CHAPTER 3

EFFECT ON THE RESISTANCE TO THE FLOW OF BILE IN THE CYSTIC DUCT

3.1 Introduction

In this chapter the effect of cystic duct geometry on the flow resistance is studied. The two types of geometry have been taken into account namely cosine shaped and composite shaped. It is assumed that the rheology of bile is characterized by Newtonian fluid. The flowing bile is incompressible, homogeneous, laminar and steady. The formation of stone in the rigid cystic duct is in axially symmetric manner and depends upon the axial distance and the height of its growth. The maximum height of the stone is much less as compared to the length and unobstructed radius of the cystic duct. The role of valves of heister is assumed to be negligible, therefore not contributing in the geometrical configuration of the cystic duct. Moreover, stone formation is assumed to be symmetrical and the flow of bile is in axial direction. Further, the effect of geometry of cystic duct on the flow characteristics like wall shear stress and resistance to flow is analyzed for both geometrical configuration namely-

a) Cosine shaped
b) Composite shaped

3.2 Governing equation

The bile flow is slow, symmetric and steady, the basic equations governing its motion are written in cylindrical co-ordinates as follows [12]

\[
\mu \left[ \frac{d^2 u_z}{dr^2} + \frac{1}{r} \frac{du_z}{dr} \right] = \frac{\partial P}{\partial z}
\]  

(3.1)

The constitutive equation for the Newtonian fluid can be written as [12]

\[
\tau = -\mu \frac{du_z}{dr}
\]  

(3.2)
In equation (3.1) and (3.2), z is the co-ordinate along the axis of the cystic duct in the flow direction, r is the co-ordinate in the radial direction and perpendicular to fluid flow, P is the pressure at any time, \( \frac{\partial P}{\partial z} \) is the pressure gradient which essentially does not vary across the radius, \( u_z \) is the axial velocity of bile and \( \mu \) is the viscosity coefficient and \( \tau \) is the shearing stress.

3.3 Boundary Conditions

The equations (3.1) and (3.2) are solved using the following boundary conditions-

a) \( u_z = 0 \) at \( r = R(z) \) \hspace{1cm} (3.3)

b) \( \frac{du_z}{dr} = 0 \) at \( r = 0 \) \hspace{1cm} (3.4)

c) \( P = P_0 \) at \( z = 0 \) and \( P = P_L \) at \( z = 0 \) \hspace{1cm} (3.5)

Case: I

3.4 Cosine Shaped Geometry

The geometry of the cystic duct (Figure: 3.1) can be written as [30]-

\[
R(z) = R_0 \left[ 1 - \frac{\delta}{2R_0} \left( 1 + \cos \frac{2\pi}{L_2} \left( z - \frac{L_1}{2} \right) \right) \right] ; \hspace{0.5cm} L_1 \leq Z \leq L_2 + L_1
\]

\[
= 1 \hspace{1cm} \text{elsewhere}
\]

(3.6)

Where \( R(z) \) is the radius of the cystic duct in the obstructed region, \( R_0 \) is the radius of cystic duct in the normal region, \( \delta \) is the maximum height of the stone assumed to be much smaller in comparison to the radius of cystic duct \( (\delta \ll R_0) \), \( L_1 \) is the length of cystic duct till the onset of stone, \( L_2 \) is the length of stone and \( L \) is the length of cystic duct.
Figure: 3.1 The cosine shaped geometry of cystic duct.
3.5 Analysis of Model

We solve the equation (3.1) using the boundary conditions (3.3) and (3.4). The axial velocity is given by

$$ u_z = \frac{1}{4\mu} \frac{\partial P}{\partial z} (r^2 - R^2(z)) $$

(3.7)

The flux is given by [12]

$$ Q = \int_0^{R(z)} 2\pi r u_z \, dr $$

(3.8)

Using equation (3.7) the equation (3.8) becomes

$$ Q = -\frac{\pi}{8\mu} \frac{\partial P}{\partial z} \left( R^4(z) \right) $$

(3.9)

The pressure gradient can be obtained by the equation (3.9) as follows

$$ \frac{\partial P}{\partial z} = -\frac{8\mu}{\pi} \frac{Q}{R^4(z)} $$

(3.10)

On integrating equation (3.10) and using condition (3.5) we arrive at

$$ \int_0^L dP = P_L - P_0 = -\frac{8\mu Q}{\pi} \int_0^L \frac{1}{R^4(z)} \, dz $$

$$ \Delta P = P_0 - P_L = \frac{8\mu Q}{\pi} \int_0^L \frac{dz}{R^4(z)} $$

(3.10 a)

3.6 Resistance to flow

Following [5, 30], the resistance to flow is defined as follows

$$ \text{Resistance} = \lambda = \frac{\Delta P}{Q} $$

Using equation (3.10 a), we get
\[
\bar{\lambda} = \frac{8\mu Q}{\pi} \int_0^L \frac{dz}{R^4(z)}
\]

(3.11)

\[
= \frac{8\mu L}{\pi R_o^4} \left[ 1 - \frac{L_e}{L} \right]
\]

\[+ \frac{L_e}{L} \left( 1 - \frac{\delta}{2R_o} \right) \left( 1 - \frac{\delta}{R_o} \right)^{-7/2} \left( \frac{1}{2R_o} \right)^2 \left( \frac{3}{2} \left( \frac{\delta}{2R_o} \right)^2 \right) \]  

(3.12)

In absence of any gallstone, the resistance to flow is given by \(\lambda_N\)

\[
\lambda_N = \frac{8\mu L}{\pi R_o^4}
\]

(3.13)

In dimensionless form, the flow resistance to flow of bile in the rigid axially symmetric cystic duct is given by

\[
\bar{\lambda} = \frac{\bar{\lambda}}{\lambda_N}
\]

\[= \left[ 1 - \frac{L_e}{L} + \frac{L_e}{L} \left( 1 - \frac{\delta}{2R_o} \right) \left( 1 - \frac{\delta}{R_o} \right)^{-7/2} \left( \frac{1}{2R_o} \right)^2 \left( \frac{5}{8} \frac{\delta^2}{R_o} \right) \right] \]

(3.14)

3.7 Shear stress

Using equation (3.7) and equation (3.10) the equation of shearing stress at the wall is given by

\[
\tau_k = \frac{4\mu Q}{\pi R^3(z)}
\]

(3.15)

In absence of any stone i.e. \(R(z) = R_0, \delta = 0\) the shearing stress is given by \(\tau_N\)

\[
\tau_N = \frac{4\mu Q}{\pi R_0^3}
\]

(3.16)

The non-dimensional expression for the shearing stress is given as
\[ \tau_1 = \tau_R = \left( \frac{R_0}{R} \right)^3 = \frac{1}{[1 - \frac{\delta}{2R_0} \{1 + \cos \frac{2\pi}{L_2}(z - L_1 - \frac{L_2}{2})\}]^3} \]  

(3.17)

At \( z = L_1 + \frac{L_2}{2} \), \( \tau_1 = \tau_2 \)

\[ \tau_2 = \frac{1}{(1 - \frac{\delta}{R_0})^3} \]  

(3.18)

The equations (3.17) and (3.18) give the value of shear stress at the wall and at the maximum height of stone.

**Case: II**

**3.8 Composite shaped geometry**

The geometry of cystic duct is composite shaped (Figure: 3.2), which is given by [9]

\[
R(z) = \begin{cases} 
R_0 - \frac{2\delta}{L_2} (z - L_4); & L_1 \leq z \leq L_1 + \frac{L_2}{2} \\
R_0 - \frac{\delta}{2} [1 + \cos \frac{2\pi}{L_2} (z - L_1 - \frac{L_2}{2})]; & L_1 + \frac{L_2}{2} \leq z \leq L_1 + L_2 \\
R_0; & \text{otherwise}
\end{cases}
\]  

(3.19)

Where \( R(z) \) is the radius of the cystic duct in the obstructed region, \( R_0 \) is the radius of cystic duct in the normal region, \( \delta \) is the maximum height of the stone assumed to be much smaller in comparison to the radius of cystic duct \( (\delta << R_0) \), \( L_4 \) is the length of cystic duct till the onset of stone, \( L_2 \) is the length of stone and \( L \) is the length of cystic duct.
Figure 3.2: The composite shaped geometry of cystic duct
3.9 Resistance to Flow

Using the geometry given by equation (3.19) in equation (3.11), the flow resistance to flow of bile in the rigid axially symmetric cystic duct, in dimensionless form, is given by

\[
\lambda = \frac{\lambda}{\lambda_N} = 1 - \frac{L_\|}{L} 
\]

\[
+ \frac{L_\|}{2L} \left[ \left(1 + 2 \frac{\delta}{R_0}\right) \left(1 - 2 \frac{\delta}{R_0}\right) \left(1 - 2 \frac{\delta}{R_0}\right)^{-\gamma/2} \right. 
\]

\[
\left. + \left(1 - 2 \frac{\delta}{R_0}\right) \left(1 - 2 \frac{\delta}{R_0}\right) \left(1 - 2 \frac{\delta}{R_0}\right)^{-\gamma/2} \right] 
\]

\[
= \left(1 - \frac{\delta}{R_0}\right)^{-\gamma/2} \left(1 - \frac{\delta}{R_0}\right)^{-\gamma/2} \left(1 - \frac{\delta}{R_0}\right)^{-\gamma/2} \left(1 - \frac{\delta}{R_0}\right)^{-\gamma/2} 
\]

\[
\] (3.20)

3.10 Shear stress

Using equation (3.7) and equation (3.10) the equation of shearing stress at the wall is given by

\[
\tau_R = \frac{4\mu Q}{\pi R^2(z)} 
\] (3.21)

In absence of any constriction (i.e.) \( \delta = 0 \), the expression for the shearing stress reads

\[
\tau_N = \frac{4\mu Q}{\pi R^2_0} 
\] (3.22)

A non-dimensional expression for the shearing stress is given as

\[
\tau_1 = \frac{\tau_R}{\tau_N} = \frac{1}{(R / R_0)^3} 
\] (3.23)

At \( z = L_\| + \frac{L_\|}{2} \), \( \tau_1 = \tau_2 \)
where

\[ \tau_2 = \frac{1}{(1 - \frac{\delta}{R_0})^3} \]  \hspace{1cm} (3.24)

The equation (3.23) and (3.24) gives the value of shear stress at the wall and at the maximum height of stone.
GRAPHICAL REPRESENTATION
Figure 3.3 (a): The variation of resistance to flow with the stone height for different values of stone length (t) for cosine shaped geometry.
Figure 3.3 (b): The variation of resistance to flow with the stone height for different values of stone length (t) for composite shaped geometry.
Figure 3.4: Flow resistance with stone height in the two different geometry of the cystic duct at different stone length (t).
Figure 3.5 (a): The variation of shear stress with the stone height for cosine shaped geometry.
Figure 3.5 (b): The variation of shear stress with the stone height for composite shaped geometry.
3.11 Discussion and Results

The variation of resistance to flow and the shear stress with the changing stone height and length has been plotted and analyzed for the following set of parameters [15]-

$L_2/L_1=.1,.2,.3$ and $.4$

$R_0=.15$ cm

Figure: 3.3 (a) and (b) illustrate the variation of resistance to flow with the stone height for different values of stone length. It is observed that the resistance to flow increases as the stone height and length increases, for both the geometries i.e. for cosine shaped geometry and for composite shaped geometry.

Figure: 3.4 show the resistance to flow of two different geometry of the cystic duct for with stone height. It shows that the resistance to the flow of bile offered by cosine shaped cystic duct is higher than the resistance given by composite shaped cystic duct.

Figure: 3.5 (a) and (b) demonstrate the variation of shear stress with the stone height, for cosine shaped geometry and for composite shaped geometry of the cystic duct. The graph shows almost the same pattern of shear stress for both the geometry. Thus, the cosine shaped geometry and composite shaped geometry of the cystic duct do not bring any significant change in the value of shear stress.
3.12 Conclusion

This analysis examined the effect of presence of gallstone on the flow behavior of bile in the cystic duct. We have studied the impact of two different geometries of stone on the flow pattern of bile in the cystic duct. The deductions are as follows:

- The resistance to flow increases as the size of stone increases.
- The resistance to the flow of bile is more in cosine shaped geometry of the cystic duct than in the composite shaped cystic duct.
- The shear stress increases with the increase in the height of stone.