CHAPTER 7

SIR EPIDEMIC MODELS WITH IMMIGRATION USING GENERALIZED SATURATED INCIDENCE RATE UNDER TREATMENT

Infectious diseases are a major health problem throughout the world. Mathematical modeling can help to understand the spread of infectious diseases and to test control strategies. Mathematical modeling is an essential tool for studying a diverse range of such diseases to gain a better understanding of transmission mechanisms and make predictions; determine and evaluate control strategies. In SIR epidemic model, transmission of a disease is a dynamical process driven by the interaction between susceptibles and infectives.Transmission means the rate of incidence. The existence and stability of the disease free and endemic equilibrium are considered through the incidence rate.

7.1 INTRODUCTION

Epidemics have ever been a great concern of human kind, because the impact of infectious diseases on human and animal is enormous, both in terms of suffering and social and economic consequences. Many authors [08, 20, 22, 40, 68] have proposed various kinds of epidemic models to understand the mechanism of disease transmission. The basic elements for the description of infectious diseases have been considered by three epidemiological classes: $S(t)$ that measures the susceptible portion of population, $I(t)$ the infected, and $R(t)$ the removed ones. Modeling and analysis of such infectious diseases have been done by many scientists [03, 15, 33, 36, 50]. Delay differential equations have been successfully used to model varying infectious period in a range of SIR, SIS, and SIRS epidemic models. Hethcote and Van den Driessche [37] have considered an SIS epidemic model with constant time delay, which accounts for duration of infectiousness. McCluskey [49] and Hai-Feng and Zhan-Ping Ma [32] have studied the effect of time delay on the stability of the endemic equilibrium. They gave some conditions for which the endemic equilibrium is asymptotically stable for all delays and also discussed the existence of
orbitally asymptotically stable periodic solutions. The mathematical analysis of epidemiological modeling is often used for the assessment of the global asymptotic stability of both the disease free and endemic equilibrium. The contact rate is often a function of population density, reflecting the fact that contacts take time and saturation occurs. Brauer [7] proposed the models for the transmission of disease with immigration of infective. Delayed SIR epidemic model with a saturated incidence rate is discussed by Fathalla A. Rihan and Naim Anwar [24]. Esteva and Matias [22] have studied a model for transmitted diseases with incidence.

Two mathematical models are constructed in this chapter. They are i) SIR epidemic model with immigration using generalized saturated incidence rate under treatment, ii) Modified SIR epidemic model with immigration using generalized saturated incidence rate under treatment. This chapter is organized as follows: in section 7.2, description of model I is given and the mathematical analysis are analyzed by a lemma and some theorems. Numerical illustrations and simulations are given in the same section. In section 7.3, description of model II and mathematical analysis are given related to this model. In the same section, the analytical results are illustrated with relevant interpretations and implications. Conclusions are presented in section 7.4.

7.2 MODEL DESCRIPTION AND ANALYSIS FOR MODEL I

7.2.1 Model Description

In Kermack and McKendrick [44] births and deaths compartment model, the SIR model is of the form

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \lambda SI + \beta R \\
\frac{dI}{dt} &= \lambda SI - (d + m)I \\
\frac{dR}{dt} &= mI - (d + \beta)R
\end{align*}
\]

where \( a \) is the recruitment rate of the population, \( d \) the natural death rate of the population, \( \lambda \) the proportionality constant, \( m \) is the natural recovery rate
of the infective individuals, $\beta$ the rate at which the recovered individuals lose immunity and return to susceptible class.

Ankit Agrawal and Saxena [4] introduced a generalized saturated incidence rate $\phi = \frac{\lambda SI}{\rho + \alpha I}$ in the SIR epidemic model. The model is

$$\frac{dS}{dt} = a - dS - \phi + \beta R$$

$$\frac{dI}{dt} = \phi - (d + m)I$$

$$\frac{dR}{dt} = mI - (d + \beta)R$$

In this chapter, SIR epidemic models with immigration is analyzed using generalized saturated incidence rate $\phi = \frac{\lambda SI}{\rho + \alpha I}$ and the treatment function

$$T(I) = \begin{cases} rI, & \text{if } 0 \leq I \leq I_0 \\ K_1, & \text{if } I > I_0 \end{cases}$$

is also considered to analyze the transmission rate.

### 7.2.2 Assumptions

- Assume that the population consists of three types of individuals. They are susceptible, infective and recovered individuals.
- Let the saturated incidence rate be $\phi = \frac{\lambda SI}{\rho + \alpha I}$.
- Assumed that the capacity of the treatment of a disease in a community is a constant treatment rate $0 \leq r \leq 1$.
- The treatment rate is $T(I) = \begin{cases} rI, & \text{if } 0 \leq I \leq I_0 \\ K_1, & \text{if } I > I_0 \end{cases}$
- Consider the susceptible rate with immigration.
- Let the immigration be constant.
SIR epidemic model with immigration under treatment: The generalized saturated incidence rate is considered with a single parametric measure \( \alpha \) and the model is discussed by taking two different cases of treatment function.

### 7.2.3 Notations

- \( S \) : Number of susceptibles
- \( I \) : Number of infectives.
- \( R \) : Number of removed or recovered individuals.
- \( a \) : Recruitment rate of the population
- \( d \) : Natural death rate of the population
- \( \lambda \) : The proportionality constant
- \( m \) : Natural recovery rate of the infective individuals
- \( \beta \) : The rate at which recovered individuals lose immunity and return to susceptible class
- \( \mu \) : Increase of susceptibles at a constant rate
- \( \alpha \) : The parameter measure of the psychological or inhibitory effect
- \( E_0 \) : Disease-free equilibrium.
- \( E^* \) : Endemic equilibrium
- \( R_0 \) : Basic Reproduction Number
- \( S^* \) : Susceptible rate at the endemic equilibrium
- \( R^* \) : Recovered or removed rate at the endemic equilibrium
- \( I^* \) : Infected rate at the endemic equilibrium
- \( \phi \) : The transmission rate
- \( \rho \) : The positive constant \( \geq 0 \)
- \( \lambda SI \) : The infection force of the disease.
7.2.4 Mathematical Model I

An SIR epidemiological model with asymptotically homogeneous incidence rate with treatment rate function is considered in this section.

The proposed model is

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m)I - T(I) \\
\frac{dR}{dt} &= mI - (d + \beta)R + T(I)
\end{align*}
\]  

(7.1)

Consider the treatment function \( T(I) \) is defined by

\[
T(I) = \begin{cases} 
  rI, & \text{if } 0 \leq I \leq I_0 \\
  K, & \text{if } I > I_0
\end{cases}
\]

This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value \( I_0 \) and the treatment is constant when the number of infective crosses the fixed value \( I_0 \). In the next section, the stability of the model is discussed by taking two different cases of treatment function.

7.2.5 Equilibrium Points and Stability

**Case (i)**

*SIR model with \( 0 \leq I \leq I_0 \)*

When \( T(I)=rI \), the model reduces to

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m)I - rI \\
\frac{dR}{dt} &= mI - (d + \beta)R + rI
\end{align*}
\]  

(7.2)

The transmission rate \( \phi = \frac{\lambda SI}{\rho + \alpha I} \) displays a saturation effect accounting for the fact that the number of contacts an individual reaches some maximal value due to spatial distribution of the population.
7.2.6 Main Results

In this section, the study of SIR epidemic model is to obtain the properties of the equilibrium points and analyze sufficient conditions under which the equilibrium points are unique or global. Then the system (7.2) becomes

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m + r)I \\
\frac{dR}{dt} &= (m + r)I - (d + \beta)R
\end{align*}
\]

(7.3)

Because of the biological meaning of the components \((S(t), I(t), R(t))\), it is focused on the model in the first octant of \(\mathbb{R}^3\).

First consider the existence of equilibrium of system (7.3). It is easy, by computations, to conclude that the system (7.3) has two equilibrium states: the disease free equilibrium state \(E_0 = \left(\frac{a + \mu}{d}, 0, 0\right)\) which exists for all parameter values and endemic equilibrium \(E^* = (S^*, I^*, R^*)\). To find the endemic equilibrium \((S^*, I^*, R^*)\), set the system (7.3) is equal to zero. Then

\[
\begin{align*}
&\quad a - dS - \phi + \beta R + \mu = 0 \\
&\quad \phi - (d + m + r)I = 0 \\
&\quad (m + r)I - (d + \beta)R = 0 \\
\Rightarrow &\quad R = \frac{(m + r)I}{d + \beta} \\
\text{Now} &\quad R^* = \frac{(m + r)I^*}{d + \beta} \\
&\quad \phi - (d + m + r)I = 0 \\
\Rightarrow &\quad \phi = (d + m + r)I
\end{align*}
\]
and \( \frac{\lambda IS}{\rho + \alpha I} - (d + m + r)I = 0 \)

\[ S = \frac{(\rho + \alpha I)(d + m + r)I}{\lambda I}, \]

\[ S^* = \frac{\phi(\rho + \alpha I^*)}{\lambda I^*} \]

Then

\[ a - \frac{d(\rho + \alpha I)(d + m + r)}{\lambda} - (d + m + r)I + \beta \frac{(m + r)I}{d + \beta} + \mu = 0 \]

\[ \lambda(a + \mu)(d + \beta) - d \rho(d + m + r)(d + \beta) - I[d \alpha (d + m + r)(d + \beta) \]

\[ + (d + m + r)(d + \beta) \lambda - \lambda \beta (m + r)] = 0 \]

\[ I = \frac{(a + \mu) \lambda(d + \beta) - d \rho(d + m + r)(d + \beta)}{d \alpha(d + m + r)(d + \beta) + \lambda(d + m + r)(d + \beta) - \lambda \beta (m + r)} \]

\[ I^* = \frac{d \rho(d + m + r)(d + \beta)[R_0 - 1]}{d \alpha(d + m + r)(d + \beta) + \lambda(d + m + r)(d + \beta) - \lambda \beta (m + r)} \]

The reproduction number is

\[ R_0 = \frac{\lambda(a + \mu)}{\rho d(d + m + r)} > 1 \quad (7.4) \]

**Lemma: 7.2.1**

*The plane \( S + I + R = \frac{a + \mu}{d} \) is a manifold of system (7.3) which is attracting in the first octant.*

**Proof**

Summing up the three equations in (7.3) and denoting

\[ N(t) = S(t) + I(t) + R(t) \]

\[ \frac{dN}{dt} = (a + \mu) - dN \quad (7.5) \]

It is clear that \( N(t) = \frac{a + \mu}{d} \) is a solution of system (7.5). For any \( N(t^{'}) \geq 0 \), solve the equation (7.5) to get the general solution of the
equation (7.5). This is the linear differential equation of first order so the general solution of (7.5) is

\[ N = \frac{1}{d} \{ (a + \mu) - [(a + \mu) - dN(t')]e^{-d(t - t')} \} \]

When \( t \to \infty \),

\[ N(t) = \frac{a + \mu}{d} \cdot \]

This implies the conclusion.

It is clear that the limit set of system (7.3) is on the plane \( S + I + R = \frac{a + \mu}{d} \).

Thus, the reduced system is

\[
\begin{align*}
\frac{dI}{dt} &= \frac{\lambda I (\frac{a + \mu}{d} - I - R)}{\rho + \alpha I} - (d + m + r)I \equiv P(I, R) \\
\frac{dR}{dt} &= (m + r)I - (d + \beta)R \equiv Q(I, R)
\end{align*}
\]  

(7.6)

**Theorem: 7.2.2**

System (7.6) does not have non-trivial periodic orbit if \( \alpha (2d + \beta + m + r) > 0 \)

**Proof**

Consider system (7.6) for \( I > 0 \) and \( R > 0 \). Take a Dulac function \( D(I, R) = \phi^{-1} \)

\[ D(I, R) = \frac{\rho + \alpha I}{\lambda IS} \]

Notice that

\[
\begin{align*}
\frac{\partial (DP)}{\partial R} + \frac{\partial (DQ)}{\partial R} &= -\frac{d(\rho + \alpha I)}{\lambda(\frac{a + \mu}{d} - I - R)^2} - \frac{(\rho + \alpha I)(d + \beta)}{\lambda I(\frac{a + \mu}{d} - I - R)^2} [R - (\frac{a + \mu}{d} - I - R)]
\end{align*}
\]
\[
\frac{\partial(DP)}{\partial R} + \frac{\partial(DQ)}{\partial R} < 0
\]

Hence the conclusion follows.

In order to study the properties of the disease-free equilibrium \(E_0\) and the endemic equilibrium \(E^*\), the system (7.6) is rescaled by

\[
x = \frac{\lambda I}{d + \beta}, \quad y = \frac{\lambda R}{d + \beta}, \quad \tau = (d + \beta)t
\]

\[
\frac{dx}{dt} = \frac{\lambda I}{d + \beta}\left[\frac{\lambda(a + \mu)}{\rho + \alpha I} - \frac{\lambda I}{d + \beta} - (d + m + r)I\right]
\]

\[
\frac{dx}{d\tau} = \frac{\lambda I}{d + \beta}\left[\frac{\lambda(a + \mu)}{d(d + \beta)} - \frac{\lambda I}{d + \beta} - \frac{\lambda I}{d + \beta} \right] - \frac{\lambda I}{d + \beta} \frac{(d + m + r)}{(d + \beta)}
\]

\[
\frac{dx}{dt} = \frac{px}{1 + qx}(A - x - y) - Tx, \quad \frac{dy}{dt} = sx - y
\]

where \(p = \rho^{-1}\), \(q = \frac{\alpha(d + \beta)}{\lambda(\rho d + \delta b)}\), \(A = \frac{(a + \mu)\lambda}{d(d + \beta)}\),

\[T = \frac{d + m + r}{d + \beta} \quad \text{and} \quad s = \frac{m + r}{d + \beta}\]

The trivial equilibrium \((0,0)\) of system (7.7) is the disease-free equilibrium of model (7.3) and the unique positive equilibrium \((x^*, y^*)\) of system (7.7) is the endemic equilibrium \(E^*\) of model (7.3) if and only if \(A - T > 0\) and \(q > 0\), where

\[x^* = \frac{Ap - T}{p(1 + s) + Tq}\]

\[y^* = sx^*\]
Now determine the stability and topological type of \((0,0)\).

The Jacobian matrix of the system (7.7) at \((0,0)\) is

\[
M_0 = \begin{pmatrix}
Ap - T & 0 \\
-1 & s
\end{pmatrix}
\]

By Theorem of Zhang et al. [14] or Theorem of Xiao and Ruan [65], \((0,0)\) is a saddle-node. Hence the following result is obtained.

**Theorem: 7.2.3**

The disease free equilibrium \((0,0)\) of system (7.3) is

i) a stable hyperbolic node if \(T - Ap > 0\)

ii) a saddle-node if \(T - Ap = 0\)

iii) a hyperbolic saddle if \(T - Ap < 0\).

**Proof**

When \(T - Ap < 0\), the system discusses the stability and topological type of the endemic equilibrium \((x^*, y^*)\). The Jacobian matrix of the system (7.7) at \((x^*, y^*)\) is

\[
M_1 = \begin{pmatrix}
px^*[qsx^* - (Aq + 1)] & -px^*(1 + qx^*) \\
(1 + qx^*)^2 & s
\end{pmatrix}
\]

\[
\det(M_1) = \frac{px^*[Aq + (1 + s)]}{(1 + qx^*)^2}
\]

Since \(q > 0\), it follows that \(\det(M_1) > 0\) and \((x^*, y^*)\) is a node or a focus or a center. The following result shows the stability of \((x^*, y^*)\).

**Theorem: 7.2.4**

Suppose \(T - Ap < 0\), then there is a unique endemic equilibrium \((x^*, y^*)\) of system (7.7) which is a saddle mode.
**Proof**

The trace of the matrix $M_1$ is

$$\text{Trace}(M_1) = px^*[qsx^*- (Aq + 1)] + (1 + qx^*)^2$$

$$(1 + qx^*)^2$$

The sign of $\text{Trace}(M_1)$ is determined by

$$S_1 = px^*[qsx^*- (Aq + 1)]$$

Substitute $x^* = \frac{Ap - T}{p(1 + s) + Tq}$ in $S_1$ and using a straight forward calculation, implies

$$S_1 = \frac{p(Ap - T)}{[p(1 + s) + Tq]^2}[-Aq(p + Tq) - (s + 1)(Tq + p)]$$

Since $q > 0$, $[p(1 + s) + Tq] > 0$

and $[-Aq(p + Tq) - (s + 1)(Tq + p)] < 0$, $\Rightarrow S_1 < 0$.

However, when $T - Ap < 0$, $\text{Trace}(M_1) > 0$.

This completes the proof.

**Case (ii)**

**SIR model with** $I > I_0$

In this case the model is

$$\begin{aligned}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m)I - K_1 \\
\frac{dR}{dt} &= mI - (d + \beta)R + K_1
\end{aligned}$$

(7.8)

Since $S + I + R = a + \mu / d$ is invariant manifold of the system, the model reduces to

$$\frac{dI}{dt} = \frac{\lambda I (\frac{a + \mu}{d} - I - R)}{\rho + \alpha I} - (d + m)I - K_1$$
\[ \frac{dR}{dt} = mI - (d + \beta)R + K_1 \]

In order to study the properties of the disease-free equilibrium \( E_0 \) and the endemic equilibrium \( E^* \), the above system is given by

\[
x = \frac{\lambda I}{d + \beta}, \quad y = \frac{\lambda R}{d + \beta}, \quad \tau = (d + \beta)t
\]

\[
\frac{dx}{dt} = \frac{\lambda}{d + \beta} \left[ \frac{\lambda I (a + \mu) d}{\rho} - (d + m)I - K_1 \right]
\]

\[
\frac{dx}{d\tau} = \frac{\lambda I}{d + \beta} \left[ \frac{\lambda(a + \mu) d}{(d + \beta) - \rho + \alpha I} - \frac{\lambda I}{d + \beta} - \frac{\lambda R}{d + \beta} \right] - \frac{\lambda I}{d + \beta} \left( \frac{d + m}{d + \beta} - \frac{\lambda}{d + \beta} \right) K_1
\]

\[
\frac{dx}{dt} = \frac{p x}{1 + qx} (A - x - y) - Tx - c
\]

\[
\frac{dy}{dt} = sx - y + c
\]

(7.9)

Where \( p = \rho^{-1}, \quad \alpha = \frac{\alpha}{\lambda} \frac{\lambda}{d + \beta}, \quad A = \frac{(a + \mu)\lambda}{d(d + \beta)}, \quad c = \frac{\lambda K_1}{(d + \beta)^2}, \quad T = \frac{d + m}{d + \beta} \) and \( s = \frac{m}{d + \beta} \)

For the equilibrium, put

\[
\frac{p x}{1 + qx} (A - x - y) - Tx - c = 0
\]

\[
sx - y + c = 0
\]

For the endemic equilibrium \( E^* (x^*, y^*) \) of (7.9) is

\[
x^* = \frac{Ap - T - c}{p(1 + s) + (T + c)(q_1 + q_2)}
\]

\[
y^* = sx^* + c
\]

Determine the stability and topological type of \((0,0)\). The Jacobian matrix of system (7.9) at \((0,0)\) is
Theorem: 7.2.5

If \( T - Ap > 0 \) the disease-free equilibrium \((0,0)\) of system (7.7) is stable hyperbolic node, \( T - Ap = 0 \) then saddle node and \( T - Ap < 0 \) then hyperbolic saddle node.

When \( T - Ap > 0 \), study the stability and topological type of the endemic equilibrium \((x^*, y^*)\).

The Jacobian matrix of the system (7.9) at \((x^*, y^*)\) is

\[
M_1 = \begin{pmatrix}
px^*[qsx^* - (Aq + 1)] & px^*[-(1 + qx^*)] \\
(1 + qx^*)^2 & (1 + qx^*)^2
\end{pmatrix}
\]

Then \( \det(M_1) = \frac{px^*[Aq + (1 + s)]}{(1 + qx^*)^2} \)

Since \( q > 0 \), it follows that \( \det(M_1) > 0 \) and \((x^*, y^*)\) is a node or a focus or a center. Now it gives the following result on the stability of \((x^*, y^*)\)

Theorem: 7.2.6

Suppose \( T - Ap < 0 \), then there is a unique endemic equilibrium \((x^*, y^*)\) of the system (7.3) which is a saddle mode.

Proof

The trace of the matrix \( M_1 \) is

\[
\text{Trace}(M_1) = \frac{px^*[qsx^* - (Aq + 1)] + (1 + qx^*)^2}{(1 + qx^*)^2}
\]

The sign of \( \text{Trace}(M_1) \) is determined by

\[
S_1 = px^*[qsx^* - (Aq + 1)]
\]
Substitute \( x^* = \frac{Ap - T}{(1 + s) + Tq} \) in \( S_1 \) and using a straightforward calculation, implies

\[
S_1 = \frac{p(Ap - T)}{[p(1 + s) + Tq]^2}[-Aq(p + Tq) - (s + 1)(Tq + p)]
\]

Since \( q > 0, \ [p(1 + s) + Tq] > 0 \)

and \([-Aq(p + Tq) - (s + 1)(Tq + p)] < 0, \Rightarrow S_1 < 0\).

However, when \( T - Ap < 0, \ \text{Trace}(M_1) > 0 \).

The proof is completed.

7.2.7 NUMERICAL RESULTS

i) Simulation and Discussions of Model I

When the treatment rate is \( \infty \), the infective is \( 0 \leq I \leq I_0 \). Taking the parameters in the model

\[
\rho = 2, \ \beta = 0.4, \ d = 0.2, \ a = 0.1, \ \mu = 0.2, \ \lambda = 0.5, \ m = 0.3, \ r = 0.3
\]

The basic reproduction number \( R_0 = 0.47 < 1 \). In this case the disease dies out. By rescaling the system \( T - Ap = 0.705 > 0 \), the disease-free equilibrium \((0,0)\) of the system (7.3) is stable hyperbolic node, when

\[
\rho = 0.3, \ \beta = 0.4, \ d = 0.2, \ a = 5, \ \mu = 0.2, \ \lambda = 0.5, \ m = 0.3, \ r = 0.3
\]

then \( R_0 = 10.83 > 1 \). In this case the disease will invade the population and if \( T - Ap = -20.34 < 0 \) then the system (7.3) is hyperbolic saddle node.
### ii) Numerical Table for Model I

**Table 7.2:1**

*Effects of $\mu$ on $R_0^*$, $S^*$, $I^*$ and $R^*$*)

$\rho = 0.3, \beta = 0.2, d = 0.2, a = 5, \lambda = 0.5, m = 0.3, r = 0.3, \alpha = 0.4$

$\mu$ varies from 0.1 to 1.5

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$R_0^*$</th>
<th>$S^*$</th>
<th>$I^*$</th>
<th>$R^*$</th>
</tr>
</thead>
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<tr>
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<td>6.5455</td>
<td>9.4773</td>
<td>14.2159</td>
</tr>
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<td>9.6667</td>
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</tr>
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<td>6.7879</td>
<td>9.8561</td>
<td>14.7841</td>
</tr>
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<td>10.0455</td>
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</tr>
<tr>
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</tr>
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</tr>
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</tr>
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<td>17.6250</td>
</tr>
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<td>8.1212</td>
<td>11.9394</td>
<td>17.9091</td>
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<td>67.7083</td>
<td>8.2424</td>
<td>12.1288</td>
<td>18.1932</td>
</tr>
</tbody>
</table>

**Interpretation:**

From table 7.2:1, it is noted that

1. The basic reproductive number $R_0^* > 1$. $S^*$, $I^*$ and $R^*$ are monotonically increases for the increasing values of $\mu$
2. This implies that when the immigration increases, the infected rate, susceptible rate and the recovered or removal rate of the individuals will increase.

**Inference:**

From this result it is concluded that, when the value of the basic reproductive number is greater than the unity, the disease spread the population. But the infected rate increases after the treatment.

### 7.3 MODEL DESCRIPTION AND ANALYSIS FOR MODEL II

#### 7.3.1 Model Description

Wang [61] proposed a treatment function

\[
T(I) = \begin{cases} 
  rI, & \text{if } 0 \leq I \leq I_0 \\
  K_1, & \text{if } I > I_0 
\end{cases}
\]

where \( K_1 = rI_0 \) for some fixed value \( I_0 \). This means that the treatment rate is proportional to the infective when the number of infective is less or equal to some fixed value \( I_0 \) and the treatment is constant when the number of infective crosses the fixed value \( I_0 \). Ankit Agrawal and Saxena [4] introduced a generalized saturated incidence rate \( \phi = \frac{\lambda SI}{\rho + \alpha I} \) in the SIR epidemic model. The model is

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R \\
\frac{dI}{dt} &= \phi - (d + m)I \\
\frac{dR}{dt} &= mI - (d + \beta)R
\end{align*}
\]

In this section, the model 7.2.1 is analyzed by introducing the incidence rate \( \phi = \frac{\lambda SI}{\rho + \alpha I + \alpha' I^2} \) with treatment rate \( T(I) \). This new system is discussed with subsequent assumptions and notations.
7.3.2 Assumptions


- Let the saturated incidence rate be $\phi = \frac{\lambda SI}{\rho + \alpha_1 I + \alpha_2 I^2}$

- Assumed that the capacity for the treatment of a disease in a community is a constant treatment rate $0 \leq r \leq 1$.

- The treatment rate is

  $$T(I) = \begin{cases} rI, & \text{if } 0 \leq I \leq I_0 \\ K_1, & \text{if } I > I_0 \end{cases}$$

- Consider the susceptible rate with immigration.

**Modified SIR epidemic model with treatment rate:**

The generalized saturated incidence rate is considered with two parametric measures $\alpha_1$ & $\alpha_2$ and the model is discussed by taking two different cases of treatment function.

7.3.3 Notations

- $a$: Recruitment rate of the population

- $d$: Natural death rate of the population

- $\lambda$: The proportionality constant

- $m$: Natural recovery rate of the infective individuals

- $\beta$: The rate at which recovered individuals lose immunity and return to susceptible class

- $\mu$: Increase of susceptibles at a constant rate

- $\alpha_1$ & $\alpha_2$: The parameter measures of the psychological or inhibitory effect

- $R_0$: Basic Reproduction Number

- $\phi$: The transmission rate
\( \rho \quad : \quad \text{The positive constant} \geq 0 \)

\( \lambda SI \quad : \quad \text{The infection force of the disease.} \)

**All other assumption and notations in this section is as in model I of section 7.2**

### 7.3.4 Basic Mathematical Model II

In this section, consider an SIR epidemiological model with asymptotically homogeneous incidence rate with treatment rate function.

The proposed model is

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m)I - T(I) \\
\frac{dR}{dt} &= mI - (d + \beta)R + T(I)
\end{align*}
\]

(7.10)

Consider the treatment function \( T(I) \) defined by

\[
T(I) = \begin{cases} 
   rI, & \text{if} \ 0 \leq I \leq I_0 \\
   K_s, & \text{if} \ I > I_0 
\end{cases}
\]

In the next section, the stability of the model is discussed by taking two different cases of treatment function.

### 7.3.5 Equilibrium Points and Stability

**Case (i)**

*SIR model with \( 0 \leq I \leq I_0 \)

When \( T(I) = rI \), the model reduces to

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m)I - rI \\
\frac{dR}{dt} &= mI - (d + \beta)R + rI
\end{align*}
\]

(7.11)
The transmission rate $\phi = \frac{aSI}{\rho + \alpha_1 I + \alpha_2 I^2}$ displays a saturation effect accounting for the fact that the number of contacts an individual reaches some maximal value due to spatial distribution of the population.

### 7.3.6 Main Results

In this section, the study of SIR epidemic model is to get the properties of the equilibrium points and analyze sufficient conditions under which the equilibrium points are unique or global. Now the system (7.11) becomes

$$
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m + r)I \\
\frac{dR}{dt} &= (m + r)I - (d + \beta)R
\end{align*}
$$

(7.12)

The components $(S(t), I(t), R(t))$ focus the model in the first octant of $\mathbb{R}^3$.

Consider the existence of equilibrium of system (7.12). It is easy, by computations, to conclude that the system (7.12) has two equilibrium states—the disease free equilibrium state $E_0 = (\frac{a + \mu}{d}, 0, 0)$ which exists for all parameter values and endemic equilibrium $(S^*, I^*, R^*)$. To find the endemic equilibrium $(S^*, I^*, R^*)$, set the system (7.12) equal to zero. Then

\[
\begin{align*}
\frac{a - dS - \phi + \beta R + \mu}{d} &= 0 \\
\phi - (d + m)I &= 0 \\
mI - (d + \beta)R &= 0 \\
\Rightarrow R &= \frac{(m + r)I}{d + \beta} \\
\Rightarrow R^* &= \frac{(m + r)}{d + \beta} I^*
\end{align*}
\]
\( \phi - (d + m + r)I = 0 \)

\[ \Rightarrow \phi = (d + m + r)I \]

and \( \frac{\lambda IS}{\rho + \alpha_1 I + \alpha_2 I^2} - (d + m + r)I = 0 \)

\[ S = \frac{(\rho + \alpha_1 I + \alpha_2 I^2)(d + m + r)I}{\lambda I} \]

\[ S^* = \frac{\phi(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda I^*} \]

Now \( a - \frac{d(\rho + \alpha_1 I + \alpha_2 I^2)(d + m + r)}{\lambda} - (d + m + r)I + \frac{\beta(m + r)I}{d + \beta} + \mu = 0 \)

\[ d\alpha_2(d + m + r)(d + \beta)I^2 + [d\alpha_2(d + m + r)(d + \beta) + (d + m + r)(d + \beta)\lambda - \beta\lambda m]I \]

\[ + \rho d(d + m + r)(d + \beta) - \lambda(a + \mu)(d + \beta) = 0 \]

\[ I = \frac{-[d\alpha_2(d + m + r)(d + \beta) + \lambda(d + m + r)(d + \beta) - \beta\lambda(m + r)]}{2d\alpha_2(d + m + r)(d + \beta)} \]

\[ \pm \frac{[d\alpha_1(d + m + r)(d + \beta) + \lambda(d + m + r)(d + \beta) - \beta\lambda(m + r)]^2 - 4d\alpha_2(d + m + r)(d + \beta)^2}{2d\alpha_2(d + m + r)(d + \beta)} \]

i.e., \( I^* = \frac{-[d\alpha_1(d + m + r)(d + \beta) + \lambda(d + m + r)(d + \beta) - \beta\lambda(m + r)]}{2d\alpha_2(d + m + r)(d + \beta)} \)

\[ \pm \frac{[d\alpha_1(d + m + r)(d + \beta) + \lambda(d + m + r)(d + \beta) - \beta\lambda(m + r)]^2 - 4d\alpha_2(d + m + r)}{(d + \beta)^2[1-R_0]} \]

The reproduction number is

\[ R_0 = \frac{\lambda(a + \mu)}{\rho d(d + m + r)} > 1 \] (7.13)
Lemma: 7.3.1

The plane $S + I + R = \frac{a + \mu}{d}$ is a manifold of system (7.12) which is attracting in the first octant.

Proof

Summing up the three equations in (7.12) and denoting $N(t) = S(t) + I(t) + R(t)$

$$\frac{dN}{dt} = (a + \mu) - dN \tag{7.14}$$

It is clear that $N(t) = \frac{a + \mu}{d}$ is a solution of system (7.14) and for any $N(t') \geq 0$, the general solution of system (7.14) is obtained by solving system (7.14). This is the linear differential equation of first order so the general solution of system (7.14) is

$$N = \frac{1}{d} \{ (a + \mu) - [(a + \mu) - dN(t')] e^{-d(t - t')} \}$$

When $t \to \infty$, it finds that

$$N(t) = \frac{a + \mu}{d}$$

This implies the conclusion.

It is clear that the limit set of system (7.12) is on the plane $S + I + R = \frac{a + \mu}{d}$.

Thus, the reduced system is

$$\begin{aligned}
\frac{dI}{dt} &= \frac{\lambda I(\frac{a + \mu}{d} - I - R)}{\rho + \alpha_1 I + \alpha_2 I^2} - (d + m + r)I \equiv P(I, R) \\
\frac{dR}{dt} &= (m + r)I - (d + \beta)R \equiv Q(I, R)
\end{aligned} \tag{7.15}$$

Theorem: 7.3.2

System (7.15) does not have non-trivial periodic orbit if $\alpha_2 (2d + \beta + m + r) > 0$
Proof

Consider system (7.15) for $I > 0$ and $R > 0$. Take a Dulac function

$$D(I, R) = \phi^{-1}$$

i.e.,

$$D(I, R) = \frac{\rho + \alpha_1 I + \alpha_2 I^2}{\lambda IS}$$

Notice that

$$\frac{\partial(DP)}{\partial R} + \frac{\partial(DQ)}{\partial R} = - \frac{d(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda(\alpha + \mu - \frac{R}{d} - I - R)^2} - \frac{(\rho + \alpha_1 I + \alpha_2 I^2)(d + \beta)}{\lambda I(\alpha + \mu - \frac{R}{d} - I - R)^2} [R - \left(\frac{\alpha + \mu}{d} - I - R\right)]$$

$$\frac{\partial(DP)}{\partial R} + \frac{\partial(DQ)}{\partial R} < 0$$

Hence the conclusion follows.

To study the properties of the disease-free equilibrium $E_0$ and the endemic equilibrium $E^*$, rescale the system (7.15) by

$$x = \frac{\lambda I}{d + \beta}, \quad y = \frac{\lambda R}{d + \beta}, \quad \tau = (d + \beta)t$$

$$\frac{dx}{dt} = \frac{\lambda I}{d + \beta} \left[ \frac{\lambda I(\alpha + \mu - I - R)}{\rho + \alpha_1 I + \alpha_2 I^2} - (d + m + r)I \right]$$

$$\frac{dx}{d\tau} = \frac{\lambda I}{d + \beta} \left[ \frac{1}{\rho + \alpha_1 I + \alpha_2 I^2} \left\{ \frac{\lambda(a + \mu)}{d(d + \beta)} - \frac{\lambda I}{d + \beta} - \frac{\lambda R}{d + \beta} \right\} \frac{(d + m + r)}{d + \beta} \right]$$

$$\frac{dx}{dt} = \frac{px}{1 + q_1x + q_2x^2} (A - x - y) - Tx$$

$$\frac{dy}{dt} = sx - y$$

(7.16)

where

$$p = \rho^{-1}, \quad A = \frac{\lambda(a + \mu)}{d(d + \beta)}, \quad q_1 = \frac{\alpha_1(d + \beta)}{\lambda}, \quad q_2 = \frac{\alpha_2(d + \beta)^2}{\lambda^2}$$

$$T = \frac{d + m + r}{d + \beta} \quad \text{and} \quad s = \frac{m + r}{d + \beta}$$
The trivial equilibrium $(0,0)$ of system (7.16) is the disease-free equilibrium of model (7.12) and the unique positive equilibrium $(x^*, y^*)$ of system (7.16) is the endemic equilibrium $E^*$ of model (7.12) if and only if $A - T > 0$ and $q > 0$, where

$$x^* = \frac{Ap - T}{p(1 + s) + Tq},$$

$$y^* = sx^*$$

Now determine the stability and topological type of $(0, 0)$.

The Jacobian matrix of system (7.16) at $(0, 0)$ is

$$M_0 = \begin{pmatrix} Ap - T & 0 \\ s & -1 \end{pmatrix}$$

**Theorem: 7.3.3**

If $T - Ap > 0$ the disease-free equilibrium $(0,0)$ of system (7.16) is stable hyperbolic node, $T - Ap = 0$ then saddle node and $T - Ap < 0$ then hyperbolic saddle node.

When $T - Ap > 0$, the stability and topological type of the endemic equilibrium $(x^*, y^*)$.

The Jacobian matrix of the system (6.12) at $(x^*, y^*)$ is

$$M_1 = \begin{pmatrix} Ap - px^*[q_1 + Aq_2 - q_2sx^*] - 2px^* & -p \\ \frac{1 + q_1x^* + q_2x^{*2}}{s} & \frac{1 + q_1x^* + q_2x^{*2}}{1} \end{pmatrix}$$

Then $\det(M_1) = px^2[q_1 + Aq_2 - q_2sx^*] + 2px^* - Ap + ps(1 + q_1x^* + q_2x^{*2})$

It follows that $\det(M_1) > 0$ and $(x^*, y^*)$ is a node or a focus or a center. The following result gives the stability of $(x^*, y^*)$. 

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Theorem: 7.3.4

Suppose \( T - Ap < 0 \), then there is a unique endemic equilibrium \((x^*, y^*)\) of system (7.16) which is a saddle mode.

Proof

The trace of the matrix \( M_1 \) is

\[
\text{Trace}(M_1) = \frac{Ap - px^{*2}[q_1 + Aq_2 - q_2sx^*] - 2px^* - (1 + q_1x^* + q_2x^{*2})^2}{(1 + q_1x^* + q_2x^{*2})^2}
\]

The sign of \( \text{Trace}(M_1) \) is determined by

\[
S_1 = Ap - px^{*2}[q_1 + Aq_2] - 2px^*
\]

Substitute \( x^* = \frac{Ap - T}{p(1 + s) + Tq_1} \) in \( S_1 \) and using a straight forward calculation

\[
S_1 = \frac{-p(Ap - T)^2(q_1 + Aq_2)}{[p(1 + s) + Tq_1]^2} - 2 \frac{p^2(Ap - T)}{[p(1 + s) + Tq_1]} + Ap
\]

Since \( q > 0 \), \( [p(1 + s) + Tq] > 0 \)

\( \Rightarrow S_1 < 0 \)

However, when \( T - Ap < 0 \), then \( \text{Trace}(M_1) < 0 \).

This completes the proof.

Case (ii)

\textit{SIR model with I > I_0}

In this case the model is

\[
\begin{align*}
\frac{dS}{dt} &= a - dS - \phi + \beta R + \mu \\
\frac{dI}{dt} &= \phi - (d + m)I - K_1 \\
\frac{dR}{dt} &= mI - (d + \beta)R + K_1
\end{align*}
\] (7.17)
Since $S + I + R = \frac{a + \mu}{d}$ is invariant manifold of the system, the model reduces to

$$\frac{dI}{dt} = \frac{\lambda I (a + \mu - I - R)}{\rho + \alpha_1 I + \alpha_2 I^2} - (d + m)I - K_I$$

$$\frac{dR}{dt} = mI -(d + \beta)R + K_I$$

The study of the properties of the disease-free equilibrium $E_0$ and the endemic equilibrium $E^*$, rescale the above system (7.17) by

$$x = \frac{\lambda I}{d + \beta}, \quad y = \frac{\lambda R}{d + \beta}, \quad \tau = (d + \beta)t$$

$$\frac{dx}{d\tau} = \frac{\lambda}{d + \beta} \left[ \frac{\lambda I (a + \mu - I - R)}{\rho + \alpha_1 I + \alpha_2 I^2} - (d + m)I - K_I \right]$$

$$\frac{dx}{dt} = \frac{\rho x}{1 + q_1 x + q_2 x^2} \left( A - x - y \right) - T x - c$$

$$\frac{dy}{dt} = s x - y + c$$

(7.18)

where $p = \rho^{-1}, A = \frac{\lambda(a + \mu)}{d(d + \beta)}, q_1 = \frac{\alpha_1(d + \beta)}{\lambda}, T = \frac{d + m}{d + \beta}$

$$q_2 = \frac{\alpha_2(d + \beta)^2}{\lambda^2}, \quad c = \frac{\lambda K_I}{(d + \beta)^2}, \quad s = \frac{m}{d + \beta}$$

For the equilibrium, put

$$\frac{\rho x}{1 + q_1 x + q_2 x^2} \left( A - x - y \right) - T x - c = 0$$

$$s x - y + c = 0$$

For the endemic equilibrium $E^*(x^*, y^*)$ of (7.18) is
\[ x^* = \frac{Ap - T - c}{p(1 + s) + (T + c)(q_1 + q_2)}, \quad y^* = sx^* + c \]

Determine the stability and topological type of \((0,0)\). The Jacobian matrix of system (7.18) at \((0,0)\) is

\[
M_0 = \begin{pmatrix}
Ap - T & 0 \\
s & -1
\end{pmatrix}
\]

**Theorem: 7.3.5**

*If \(T - Ap > 0\) the disease-free equilibrium \((0,0)\) of system (7.18) is stable hyperbolic node, \(T - Ap = 0\) then saddle node and \(T - Ap < 0\) then hyperbolic saddle node.*

When \(T - Ap > 0\), the stability and topological type of the endemic equilibrium \((x^*, y^*)\). The Jacobian matrix of the system (7.18) at \((x^*, y^*)\) is

\[
M_1 = \begin{pmatrix}
Ap - px^*\left[q_1 + Aq_2 - q_2sx^*\right] - 2px^* & -p \\
(1 + q_1x^* + q_2sx^*)^2 & (1 + q_1x^* + q_2sx^*)
\end{pmatrix}
\]

Then

\[
\det(M_1) = \frac{px^*\left[q_1 + Aq_2 - q_2sx^*\right] + 2px^* - Ap + ps(1 + q_1x^* + q_2sx^*)}{(1 + q_1x^* + q_2sx^*)}
\]

It implies that \(\det(M_1) > 0\) and \((x^*, y^*)\) is a node or a focus or a center. The following result gives the stability of \((x^*, y^*)\).

**Theorem: 7.3.6**

*Suppose \(T - Ap < 0\), then there is a unique endemic equilibrium \((x^*, y^*)\) of system (7.18) which is a saddle mode.*

**Proof**

The trace of the matrix \(M_1\) is
\[
\text{Trace}(M_1) = \frac{Ap - px^*[q_1 + Aq_2 - q_2sx^*] - 2px^* - (1 + q_1x^* + q_2x^{*2})^2}{(1 + q_1x^* + q_2x^{*2})^2}
\]

The sign of \(\text{Trace}(M_1)\) is determined by

\[S_1 = Ap - px^*[q_1 + Aq_2] - 2px^*
\]

Substituting \(x^* = \frac{Ap - T - c}{p(1 + s) + (T + c)(q_1 + q_2)}\) in \(S_1\) and using a straight forward calculation

\[S_1 = -\frac{p(Ap - T - c)^2}{[p(1 + s) + (T + c)(q_1 + q_2)]^2} - 2p^2\frac{(Ap - T - c)}{[p(1 + s) + (T + c)(q_1 + q_2)]} + Ap
\]

\[\frac{p(1 + s) + (T + c)(q_1 + q_2)}{]} > 0
\]

\[\Rightarrow S_1 < 0.
\]

However, when \(T - Ap < 0\), \(\text{Trace}(M_1) < 0\).

This completes the proof.
7.3.7 NUMERICAL RESULTS

i) Numerical Table for Model I

Table 7.3:1
(Effects of $\mu$ on $R_0^*, S^*, I^*$ and $R^*$)

$\rho = 0.3, \beta = 0.2, d = 0.2, a = 5, \lambda = 0.5, m = 0.3, r = 0.3, \alpha = 0.4$

$\mu$ varies from 0.1 to 1.5

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$R_0^*$</th>
<th>$S^*$</th>
<th>$I^*$</th>
<th>$R^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>53.1250</td>
<td>488.8372</td>
<td>38.3227</td>
<td>57.4841</td>
</tr>
<tr>
<td>0.2</td>
<td>54.1667</td>
<td>498.3206</td>
<td>38.7001</td>
<td>58.0502</td>
</tr>
<tr>
<td>0.3</td>
<td>55.2083</td>
<td>507.8121</td>
<td>39.0743</td>
<td>58.6115</td>
</tr>
<tr>
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<td>56.2500</td>
<td>517.3115</td>
<td>39.4453</td>
<td>59.1679</td>
</tr>
<tr>
<td>0.5</td>
<td>57.2917</td>
<td>526.8187</td>
<td>39.8132</td>
<td>59.7198</td>
</tr>
<tr>
<td>0.6</td>
<td>58.3333</td>
<td>536.3334</td>
<td>40.1780</td>
<td>60.2671</td>
</tr>
<tr>
<td>0.7</td>
<td>59.3750</td>
<td>545.8555</td>
<td>40.5400</td>
<td>60.8099</td>
</tr>
<tr>
<td>0.8</td>
<td>60.4167</td>
<td>555.3848</td>
<td>40.8990</td>
<td>61.3485</td>
</tr>
<tr>
<td>0.9</td>
<td>61.4583</td>
<td>564.9211</td>
<td>41.2552</td>
<td>61.8829</td>
</tr>
<tr>
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<td>62.5000</td>
<td>574.4643</td>
<td>41.6087</td>
<td>62.4131</td>
</tr>
<tr>
<td>1.1</td>
<td>63.5417</td>
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</tr>
<tr>
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<td>593.5705</td>
<td>42.3078</td>
<td>63.4616</td>
</tr>
<tr>
<td>1.3</td>
<td>65.6250</td>
<td>603.1333</td>
<td>42.6534</td>
<td>63.9801</td>
</tr>
<tr>
<td>1.4</td>
<td>66.6667</td>
<td>612.7024</td>
<td>42.9965</td>
<td>64.4948</td>
</tr>
<tr>
<td>1.5</td>
<td>67.7083</td>
<td>622.2776</td>
<td>43.3372</td>
<td>65.0058</td>
</tr>
</tbody>
</table>
**Interpretation:**

From table 7.3:1, the following is noted

1. The basic reproductive number $R_0 > 1$. $S^*$, $I^*$ and $R^*$ are monotonically increases for the increasing values of $\mu$.

2. When the immigration increases, the infected rate, susceptible rate and the recovered or removal rate of the individuals will increase.

**Inference:**

From this table, it is concluded that, when the value of the basic reproductive number is greater than the unity, the disease spread the population. So that $I^*$ increases, when the susceptible rate $S^*$ increases.

**ii) Simulation and Discussions of Model II**

When the treatment rate is $\infty$, the infective is $0 \leq I \leq I_0$. Taking the following parameters in the model

$p = 2, \beta = 0.2, d = 0.2, a = 0.1, \lambda = 0.5, m = 0.3, r = 0.3, \mu = 0.2, \alpha = 0.4$

The basic reproduction number $R_0 = 0.47$ is less than one. In this case the disease dies out. By rescaling the system, finds that $T - Ap > 0$ the disease-free equilibrium $(0,0)$ of the system (7.17) is stable hyperbolic node.

When

$p = 0.3, \beta = 0.2, d = 0.2, a = 5, \lambda = 0.5, m = 0.3, r = 0.3, \alpha = 0.4, \mu = 0.2$

then $R_0 = 54.17$ which is greater than one. In this case the disease will invade the population and if $T - Ap < 0$ then the system (7.17) is hyperbolic saddle node.
7.4 CONCLUSION

In this chapter, two models which are SIR epidemic model with immigration and a modified SIR model with immigration are considered using generalized saturated incidence rate $\phi$. The global stability of the endemic equilibrium point $E^* = (S^*, I^*, R^*)$ depends on the basic reproduction number $R_0$. The basic reproduction number $R_0$ is the special kind of transmission rule. It plays an important role to control the disease.

In both the models, when $R_0 \leq 1$ the disease free equilibrium $E_0 = (\frac{a+u}{d}, 0, 0)$ is globally attractive in the first octant and it is globally stable, that is the disease dies out. When $R_0 > 1$ the endemic equilibrium $E^*$ exists and is globally stable in the interior. Results and parametric conditions help to develop social consciousness about the disease among susceptibles.

From the numerical tables 7.2:1 and 7.3:1, it is noted that, the susceptible rate of model II is comparatively larger than the susceptible rate of model I. So the endemic equilibrium $I^*$ and $R^*$ of model II is also large.