CHAPTER 9

AN ANALYSIS OF THE EFFECT OF THE PERIPHERAL VISCOSITY ON THE BILE FLOW CHARACTERISTICS THROUGH CYSTIC DUCT WITH STONE: STUDY OF TWO LAYER MODEL WITH SQUEEZING

9.1 Introduction

In this chapter, the effect of an overlapping stone in the cystic duct on the flow characteristics of bile is observed. The mathematical model here, considers a two-fluid model of bile, consisting of a core region of suspension of bile salts, cholesterol and bilirubin and peripheral layer of mucosa. The theoretical model used enables one to observe the simultaneous effect of bile suspension in the core region and of the peripheral layer, on the flow characteristics of bile due to the presence of an overlapping constriction in the cystic duct. The rigid cystic duct is having axially symmetric stones and the role of valves of heisters is considered to be negligible. The maximum height of the stone is much less as compared to the length and the uniform radius of the cystic duct. Cholecystokinin (CCK) stimulation not only causes the gallbladder to contract, but also allows the cystic and common bile ducts to contract. The squeezing of the gallbladder causes bile to flow from gallbladder to the cystic duct and finally into the common bile duct. Here, the flow of bile through the cystic duct due to the squeezing of the gallbladder is investigated.

The flowing bile is laminar, steady and axially symmetric. The motion is slow, so the inertia effect can be neglected. The variation of cross-section of the duct is considered to be very small and the fluid is incompressible.

Further, the effect of axially symmetric overlapping stone on the core fluid and peripheral mucosa is investigated for parameters like flow resistance and wall shear stress. In view of the above considerations the basic equation governing the flow of bile in the rigid cystic duct is given by:
9.2 Governing equation

The general equation for the flow of bile is [12]

\[
\frac{-dP}{dz} + \frac{1}{r} \frac{d}{dr} \left( \mu(r) r \frac{du_z}{dr} \right) = 0 \tag{9.1}
\]

The consistency function \( \mu(r) \) may be written as

\[
\mu(r) = \mu_i \text{ for } 0 \leq r \leq R_i(z) \]

\[
\mu(r) = \mu_z \text{ for } R_i(z) \leq r \leq R(z) \tag{9.2}
\]

In equation (9.1) and (9.2), \( z \) is the co-ordinate along the axis of the cystic duct in the flow direction, \( r \) is the co-ordinate in the radial direction and perpendicular to flow, \( \mu(r) \) represents the viscosity function; \( P \) is the pressure at any point, \( \frac{dP}{dz} \) is the pressure gradient, \( u_z \) stand for the axial velocity of fluid, \( \mu_i \) and \( \mu_z \) are the value of viscosity function at the central core region and the peripheral region. Also \( R_i(z) \) and \( R(z) \) are the radius of the central core region and of the duct in the constricted region, (i.e.) the peripheral layer is of thickness \( (R-R_1) \).

9.3 Boundary conditions and the Matching condition

The equation (9.1) is solved using the following boundary condition:

a) \( \frac{du_z}{dr} = 0 \) at \( r=0 \) \tag{9.3}

b) \( u_z = 0 \) at \( r=R(z) \) \tag{9.4}

c) \( P=P_0 \) at \( z=0 \) and \( P=P_L \) at \( z=L \) \tag{9.5}

d) \( u_1 = u_2 \) at \( r=R_1(z) \) \tag{9.6}

The equation (9.3) and (9.6) are known as no-slip and matching condition respectively.
9.4 Geometry Used

The overlapping geometry of the cystic duct and the shape of the central layer in Figure 9.1 and Figure 9.2 respectively are written as [27] –

\[
\frac{R(z)}{R_0} = 1 - \frac{3}{2} \frac{\delta}{R_0} \frac{1}{L_0^4} \left[ 11(z - d)L_0^3 - 47(z - d)^2L_0^2 + 72(z - d)^3L_0 - 36(z - d)^4 \right] , \quad d \leq z \leq d + L_0
\]

\[= 1, \quad \text{Otherwise} \quad (9.7)\]

and

\[
\frac{R(z)}{R_0} = \alpha - \frac{3}{2} \frac{\delta}{R_0} \frac{1}{L_0} \left[ 11(z - d)L_0^3 - 47(z - d)^2L_0^2 + 72(z - d)^3L_0 - 36(z - d)^4 \right] , \quad d \leq z \leq d + L_0
\]

\[= \alpha, \quad \text{Otherwise} \quad (9.8)\]

Where \( R_0 \) the radius of the duct without stones, \( L_0 \) is the length of stone(constricted region), \( L \) is the duct length, \( d \) indicates the location of constriction, \( \alpha \) is the radius of the central core radius to tube radius in the unobstructed region and ( \( \delta, \delta \) )are the maximum height of the stone and bulging of the interface in the constricted region at two locations i.e. at \( z = d + \frac{L_0}{6} \) and \( z = d + \frac{5L_0}{6} \). The stone height located at \( z = d + \frac{L_0}{2} \), called critical height, is \( \frac{3\delta}{4} \) [27].
**Figure 9.1:** The geometry of the overlapping stone in the cystic duct.

**Figure 9.2:** The shape of the central layer
9.5 Analysis of the Model

We solve the model (9.1) and (9.2) using the boundary conditions (9.3) and (9.4).

On integrating (9.1) and using (9.3) we get:

\[ \frac{du_z}{dr} = \frac{r}{2\mu(r)} \frac{dP}{dz} \]

The flow flux, \( Q \) is defined as [12]

\[ Q = \pi \int_0^R r^2 \left( -\frac{\partial u_z}{\partial r} \right) dr \tag{9.9} \]

Using (9.9) and (9.2) we get

\[ Q_1 = \frac{\pi}{8\mu_1} p R_1^s(z), \quad \text{where} \quad -\frac{dP}{dz} = p \]

and,

\[ Q_2 = \frac{\pi p}{8\mu_2} (R^s(z) - R_1^s(z)) \]

Where \( Q_1 \) and \( Q_2 \) are the flux in the core and peripheral region, also \( R(z) = R \) and \( R_1(z) = R_1 \).

Therefore, the total flux \( Q = Q_1 + Q_2 \) is given by

\[ Q = \frac{\pi p}{8\mu_2} (R^s(z) - (1-\mu)R_1^s(z)) \tag{9.10} \]

where \( \mu = \frac{\mu_2}{\mu_1} \)

From equation (9.10), the pressure gradient is written as follows:
\[ p = \frac{8Q \mu_2}{\pi \left[ R^4(z) - (1 - \mu)R_i^4(z) \right]} \]

Integrating along the length of the duct and using the conditions (9.5) we get

\[ P_0 - P_L = \frac{8Q \mu_2}{\pi R_0^4} \int_0^L \frac{dz}{\left( \frac{R}{R_0} \right)^4 - (1 - \mu) \left( \frac{R_i}{R_0} \right)^4} \quad (9.11) \]

### 9.6 Resistance to flow

Thus the resistance to flow is defined by [5, 30]

\[
\bar{\lambda} = \frac{P_0 - P_L}{Q} = \frac{8\mu_2}{\pi R_0^4} \left[ \int_0^d \frac{dz}{\left( \frac{R}{R_0} \right)^4 - (1 - \mu) \left( \frac{R_i}{R_0} \right)^4} \right. \\
+ \int_d^{d+L_0} \frac{dz}{\left( \frac{R}{R_0} \right)^4 - (1 - \mu) \left( \frac{R_i}{R_0} \right)^4} \\
+ \int_{d+L_0}^l \frac{dz}{\left( \frac{R}{R_0} \right)^4 - (1 - \mu) \left( \frac{R_i}{R_0} \right)^4} \right] 
\]

Using (9.7) and (9.8) we get

\[
\bar{\lambda} = \frac{8\mu_2 L}{\pi R_0^4 \left[ 1 - (1 - \mu) \alpha^4 \right]} \left[ \int_0^{\frac{L_0}{L}} \frac{dz}{\left( \frac{R}{R_0} \right)^4 - (1 - \mu) \left( \frac{R_i}{R_0} \right)^4} + \int_{\frac{d+L_0}{L}}^{\frac{l}{L}} \frac{dz}{\left( \frac{R}{R_0} \right)^4 - (1 - \mu) \left( \frac{R_i}{R_0} \right)^4} \right] 
\]

(9.13)
In the absence of any constriction, the resistance to flow $\lambda_n$ is given by -

$$\lambda_n = \frac{8\mu L}{\pi R_0^2 \mu} = \frac{8\mu L}{\pi R_0^2} \quad (9.14)$$

In dimensionless form the flow resistance may be expressed as

$$\lambda = \frac{\lambda_n}{L} = \frac{\mu_2}{\mu_1 (1-(1-\mu)\alpha^4)} \left[ 1 - \frac{L}{L_0} \right]$$

$$+ \frac{(1-(1-\mu)\alpha^4)}{L} \int_0^{L_0} \frac{dz}{\{(R/R_0)^4 - (1-\mu)(R_0/R_0)^4\}}$$

Using now, the fact that the total flux is equal to the fluxes across the two region (central and peripheral) one determines the relations $R_i = \alpha R$ and $\delta_i = \alpha \delta$ [26]. Therefore, we have

$$\lambda = \frac{\mu}{\beta} \left[ 1 - \frac{L}{L_0} + \frac{1}{L} \int_0^{L_0} \frac{dz}{(R/R_0)^4} \right] \quad (9.15)$$

Where $\beta = 1 - (1-\mu)\alpha^4$. The equation (9.15) gives the flow resistance in the cystic duct.

### 9.7 Shear Stress

The shearing stress at the wall can be defined as [26]-

$$\tau_r = -\frac{R}{2} \frac{dP}{dz}$$

Using equation (9.10) we get
\[ \tau_r = \frac{4 \mu \zeta Q}{\pi \beta R^3} \]  

(9.16)

And, the value of shearing stress when there are no stone is -

\[ \tau_N = \frac{4Q\mu_z}{\pi R_0^3 (1 - (1 - \mu))} = \frac{4\mu Q}{\pi R_0^3} \]  

(9.17)

The non-dimensional shearing stress at the wall is given by

\[ \tau_w = \frac{\tau_r}{\tau_N} = \frac{\frac{\mu}{\beta (R/R_0)^3}}{\frac{4\mu Q}{\pi R_0^3}} \]  

(9.18)

Again the shearing stress at the constriction throat is given by [27] (i.e.) when

\[ z = d + L_0 \frac{L_0}{6} \text{ or } d + \frac{5L_0}{6} \]  

is -

\[ \tau_T = \frac{4Q\mu_z}{\pi \beta R_0^3 \left( \frac{R}{R_0} \right)^3} \]  

(9.19)

where

\[ \frac{R}{R_0} = 1 - \frac{5}{4} \frac{\delta}{R_0} \]

Also, the shearing stress at the critical height is given by i.e. when \( z = d + \frac{L_0}{2} \)

\[ \tau_c = \frac{4Q\mu_z}{\pi \beta R_0^3 \left( 1 - \frac{3\delta}{4R_0} \right)} \]  

(9.20)
where \( \frac{R}{R_0} = 1 - \frac{3}{4} \frac{\delta}{R_0} \)

The non-dimensional shearing stress at constriction throat is given by

\[
\tau_1 = \frac{\tau_T}{\tau_N} = \frac{\mu}{(1-(1-\mu)\alpha^3)^3} \left(1 - \frac{5}{4} \frac{\delta}{R_0}\right)
\]

(9.21)

Similarly, the non-dimensional shearing stress at the critical height of stone is-

\[
\tau_2 = \frac{\tau_C}{\tau_N} = \frac{\mu}{\beta \left(1 - \frac{3}{4} \frac{\delta}{R_0}\right)^3}
\]

(9.22)

The expression (9.18), (9.21) and (9.22) exhibits the non-dimensional values of shearing stress at the wall, at the throat of stone and at the critical height of stone.
GRAPHICAL REPRESENTATION
Figure 9.3 (a): The variation of resistance to flow with stone height for different values of viscosity $\mu$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure9_3a}
\caption{The variation of resistance to flow with stone height for different values of viscosity $\mu$.}
\end{figure}
Figure 9.3 (b): The variation of resistance to flow with stone height for different values of parameter $\alpha$. 
Figure 9.3 (c): The variation of resistance to flow with stone for different values of stone length.
Figure 9.4(a): The variation of wall shear stress with the axial distance for different values of viscosity $\mu$. 
Figure 9.4 (b): The variation of wall shear stress with the axial distance for different values parameter $\alpha$. 
Figure 9.4 (c): The variation of wall shear stress with the axial distance for different values of stone height (h).
Figure 9.5 (a): The variation of wall shear stress at the throat of constriction with stone height for different values of viscosity.
Figure 9.5 (b): The variation of wall shear stress at the throat of constriction with stone height for different values of parameter $\alpha$. 
Figure 9.6 (a): The variation of wall shear stress at the critical height of stone for different values of viscosity $\mu$. 
Figure 9.6 (b): The variation of wall shear stress at the critical height of stone for different values of parameter $\alpha$. 
Figure 9.7 (a): Variation of wall shear stress at the throat of stone and the critical height with the stone height for different values of viscosity.
Figure 9.7 (b): Variation of wall shear stress at the throat of stone and the critical height with the stone height for different values of parameter \( \alpha \).
9.8 Discussion and Results

The analytical result obtained in equations (9.15), (9.18),(9.21) and (9.22) are evaluated using computer code, in order to observe the sensitivity of bile flow characteristics due to the presence of the overlapping stones, on the geometrical parameters. The value of the parameter are selected as [15, 27]-

\[ L = 3 \text{ cm} \]

\[ L_0 = 1 \text{ cm} \]

\[ \delta/R_0 = 0, .01, .05, .10, .15, .20 \]

\[ \alpha = .90, .95, .98, 1 \]

\[ \mu = .1, .3, .5, 1 \]

Various graphs have been plotted for the above values. Figure 9.3 (a), shows the variation of resistance with the changing stone height with varying viscosity. It is observed that the flow resistance increases with increasing height of stones. Also the resistance to flow decrease as viscosity \( \mu \) decreases (when peripheral viscosity decreases). Figure 9.3 (b), illustrates that for any given parameter \( \mu \),the flow characteristics \( \lambda \) decreases with decreasing values of the parameter \( \alpha \). Figure 9.3 (c), exhibits the variations of flow resistance with stone height for different values of the length of the stone. It is observed that as the stone length increases, the value of \( \lambda \) also increases.

Figure 9.4(a) ,(b) and (c) demonstrates the variation of wall shear stress with the axial distance with different parameters in the region of stone length.It shows that the wall shear stress increases with the increasing viscosity. Moreover, we can say that wall shear stress decreases with the decreasing values of the parameter \( \alpha \). At the axial distance in the region of stone, \( \tau_w \) decreases with the decreasing value of the parameter \( \alpha \),but increases with increasing value of the parameter \( \delta/R_0 \). In the above case, the wall shear stress in the non-uniform region, \( \tau_w \) rapidly increase from its values at \( z=0 \) to the peak
value in the upstream of the first throat at \( z = d + \frac{L_0}{6} \), it then decreases steeply in the downstream of the first throat, occupying minimum value at the critical height, i.e. at \( z = d + \frac{L_0}{2} \). The wall shear stress increases steeply in the upstream of the second stone throat and attains its peak magnitude (with the same value as at the first throat of the stones) at the second throat of the stones, \( z = d + \frac{5L_0}{6} \), it then decreases rapidly to the same magnitude as its value at \( z=0 \) (i.e. at the end point of the constriction profile).

Figure 9.5(a) and (b) illustrates the variation in wall shear stress with stone height at the throat of constriction. It clearly shows the decrease in the value of shear stress as there is decrease in viscosity. On the other hand, stress increases as the stone height increases. Further, it shows the variation of shear stress with stone height for different values of parameter \( a \). As the value of \( a \) decreases, the shear stress at the throat of constriction also decreases.

Figure 9.6 (a) and (b) illustrates the variation in the shear stress at the critical height with stone height for different parameters. The shear stress decreases with the decreasing value of the viscosity and parameter \( a \). As the stone height increases the shear stress also increases.

Figure 9.7 (a) and (b) shows the comparison between the wall shear stress at two points (i.e.) at the throat of constriction and at critical height of stone. It shows the diminishing value of shear stress at the critical height of stone than at the throat of constriction for the value of parameter like viscosity and \( a \). It is easy to say that shear stress at the constrictions critical height assumes significantly lower value (magnitude) than its corresponding value of the shear stress at the throat of stone.


9.9 Conclusion

The effect of peripheral layer viscosity on the flow characteristics of bile in the presence of overlapping constriction (stone) has been investigated. A two layered model of bile, assuming the central core region to be a suspension of bile salts, cholesterol and bilirubin and peripheral layer of mucosa. The characteristics of flow (resistance and wall shear stress) have been studied. The following are the observations from the analysis-

- The flow resistance increases with the increasing stone height.
- As the viscosity increases the resistance to flow also increases.
- The flow characteristics i.e. resistance decreases with the decreasing value of parameter $\alpha$.
- As the length of stone increases, it also increases the resistance to flow.
- The wall shear stress increases with the increase in viscosity of bile.
- As the stone height increases it increases the wall shear stress.
- With the decreasing value of $\alpha$ the wall shear stress also decreases.
- The shear stress at the throat of constriction increases with the increasing value of viscosity. Also, as stone height increases the stress also increases.
- The shear stress at the throat of constriction decreases with decreasing value of $\alpha$.
- The shear stress at the critical height shows an increase with increasing value of stone height and decreases with the decreasing value of viscosity and $\alpha$.
- At the two places of the throat of constriction the wall shear stress shows same magnitude.
- The shear stress at the critical height of the stone assumes significantly smaller value than at the stone throat.

The flow characteristics assume lower value in the two layer model. This observation indicates that the peripheral layer of mucous helps in the functioning of the diseased cystic duct.