2\lambda^4 - \lambda^2\rho(5^2 - 9\rho^2\rho + 3\rho^2))\]

\[h_o = \frac{2\lambda^6}{d^5}(k\alpha\beta(14 + 237\lambda + 480\lambda^2 + 738k^2)\lambda^3 - (178k^2 + 30k + 1 + 292k^3 + 120k^4)\lambda^4\]

\[- \alpha^2\beta(720^3\beta + 600^2\beta - 94^2 + 71\beta - 79^2 + 9\lambda^2 + 3\rho^2(5^2 - 9\rho^2\rho^2)\]

\[-188^2\beta + 16^2 - 59\beta^2 + 2\rho^2(5\rho - \rho^2)(3\rho - \rho)(4\rho - \rho)\] (8.9)

where \(\beta = \frac{\beta^+}{\beta^-}\).

Then an approximation to \(H\) as follows

\[= \frac{\lambda^2}{2} + \frac{\lambda^3}{2}(-2\alpha\beta + 2\lambda + \lambda^2)\]

\[+ 2\alpha^2\beta(3\beta - 2\lambda)\]

\[+ 2\alpha^2\beta(72\beta - 14 + 26\beta - 5\beta) = 2\alpha^2\beta(3\beta - \rho)(4\beta - \rho)\]

\[+ \frac{\lambda^6}{5}(\alpha\beta(14 + 237\lambda + 480\lambda^2 + 738k^2)\lambda^3 - (178k^2 + 30k + 1 + 292k^3 + 120k^4)\lambda^4\]

\[- \alpha^2\beta(720^3\beta + 600^2\beta - 94^2 + 71\beta - 79^2 + 9\lambda^2 + 3\rho^2(5^2 - 9\rho^2\rho^2)\]

\[+ \alpha^3\beta(5^2 + 480^3\beta^2 - 188^2\beta + 16^2 - 59\beta^2)\]

\[-2\alpha^4\beta(5\beta - \beta)(3\beta - \rho)(4\beta - \rho)\] (8.10)

Substituting (8.7) into the second equation of system (8.8) leads to the vector field reduced to the center manifold

\[z = 2(\lambda - \alpha\beta)\lambda^3 + 3\alpha^2\beta - 2\lambda^2\rho^2(\rho + 3\rho^2)\]

\[+ 2(\lambda - \alpha\beta)\lambda^3 + 3\alpha^2\beta - 2\lambda^2\rho^2(\rho + 3\rho^2)\]

\[+ 2(\lambda - \alpha\beta)\lambda^3 + 3\alpha^2\beta - 2\lambda^2\rho^2(\rho + 3\rho^2)\]
Then, by the center manifold theorem, the following result shows about the non-hyperbolic equilibrium $0$. Next, to study the stability of the endemic equilibrium $2$ under the conditions:

$$0 = 1, \ R_1 > 1$$

(8.12)

Clearly, (8.12) leads to

$$\lambda < \alpha, \quad + \mu > \frac{\lambda}{\lambda} \cdot \beta$$

(8.13)

The Jacobian matrix of system (8.4) at $2$ has the form

$$\mathbf{J}_2 = \begin{pmatrix}
-1 & -\lambda^* & -\lambda^* \\
\lambda^* & -\alpha - \beta & \alpha \\
\lambda^* & \alpha & \alpha^2
\end{pmatrix}
$$

It is easy to obtain the characteristic equation of $M_{\mathbf{J}_2}$ is

$$\theta^2 + \frac{\theta}{\theta} = 0$$

(8.14)

where $\theta$ is a complex number and

$$a_1 = \alpha n d (\lambda (a + \mu) - n d) (\lambda - \alpha d)^3$$

$$2 = 1 (\alpha + \mu)^2 + 2 (\alpha + \mu)$$

$$3 = (\alpha - \lambda) (-\alpha - \lambda (a + \mu) + \alpha^2 + \lambda^2 (\alpha + \mu))^2$$

$$1 = -\lambda^2 (\lambda - \alpha)^3$$

$$2 = \lambda (\lambda - \alpha) (\lambda - \alpha)^2 + 2 \alpha$$
By using (8.12) & (8.13), implies

\[ a_1 > 0, \quad a_3 > 0, \quad b_1 > 0, \quad b_2 > 0. \]

**Theorem: 8.3.2**

If \( K < 0 \), the periodic solution is stable, while if \( K > 0 \), the periodic solution is unstable. The case \( K < 0 \), is referred to as a supercritical bifurcation, and the case \( K > 0 \) is referred to as a subcritical bifurcation.

In the following, the dynamic property of \( y_4 \) is focused on under the conditions

\[ 0 < 1, \quad 1 > 1, \quad \Delta = 0 \tag{8.15} \]

The Jacobian matrix of system (8.4) at \( y_4 \) is

\[
E_* = \begin{pmatrix}
-\lambda & \frac{-\lambda}{4} \\
\frac{\alpha \beta}{4} & \lambda
\end{pmatrix}
\]

By using the relationship

\[
\left( \begin{array}{c}
y_4 \\
\lambda \end{array} \right) \cdot \frac{1}{\lambda} \begin{pmatrix}
\alpha + \lambda \\
\beta - \lambda \alpha - \lambda \mu \alpha
\end{pmatrix} = \begin{pmatrix}
\alpha \\
\beta - \lambda \alpha - \lambda \mu \alpha
\end{pmatrix} \cdot \begin{pmatrix}
y_4 \\
\lambda \end{pmatrix}
\]

the characteristic equation of \( M_{E_*} \) is as follows

\[ \theta^2 + \theta + = 0, \tag{8.16} \]

where

\[
= \frac{1}{(1 + \alpha \lambda)^2} \left[ -\lambda (\alpha + \mu) + \frac{\alpha \beta}{\lambda} - \frac{\alpha}{\lambda} \right] - \lambda (\alpha + \mu)^2 + \frac{\alpha \beta}{\lambda} - \lambda (\alpha + \mu)
\]
\[ = \frac{\alpha}{1 + \alpha^2} \{ (\alpha - 1) \theta^2 + (\alpha^2 - 1) \theta + \alpha - 1 \} + \alpha[(\beta + \mu) - \lambda(\alpha + \mu)] + \lambda(\alpha - \alpha^2) \]  

(8.17)

and \( \theta \) is a complex number. Noticing \( \alpha > 1 \)

\[ + \mu > \frac{\alpha + (\beta + \mu)\lambda}{\alpha \lambda} \]

For convenience, let \( + \mu = \frac{\alpha + (\beta + \mu)\lambda}{\alpha \lambda} + \epsilon \), where \( \epsilon > 0 \). By means of \( \Delta = 0 \), then

\[ = \frac{(\alpha \epsilon + 2 \alpha^2)(\alpha \epsilon + 2(\beta + \mu)) - 2\epsilon \beta}{4 \lambda^2(1 + \alpha)}[\alpha^2 \epsilon + 4\beta^2 + 4 \epsilon^2 + 4 \rho \alpha \epsilon] \]

\[ q = \frac{2 \alpha \beta d I_4^* (\alpha - \beta)(\alpha \epsilon + 2 \alpha^2)}{(1 + \alpha I_4^*)^2[4 \beta n + (\alpha \epsilon + 2 \alpha^2)^2 + 4 n \beta]} \]

\[ \lambda = \frac{4 n d \alpha \beta}{\alpha^2 \epsilon^2 + 4 n \beta + 4 n \epsilon^2 + 4 n \epsilon \alpha} \]

\[ I_4^* = \frac{\epsilon}{2n} \]

Let \( d^* = \frac{2 \epsilon \alpha \beta n [\alpha \epsilon + 2 \alpha n^2 + 4 \beta n]}{(\alpha \epsilon + 2 \alpha n)^3[\alpha \epsilon + 2(\beta + n)]} \). Then, under the conditions of (8.15), the theorem follows.

**Theorem: 8.3.3**

i) If \( \alpha > \beta \) and \( > ^* \), \( 4 \) is an asymptotically unstable node or focus.

ii) If \( \alpha > \beta \) and \( < ^* \), \( 4 \) is an unstable node or focus.

iii) If \( \alpha = \beta \) or \( = ^* \), \( \alpha > \beta \), \( 4 \) is a non-hyperbolic equilibrium.

iv) If \( \alpha > \beta \), \( 4 \) is a saddle.

Clearly (8.16) has a negative root \( \_ \) except 0 if \( \alpha = \beta, d > d^* \).

(8.18)
In the sequel, discuss the stability of \( \dot{x} \) under the assumptions of (8.17). For the purpose, first transfer \( \dot{x} \) to the origin by

\[
\begin{align*}
\dot{x} &= -\dot{y} + \dot{z} = -\dot{y} + \dot{z} \\
\dot{y} &= \frac{[2(\beta + 2) + \beta \varepsilon](\beta + \varepsilon)}{(\beta \varepsilon + 2)^2 + 4 \beta \varepsilon} \dot{y} + \frac{[2(\beta + 2) + \beta \varepsilon]}{(\beta \varepsilon + 2)^2 + 4 \beta \varepsilon} \dot{z} \\
d\dot{z} &= 2d^2 \beta^2 \varepsilon (\beta \varepsilon + 2n)^2 + 4n \beta x_1 + \frac{2 \beta \varepsilon n}{(2n + \varepsilon \beta)^2} x_2 + f_2(x_1, x_2)
\end{align*}
\]

Therefore

\[
\begin{align*}
\dot{x} &= \frac{2d \beta^2 \varepsilon}{(\beta \varepsilon + 2n)^2 + 4n \beta} x_1 + \frac{2 \beta \varepsilon n}{(2n + \varepsilon \beta)^2} x_2 + f_2(x_1, x_2)
\end{align*}
\]

where

\[
\begin{align*}
\begin{pmatrix}
1 \\
2
\end{pmatrix}
&= \begin{pmatrix}
1 \\
2
\end{pmatrix}
= \begin{pmatrix}
1 & -2 \\
2 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix}
= \begin{pmatrix}
11 & 12 \\
21 & 22
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
11 & 12 \\
21 & 22
\end{pmatrix}
&= \begin{pmatrix}
-2d \varepsilon^2 \\
2 \beta \varepsilon^2
\end{pmatrix}
\end{align*}
\]

Secondly, we define

\[
\begin{align*}
\begin{pmatrix}
1 & 2 \\
1 & 2
\end{pmatrix}
&= \begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix}
\end{align*}
\]

where

\[
\begin{align*}
11 &= -[(\beta \varepsilon + 2)^2 + 4 \beta \varepsilon], \\
12 &= -[(\beta \varepsilon + 2)^2 + 2 \beta (\beta \varepsilon + 2)], \\
21 &= d(2n + \beta \varepsilon), \\
22 &= 2 \beta \varepsilon^2
\end{align*}
\]

It isn’t difficult to get the inverse of \( \begin{pmatrix}1 & 2 \\ 2 & 2\end{pmatrix} \) as follows

\[
\begin{pmatrix}
11 & 12 \\
21 & 22
\end{pmatrix}^{-1} = \begin{pmatrix}
11 & 12 \\
21 & 22
\end{pmatrix}
\]

where

\[
t_{11} = \frac{2 \varepsilon \beta^2}{\det(S)}
\]
\[ t_{21} = \frac{d(2n + \varepsilon \beta)^2}{\det(S)} \]
\[ 22 = \frac{-\left(4 + 4 \varepsilon \beta + \varepsilon^2 \beta^2 + 4 \beta\right)}{\det(S)} \]

Corresponding, system (8.18) becomes

\[
\begin{pmatrix}
1 \\
2
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
1 \\
2
\end{pmatrix} + \begin{pmatrix}
(\ldots) \\
(\ldots)
\end{pmatrix}
\]  

(8.20)

where

\[
1 = -1 \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix} + 3 \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix}^2 + \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix}^3 + \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix}^4
\]

\[
2 = -1 \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix} + 3 \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix}^2 + \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix}^3 + \begin{pmatrix}
11 & 12 \\
12 & 22
\end{pmatrix}^4
\]

By the existence theorem in the center manifold theory, there exists a center manifold for system, which can be expressed locally as follows

\[ c(0) = \{ (1', 2) \in \mathbb{R} | 2 = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \| 2 \| < \delta, -\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0, -\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 \} \]

where \( \delta \) is sufficiently small and \( D\bar{H} \) is the derivative of \( \bar{H} \) with respect to \( y_1 \).

Obviously, the first task is to compute the center manifold \( c(0) \).

For the purpose, let \( -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \) be

\[
2 = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\begin{pmatrix} 2 \\ 3 \end{pmatrix} + -\begin{pmatrix} 4 \\ 5 \end{pmatrix} + ... 
\]  

(8.21)
By the invariance of $W^c(0)$ under the dynamics of (8.22), the center manifold satisfies

$$1(1, \overline{\varphi}(1)) + \overline{\varphi}(1) - \overline{\varphi}(1) = 0$$

(8.22)

Balance equation (8.22) with respect to $\tau$ yields

$$2 = \frac{-21}{\det(\overline{\varphi})}(\frac{32}{22 - 21} - \frac{2}{21} + \frac{1}{21} + 1)$$

$$3 = \frac{-21}{2}(\frac{1}{21} + \frac{2}{22} + \frac{1}{21} + 1)$$

with

$$c_1 = -2t_{22}t_{11}(\overline{a}_1 - b_2)^2$$

$$c_2 = 2(\overline{a}_1 - b_2)^2s_{22}t_{22}^2 + t_{22}(b_3p - 2b_2a_1s_{11}t_{11} - \overline{a}_1^2s_{12}t_{12} + 2\overline{a}_1^2s_{11}t_{11}$$

$$+ 2t_{12}t_{21}) + \overline{\varphi}(2)$$

Substituting (8.21) into the first equation of system (8.22) leads to

$$1 = \frac{2}{21} + \frac{3}{2} + \frac{4}{1}$$

$$2 = \frac{-21}{\det(\overline{\varphi})}[(\frac{32}{21} - \frac{2}{22} + \frac{1}{22} + 1)$$

$$+ (\frac{4}{22} - \frac{1}{21} + \frac{2}{22} + \frac{1}{22} + 1)]$$

$$3 = \frac{-21}{2}[(\frac{32}{21} - \frac{2}{22} + \frac{1}{22} + 1)$$

$$+ (\frac{4}{22} - \frac{1}{21} + \frac{2}{22} + \frac{1}{22} + 1)]$$

$$+ 2t_{22}a_1^2s_{22} + 2t_{22}b_2^2s_{22} - 2t_{12}b_2t_{21}\overline{a}_1s_{11} + 2t_{12}\overline{a}_1^2t_{21}s_{11})s_{21}^2$$

$$+ (\overline{a}_1^2s_{12}^2t_{11} - 3t_{21}\overline{a}_1s_{11}s_{22}t_{22} + t_{21}\overline{a}_1^2s_{11}s_{11})$$
Notice that

\[ \bar{a}, b, s_{21}, s_{11}, \det(S)t_{12}, Ht_{11} > 0 \]

Then

\[ \det( ) > 0. \]

Under the conditions of (8.15) and (8.18), the endemic equilibrium is asymptotically stable if \( \det( ) < 0 \), and unstable if \( \det(S) > 0 \).

Suppose both \( _1 \) and \( _3 \) are hyperbolic equilibria. In the following, under the conditions of

\[ _0 < 1, _1 > 1, \Delta > 0 \]

investigate the stability of \( _1 \) and \( _3 \). Since, the disease free equilibrium \( _0 \) is asymptotically stable if \( _0 < 1 \).

**Theorem: 8.3.4**

Under the conditions of (8.23), either one of the two endemic equilibria (i.e., \( _1, E_3 \)) is asymptotically stable and the other endemic equilibrium is unstable or both of them are unstable.

Finally, discuss the stability of the endemic equilibrium \( _1 \) under the condition of \( _0 > 1 \). Clearly, the Jacobian matrix of system (8.4) at \( _1 \) has the form

\[
\begin{pmatrix}
- \bar{a} & \alpha \bar{a} + \beta \\
\alpha & \alpha \beta \\
\end{pmatrix}
\]

where

\[ _1(I^*_1) = 1 - \frac{\alpha \beta( + \bar{a})^2}{\bar{a}^2( + \mu)(1 + \bar{a})^2} \]

Hence \( _1 \) is decrease with respect to \( _1 \) when \( \bar{a} > 0 \) and increase \( \bar{a} < 0 \).
**Theorem: 8.3.5**

If one of the following is satisfied:

i) \( \lambda \geq \alpha \) and \( + \mu \leq \frac{\beta}{\alpha} \)

ii) \( \lambda < \alpha \) and \( + \mu \leq \frac{(\beta + )}{\alpha\beta} \)

then \( \mathbf{1} \) is asymptotically stable.

**Proof**

i) By using \( 0 > 1 \), then \( \mathbf{1}(0) > \frac{\lambda (\beta + ) - \alpha\beta}{\lambda( + \mu)} \). If \( \lambda \geq \alpha \), then

\[
\mathbf{1}(0) > \frac{2\alpha}{\lambda^2 ( + \mu)} > 0. \quad \text{Then} \quad > 0 \text{ if } \lim_{1 \to \infty} \mathbf{1}(1) = 1 - \frac{\beta}{\lambda( + \mu)} > 0
\]

So, \( \mathbf{1} \) is asymptotically stable if \( \lambda \geq \alpha \) and \( ( + \mu) \leq \frac{\beta}{\alpha} \).

ii) Obviously, \( 0 > 1 \), leads to \( < \frac{\lambda ( + \mu)}{\beta + } \). Then, \( \mathbf{1}(0) > 1 - \frac{( + \mu)\alpha\beta}{(\beta + )^2} \)

and \( > 0 \) if \( ( + \mu) \leq \frac{(\beta + )^2}{\alpha\beta} \)

By Bendixion-Poincare theorem, it is not difficult to obtain the following theorem.

**Theorem: 8.3.6**

If \( > 0 \) and \( > 0 \), and further \( \lambda < \frac{3}{\alpha( + \mu)^2} \), the endemic equilibrium \( \mathbf{1} \) is globally asymptotically stable, while \( < 0 \) and \( > 0 \) it is unstable and there at least exists one periodic orbit.

**8.4 Numerical Results:**

i) **Simulation and Discussions**

In this section, the numerical simulations are carried out to understand the results more intuitively.

When using the values \( = 75.5, d = 0.5, \mu = 1, i = 0.5, n = 7.5, \beta = 20, \alpha = 0.5 \)
This gives $\lambda = 0.5, \alpha = 0.25$ and $\mu = 76.5 \frac{\beta - 40}{\alpha} \Rightarrow \lambda \geq \alpha \Rightarrow \mu > \frac{\beta}{\alpha}$.

Therefore it satisfies one of the condition of theorem 8.3.5. Hence it is asymptotically stable. Also using the above values, it gives $\lambda > 0$ and $\alpha > 0$ and further $\lambda < \frac{3}{\alpha(\alpha + \mu)^2}$. Hence the endemic equilibrium is globally asymptotically stable. It satisfies the theorem 8.3.6.

ii) Numerical Table

Table 8.4:1

(Effects of $\mu$ on $R_0$, $R_1^*$, $I_1^*$, $I_2^*$, $S_1^*$, and $S_2^*$)

$= 75.5, d = 0.5, \lambda = 0.5, = 7.5, \beta = 20, \alpha = 0.5$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$R_0$</th>
<th>$R_1^*$</th>
<th>$I_1^*$</th>
<th>$I_2^*$</th>
<th>$S_1^*$</th>
<th>$S_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2.7818</td>
<td>1.2236</td>
<td>10.8813</td>
<td>1.8617</td>
<td>12.8774</td>
<td>53.4647</td>
</tr>
<tr>
<td>2.0</td>
<td>2.8182</td>
<td>1.2396</td>
<td>11.0234</td>
<td>1.9947</td>
<td>12.8915</td>
<td>51.7584</td>
</tr>
<tr>
<td>3.0</td>
<td>2.8545</td>
<td>1.2556</td>
<td>11.1654</td>
<td>2.1277</td>
<td>12.9054</td>
<td>50.1973</td>
</tr>
<tr>
<td>4.0</td>
<td>2.8909</td>
<td>1.2716</td>
<td>11.3073</td>
<td>2.2606</td>
<td>12.9192</td>
<td>48.7635</td>
</tr>
<tr>
<td>5.0</td>
<td>2.9273</td>
<td>1.2876</td>
<td>11.4491</td>
<td>2.3936</td>
<td>12.9327</td>
<td>47.4420</td>
</tr>
<tr>
<td>6.0</td>
<td>2.9636</td>
<td>1.3036</td>
<td>11.5907</td>
<td>2.5266</td>
<td>12.9461</td>
<td>46.2202</td>
</tr>
<tr>
<td>7.0</td>
<td>3.0000</td>
<td>1.3196</td>
<td>11.7322</td>
<td>2.6596</td>
<td>12.9592</td>
<td>45.0872</td>
</tr>
<tr>
<td>8.0</td>
<td>3.0364</td>
<td>1.3356</td>
<td>11.8736</td>
<td>2.7926</td>
<td>12.9722</td>
<td>44.0337</td>
</tr>
<tr>
<td>9.0</td>
<td>3.0727</td>
<td>1.3516</td>
<td>12.0149</td>
<td>2.9255</td>
<td>12.9851</td>
<td>43.0515</td>
</tr>
<tr>
<td>10.0</td>
<td>3.1091</td>
<td>1.3676</td>
<td>12.1561</td>
<td>3.0585</td>
<td>12.9977</td>
<td>42.1337</td>
</tr>
<tr>
<td>11.0</td>
<td>3.1455</td>
<td>1.3836</td>
<td>12.2972</td>
<td>3.1915</td>
<td>13.0102</td>
<td>41.2741</td>
</tr>
<tr>
<td>12.0</td>
<td>3.1818</td>
<td>1.3996</td>
<td>12.4382</td>
<td>3.3245</td>
<td>13.0225</td>
<td>40.4674</td>
</tr>
<tr>
<td>14.0</td>
<td>3.2545</td>
<td>1.4315</td>
<td>12.7199</td>
<td>3.5904</td>
<td>13.0467</td>
<td>38.9942</td>
</tr>
<tr>
<td>15.0</td>
<td>3.2909</td>
<td>1.4475</td>
<td>12.8606</td>
<td>3.7234</td>
<td>13.0586</td>
<td>38.3198</td>
</tr>
</tbody>
</table>
Interpretation:
From table 8.4:1, it is found the following
1. When $R_0 > 1$, there exists a unique endemic equilibrium $E_1 = (I_1^*, S_1^*)$
2. When $I_0 > 1$ there exists a unique endemic equilibrium $E_2 = (S_2^*, I_2^*)$
3. When $I_0 > 1$, then it is noted that $I_1^*$, $I_2^*$, and $S_1^*$ increases as $\mu$ increases.

Inference:
From these results, it is found that when the immigration increases, the disease will spread into the population. But $S_2^*$ decreases as $\mu$ increases

iii) Graphical Representation and Their Results
Result (i):
\[ R_0 > 1, \quad a = 5.5, \quad \mu = 2.15, \quad d = 0.5, \quad \lambda = 0.5, \quad n = 0.75, \quad \beta = 2, \quad \alpha = 0.5. \]

![Figure 8.1: $E_1$ is asymptotically stable](image)

Result (ii):
\[ R_0 > 1, \quad a = 45.0, \quad \mu = 31.5, \quad d = 0.5, \quad \lambda = 0.5, \quad n = 7.5, \quad \beta = 20, \quad \alpha = 0.5. \]

![Figure 8.2: $E_1$ is unstable and there at least exists a periodical orbit in the first quadrant of $(S, I)$ plane.](image)
Result (iii):

\[ R_0 = 1, R_1 > 1, \quad 500.13, \mu = 23.5, d = 0.5, \lambda = 0.02, n = 11.1, \beta = 9.85, \alpha = 0.5. \]

Figure 8.3: \( E_2 \) is unstable, and the trajectories around it approach to the disease free equilibrium \( E_0 \).

Result (iv):

\[ R_0 = 1, R_1 > 1, \quad 1.02, \mu = 0.15, d = 0.005, \lambda = 0.00019, n = 0.01, \beta = 0.035, \alpha = 0.05. \]

Figure 8.4: \( E_2 \) is asymptotically stable

Result (v):

\[ R_0 < 1, R_1 > 1, \quad 12.36, \mu = 2.3, d = 1, \lambda = 0.2, n = 2, \beta = 1, \alpha = 0.35, \Delta = 0. \]
Both the unique endemic equilibrium and the disease free equilibrium are asymptotically stable.

8.5 CONCLUSION

In this chapter, the qualitative analysis of SIR epidemic model with immigration is formulated with saturated treatment rate, which characterizes the effect of limited treatment capacity on the spread of epidemic disease. The system (8.4) has bi-stable equilibria. Since the traditional SIR epidemic model and the multiple endemic equilibria co-exist in the system (8.4). From the table 8.4:1, it is shown that as increases, $\mu$, $R_0$, $I_{1}^*$, $I_{2}^*$, $S_{1}^*$, and $S_{2}^*$ are also increases. Thai is when $R > 1$, there exists a unique endemic equilibrium.

The figures exhibit the dynamic behaviors of system (8.4), such as the stability of the endemic equilibria, the existence of the periodic orbit and the bi-stable equilibria, respectively. First consider $R > 1$. Then there exist an unstable disease free equilibrium and a unique endemic equilibrium. In Fig. 8.1, $E_1$ is asymptotically stable. However, in Fig. 8.2, it is unstable and at least one periodic orbit occurs.
Next consider $R_0 = 1$. Then there exist a unique endemic equilibrium $E_1 > 1$, and its stability depends on the parameters which are sketched in Figures 8.3 & 8.4, respectively.

Finally, let $0 < 1$ and $0 > 1$. By the theorem, the existence of the endemic equilibrium depends on the sign of $\Delta$, i.e. system (8.4) has two endemic equilibria $E_1$, $E_3$ if $\Delta > 0$ and a unique endemic equilibrium $E_4$ if $\Delta = 0$ The dynamic behaviors of the endemic equilibria is showed in Fig. 8.5.

From the above five figures, it is shown that

i) When $R_0 > 1$, $E_1$ is asymptotically stable

ii) When $R_0 > 1$, $E_1$ is unstable

iii) When $R_0 = 1$, $R_i > 1$, $E_2$ is unstable

iv) When $R_0 = 1$, $R_i > 1$, $E_2$ is asymptotically stable

v) When $R_0 < 1$, $R_i > 1$, unique endemic equilibrium $E_4$ and the disease free equilibrium $E_0$ are asymptotically stable.