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CHAPTER-I:

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INTRODUCTION

Preliminary Remarks:

The subject Numerical Analysis essentially covers all computational methods in its purview and is as old as Mathematics itself [Apostol 1962]. In early ages the numerical methods were used for astronomical calculations, evaluation of certain mathematical entities like π, e etc. But at that time these investigations were simply sporadic and being not systematic did not give rise to an independent mathematical discipline. Later on these methods developed into an independent discipline out of the necessity for knowing the numerical magnitude for a mathematical expression.

The ultimate aim of the subject Numerical Analysis is to develop suitable methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions, which are not expressed in convenient mathematical forms. Such problems may be formulated, for example, in terms of an algebraic or transcendental equation, an ordinary or partial differential equation, an integral or integro-differentiable equation or in terms of a set of such equations which are simply the media of expression in the realm of mathematics.
The gradual development of Physics, Technology or of Science in general takes the form of very complicated mathematical equations describing such processes, so much so that the methods of formal analysis fail often to handle such mathematical problems, but Numerical Analysis do often provide an answer to such problems and to desired degree of accuracy.

The role of Numerical Analysis is thus to provide approximate solutions of complicated mathematical problems whose exact solutions either cannot be found or can be found with much difficulty. The adjective "approximate" has been appended to indicate that the solution obtained by numerical methods may not always agree to the exact solution, but always differs from exactness by less than a specific tolerance.

The development of Numerical Analysis in abstract spaces, or in other words the research on Numerical Functional Analysis is a phenomenon of the last two or three decades. Research on Numerical methods in abstract space have vastly changed the pattern of thinking of a numerical analyst and have yielded a compact and unified representations to many methods which were previously discussed in different forms for different types of equations.
Numerical Analysis in abstract spaces is of abiding interest to Mathematicians, because of its abstract content and is also of immense practical value, since many of its methods can be programmed in an electronic computer to solve many otherwise intractable problems.

The methods of solving equations in abstract spaces can be divided into two classes. The methods of one class are called Direct methods in contrast to "Iterative methods" which are known to belong to the other class. According to Sobolev direct methods are those methods for the approximate solution of the problems of the theory of differential and integral equations which reduce these problems to finite systems of algebraic equations \[\text{Mikhlin, 1964}\]. In many cases the problem of integrating a differential equation is replaced by the equivalent problem of seeking a function which gives a minimum value to some integral. Such problems are known as Variational problems. It happens that many direct methods are conveniently applied not immediately to the differential equation but to the variational problem equivalent to it. Among the approximate methods for the solution of variational problems Ritz method and Galerkin method are well known. To name a few of the other direct methods, are methods of least square and
methods of moments. A powerful method related to direct methods was also developed by Murray Petryshyn, 1962, Murray Petryshyn, 1962, Mikhlin Mikhlin, 1964 in his elegant monograph "Variational methods in Mathematical Physics" has made an elaborate treatment of direct method.

Iterative processes belong to another important class of methods for solving equations. Spoken in its simplest term we are given an equation \( f(u) = 0 \) whose solution in exact form either can not be found or can be found with much difficulty. To solve the equation iteratively means to build up a recurrence equation in which any member say \( u_{k+1} \) of the sequence \( \{ u_k \} \) is expressed in terms of \( u_k \) or other members of the sequence for lower subscripts. Or in other words we build up equations of the form \( u_{k+1} = \phi_k(u_k, u_{k-1}, \ldots, u_0) \), such that, the more value \( k \) takes up, the more \( u_k \) tends to the solution of the given equation Berezin & Zhidkov, 1965. Here \( \phi_k \) depends on \( f \), the number of iterations \( n \) and the elements \( u_0, u_1, \ldots, u_k \). We call the iterative method a first order method if \( \phi_k \) depends only on \( u_k \) and is independent of \( u_0, u_1, \ldots, u_{k-1} \). Furthermore, the method is stationary if \( \phi_k \) is independent of \( k \). \( \phi_k \) is in its simplest form when it is linear.

The development of iterative methods in abstract spaces for operator equations have immensely increased the usefulness of the
Functional Analysis has helped in treating the so-called different methods for different types of equations like matrix equations, differential equations, integral equations from a single uniform viewpoint. Discovery of new iterative methods in abstract spaces for operator equations when interpreted for specific types of equations become meaningful generalizations of existing methods in the particular area.

To speak a few words about the growth of Functional Analysis, it may be said that the essence of Functional Analysis lies in the generalization of the concept and methods of elementary Analysis, as well as of related domains of Algebra and Geometry. Such a generalization permits us to deal with problems previously treated in different manner in different branches of Analysis from a single uniform viewpoint.

The generalization of Geometry side by side with that of Analysis made it possible to find deep analogies between Analysis and Geometry.

A set in which a limit of a sequence is defined is called an abstract space. Abstract spaces whose elements are functions or numerical sequences are called functional spaces. The main object of Functional Analysis is the study of certain classes of operators defined in functional spaces.
Discovery of Iteration:

To trace back the situation that led to the discovery of the iterative principle \(^\text{cf. Whittaker & Robinson, 1937}\), it was originally the school of Hindu Mathematicians who discovered a method for the extraction of the square and cube roots of number, digit by digit. The method was communicated to the Arabs, who transmitted it to Europe. The method was extended by Vieta in 1600, so as to furnish the roots of algebraic equations in general.

It was only in 1674 that the principle of iteration was discovered and was communicated in a letter from Gregory to Collins, and independently a few months later, in a letter from Michael Davy to Newton.

In this connection the algorithm suggested by Newton for the determination of square roots may be described as follows:

Let \( N \) be the number whose square root is required. Let us take any number \( x_0 \) and from it \( x_1 \) is formed according to the equation \( x_1 = \frac{1}{2} \left( x_0 + \frac{N}{x_0} \right) \). From \( x_1 \), \( x_2 \) is formed according to the equation \( x_2 = \frac{1}{2} \left( x_1 + \frac{N}{x_1} \right) \). From
\( x_2, x_3 \) is formed according to the equation 
\[ x_3 = \frac{1}{2}(x_2 + \sqrt{N/2}) \]
and so on. Then it can be proved that, the sequence of numbers 
\( x_0, x_1, x_2, \ldots \), tends to a limit which is \( \sqrt{N} \).

The nature of iterative methods readily admits of a geometrical interpretation. Let \( f(x) = 0 \) be the equation. We write it in the form \( f_1(x) = f_2(x) \) as may easily be done in many ways. We draw the curves \( y = f_1(x) \) and \( y = f_2(x) \); the real roots of \( f(x) = 0 \) are evidently the abscissae of the points of intersection of these two curves. An iterative process for finding them may be devised as follows. We select any point \( x_0 \) on the axis of \( x \) so that the value of \( x_0 \) is nearly equal to that of the abscissa of one of the points of intersection of the curves. From \( x_0 \), draw a straight line, parallel to the axis of \( y \) until it meets the curve which has the slope of lesser magnitude. Suppose, for example, when \( x = x_0 \) that \( |f'(x)| < |f_2'(x)| \) and that the line \( x = x_0 \) meets the curve \( y = f_1(x) \) at the point \( (x_0, y_0) \). From this second point draw a line parallel to the axis of \( x \) until it meets \( y = f_2(x) \) in the point \( (x_1, y_1) \). From the third point draw a line parallel to the axis of \( y \) until it meets the curve \( y = f_1(x) \) in \( (x_2, y_2) \) and from this fourth point a line parallel to the axis of \( x \) and so on. Then the abscissa of the first and the second points is \( x_0 \),
that of the third and the fourth points is $x_1$ and in general $x_n$ approaches nearer to the point of intersection of the two curves for increasing value of $n$ i.e. $x_n$ converges to a root of the original equation. There are two main types of diagrams resulting from this process according as the slopes of the two curves have the same or different signs for the abscissae $x_0, x_1, x_2, \ldots$.

\begin{figure}[h]
\centering
\begin{subfigure}[b]{0.45\textwidth}
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\includegraphics[width=\textwidth]{fig11}
\caption{Fig 11}
\end{subfigure}\hfill
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig12}
\caption{Fig 12}
\end{subfigure}
\end{figure}

Jacobi, Gauss–Seidel and SOR methods:

With the advent of modern digital computer the iterative methods of solving linear algebraic equations or matrix equations have gained much importance. This is because the computer is
able to tackle only a discrete system. Now that under certain conditions linear partial differential equations can be approximated by matrix equations, the problem of solving certain types of linear partial differential equations can be reduced to the problem of solving linear algebraic equations. There is a list of methods for solving linear algebraic equations like, Gaussian elimination method, Gauss-Jordan method, method of Conjugate-Gradient, Gauss-Seidel method, methods of Successive over or under relaxation, the method of Steepest descent and many others [Berezin, I. & Zhidkov, N. 1965].

We discuss in some detail the Jacobi method, Gauss-Seidel method, the methods of Successive over and under relaxation [Varga, 1962] because they are relevant to our study of the linear operator equations. Let \( A = (a_{ij}) \) be a non-singular \((n \times n)\) matrix with all its diagonal elements non-zero. In an effort to solve the matrix equation \( Au = f \) we express \( A \) as \( A = D + E + F \), \( D = \text{diag}\{a_{11}, a_{22}, \ldots, a_{nn}\} \), \( E \) and \( F \) are respectively lower and upper triangular \((n \times n)\) matrices. Starting from \( u^{(0)}_i, i = 1, 2, \ldots, n \) some initial estimates of the components of the unique solution \( u \) of \( Au = f \), the sequence \( \{u^{(m)}_i\}, i = 1, 2, \ldots, n \) are defined as

\[
u_i^{(m+1)} = - \sum_{j=1, j \neq i}^{n} \left( \frac{a_{ij}}{a_{ii}} \right) u_j^{(m)} + \frac{f_i}{a_{ii}}, \quad 1 \leq i \leq n, \quad m \geq 0.
\]
In matrix notation the above system of equations becomes,

\[ u^{(m+1)} = -D^{-1}(E+F)u^{(m)} + D^{-1}f, \quad m \geq 0 \quad (1) \]

The matrix \( B = -D^{-1}(E+F) \) is called the point-Jacobi matrix associated with and the above \( (1) \) is known as, the point Jacobi method or point full step iterative method.

The iterative method known as point Gauss-Seidel or point one step iterative method is given by

\[ u^{(m+1)} = -(D + E)^{-1}Fu^{(m)} + (D + E)^{-1}f, \quad m \geq 0 \quad (2) \]

and \( \{u^n\} \) are prechosen.

The matrix \( B = -(D + E)^{-1}F \) is called the point Gauss-Seidel matrix associated with .

Setting \( L = -D^{-1}E \) and \( U = -D^{-1}F \) the point successive iterative method is given by,

\[ u^{(m+1)} = (I - \omega L)^{-1}\{(1 - \omega)I + \omega U\}u^{(m)} + \omega (I - \omega L)^{-1}D^{-1}f, \quad m \geq 0 \quad (3) \]

and \( \{u^n\} \) are prechosen.
\( \omega \) is called the relaxation factor. The method is known as successive overrelaxation or under relaxation according as \( \omega > 1 \) or \( \omega < 1 \). If \( \omega = 1 \) the above method reduces to the Gauss-Seidel iterative method.

The above class of methods can be unified into a general iterative method of the form

\[
\begin{align*}
u^{(m+1)} &= M^{-1}N u^{(m)} + M^{-1} f, \\
m > 0, \{ u^0 \} & \text{ are prechosen}
\end{align*}
\]

and \( A = M - N \).

Writing \( T = \hat{M}^{-1}N = M^{-1}(M - A) = I - M^{-1}A \), the sequence \( \{ u^{(m+1)} \} \) can be written as

\[
\begin{align*}
u^{(m+1)} &= Tu^{(m)} + M^{-1} f \\
&= T(Tu^{(m-1)} + M^{-1} f) + M^{-1} f \\
&= T^2u^{(m-1)} + TM^{-1} f + M^{-1} f \\
&\vdots \\
&= T^{m+1}u^{(0)} + T^mM^{-1} f + T^{m-1}M^{-1} f + \ldots + M^{-1} f \\
&= T^{m+1}u^{(0)} + \sum_{i=0}^{m} T^i (M^{-1} f) 
\end{align*}
\]
Here \( T \) is a finite matrix. We know that if spectral radius of \( T \) is less than one or \( \rho ( T ) < 1 \), then \( T^n \to 0 \) as \( n \to \infty \) and \( ( I - T )^{-1} = \sum_{l=0}^{\infty} T^l \) \( \leq \) Varga, 1962.

Thus if \( \rho ( T ) < 1 \),

\[
\lim_{m \to \infty} u^{(m+1)} = ( I - T )^{-1} f
\]

or \( ( I - T ) u^* = M^{-1} f \)

or \( ( I - M^{-1} N ) u^* = M^{-1} f \)

or \( ( M - N ) u^* = Au^* = f \),

i.e., \( \{ u^{(m+1)} \} \) converges to the unique solution of the equation \( Au = f \).

Varga in his excellent monograph \( \leq \) Varga, 1962 \( \leq \) has made an elaborate treatment of Point-Jacobi, Gauss-Seidel and successive overrelaxation iterative methods. He has made an extension of the concept of \( M \)-matrix in the real field (i.e., the real matrix with negative off-diagonal elements and with \( M^{-1} \geq 0 \)) to the \( H \)-matrix in the complex field and along with Alefeld, G., proved some convergence theorems for the
symmetric successive over-relaxation method \cite{Varga, 1976}. By introducing the concept of generalized diagonal dominance K. R. James and W. Riha proved some convergence theorems for successive overrelaxation \cite{James & Riha, 1975}.

One of the most powerful methods of solving an equation $f(x) = 0$ where $f(x)$ is a real or complex function, is known as the Newton-Raphson method \cite{Berezin & Zhidkov, 1965}. Let the function $f(x)$ be continuously twice-differentiable in the closed interval $J$ and let in particular $f'(x)$ does not vanish in $J$. Starting from $x_i \in J$, an initial approximation to the solution

\begin{figure}[h]
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\includegraphics[width=\textwidth]{fig13}
\caption{Fig. 13}
\end{figure}
the method of setting the sequence \( \{ x_n \} \) as

\[
\begin{align*}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)}, \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)},
\end{align*}
\]

is known as the Newton-Raphson method. The method has a simple geometrical interpretation. The value of \( x_{n+1} \) is the abscissa of the point of intersection of the tangent to the curve \( y = f(x) \) at the point \( x = x_n \) with the \( x \)-axis (Fig. 1.3). The Newton-Raphson method is therefore often known as the tangent method. For solving systems of non-linear equations Newton-Raphson method, method of steepest descent etc. can be used [Berezin & Zhdikov, 1965].

Fixed point theorems have a great role in proving the existence of the solution of equation or equations. Those fixed point theorems which not only guarantee the existence of solution but also indicate methods of finding them out are known as constructive fixed point theorems. In a later stage we deal in some details about the fixed point theorems.

**Iterative methods for bounded linear operator equations in abstract spaces:**

Denoting \( M^{-1} \) by \( P \) the iterative method (4) discussed earlier takes the form
Many authors have studied prototypes of the above iterative procedure, either in \( \mathbb{R}^n \) or in abstract spaces like Hilbert and Banach spaces (Bialy, (1959); Keller, (1968); Petryshyn, (1963(i)); Rall (1965); etc. all \( \mathcal{J} \) for the solution of the equation \( Au = f \). The method was thoroughly treated by Householder (1965) \( \mathcal{J} \) for matrix equations. Many authors Neumann (Courant & Hilbert (1931) \( \mathcal{J} \), Wiarda (Petryshyn, 1963(ii) \( \mathcal{J} \), Bückner (Petryshyn, (1963(ii)) \( \mathcal{J} \) discussed methods of the type (6) for solving Fredholm integral equation. Young (Young, (1954) \( \mathcal{J} \) discussed the method for solving elliptic partial differential equation. In a Hilbert space setting Petryshyn defined the concept of K-positive definiteness (Petryshyn, (1962(i)) \( \mathcal{J} \) in order to get a guarantee that a K-positive definite operator has a bounded inverse. Petryshyn studied this method in his dissertation (Petryshyn, 1962(i)) \( \mathcal{J} \) and lately in a series of papers (Petryshyn (1962(ii));(1963(i)); (1965);(1970) \( \mathcal{J} \). In a Banach space Petryshyn (Petryshyn,1965(ii) \( \mathcal{J} \)
considered a method which is somewhat more general than the method (6). He proves that if for every $f$ in a Banach space $X$, equation $Au = f$ possesses a unique solution $u^*$ in $X$, then the sequence of iterates $\{u^{(m+1)}\}$ determined by the process.

$$C u^{m+1} = Lu^m + f, \quad m = 0, 1, 2,$$

where $C$ is a continuously invertible operator, $B$ a bounded linear operator, $T = C^{-1}L$, $L = C - BA$, $f_0 = Bf$, converges to the solution $u^*$ for any initial approximation $u_0$ in $X$ if and only if $\sum_{n=0}^{\infty} T^n \varrho$ converges for every $\varrho$ in $X$.

In this way he has unified a number of methods for the solution of matrix equations, integral equations.

Koliha has discussed the iterative procedure (6) in a Hilbert space where $A$, the restriction of $A$ to a subspace $\mathcal{Y}$ of $\mathcal{H}$, has an inverse in $\mathcal{Y}$ which is not necessarily bounded and discusses the "total convergence" of the sequence $\{u^n\}$ to a solution of the equation (not necessarily to the same limit) for every $u^* \in \mathcal{H}$ and for a fixed $f \in \mathcal{H}$. Koliha has considered singular matrix equations.

Let $X$ be a Banach space and $A$ a bounded linear operator from $X$ to $X$. Suppose we want to solve $u - Au = f$ by the
Browder \cite{Browder, 1958} studied the existence of a solution of the above equation for a reflexive Banach space and $A$ asymptotically bounded i.e. $(\|A_k\| \leq M$ for some $M > 0$ and all $k \geq 1$). In a later paper Browder and Petryshyn \cite{Browder & Petryshyn, 1966(i)} studied the same equation when $A$ is asymptotically convergent i.e. $A^k u$ converges in $X$ as $k \to \infty$ for each $u$ in $X$. Koliha \cite{Koliha, 1973, 1974} has defined a bounded linear operator $A$ mapping $X \to X$ as convergent if $\sum A^n$ converges, power convergent \cite{Koliha, 1974} if the sequence $\{A^n\}$ converges, power regular if $\{A^n - A^{n+1}\} \to 0$. (The convergence is understood in one of the usual operator topologies, uniform, strong or weak). Koliha has given a characterisation of uniform convergent and uniformly power convergent operators on Hilbert space \cite{Koliha, 1973}. He has shown \cite{Koliha, 1974} for a bounded linear operator $A$ mapping $X \to X$ the following conditions are equivalent.

1) $A$ is a strongly and absolutely convergent;

2) $A$ is uniformly convergent;

3) $A^n \to 0$ uniformly;
iv) \[ \left\| A^p \right\| < 1 \] for some positive integer \( p \).

v) \[ \rho(A) = \lim_{n \to \infty} \left\| A^n \right\|^{1/n} < 1 \]

vi) \( \sigma(A) \) i.e. the spectrum of \( A \) lies in the open unit disc.

In framing our convergence theorems we are motivated by some of the above generalizations and have tried to give partial generalizations to some of those concepts in a supermetric space.

For more methods of solving bounded linear operator equations we may mention that Altman has used the orthogonal projection theorem in a Hilbert space \( \overline{\text{Lusternik & Sobolov, (1965)}} \) to build a sequence of iterates for the solution of linear algebraic equations in a finite dimensional linear space \( \overline{\text{Altman, (1957(1))}} \). Sen \( \overline{\text{Sen, (1965)}} \) has extended the method to \( l_\infty \), which is of infinite dimension.

Polsky \( \overline{\text{Polsky, (1962)}} \) has considered the orthogonal projection of a Hilbert space \( H \). onto its closed subspace \( M_n \). According to him a sequence of closed subspaces \( M_n \) of \( H \) will be called projectionally complete if \[ \left\| f - P_n f \right\| \to 0 \] as \( n \to \infty \) for every \( f \) in \( H \), where \( P_n \) is the
orthogonal projection of $H$ onto $M_n$. Polsky has used projectional methods for the solution of operator equations in Hilbert spaces.

According to Petryshyn (1967), a reflexive Banach space $X$ is said to possess the property $(\mathcal{R})_C$ if the space admits of a sequence \{${X_n}$\} of finite dimensional subspaces $X_n$ of $X$ and a constant $C > 0$ such that $P_n X = X_n$, $X_n \subseteq X_{n+1}$, $n = 1, 2, 3, \ldots$, $\bigcup_{n=1}^{\infty} X_n = X$, $\|P_n\| \leq C$, $n = 1, 2, \ldots$, $P_n P_j = P_j$ for $n \geq j$.

The criterion for the existence and uniqueness of solutions of linear functional equations have been investigated by Petryshyn by applying projection methods (Petryshyn, 1967), Gavurin (Gavurin, 1971) has devoted a chapter on "projectional methods" in his book on computational methods.

The method of steepest descent for a certain class of linear functional equations was studied by Kantorsvich (Todd, 1962).

For a symmetric and positive definite matrix of finite order, the method with minimal residuals has been intensively studied by Krasnoselskii and Krein (Krasnoselskii & Krein, 1952).
Iterative method for unbounded linear operator equations:

Kantorovich [Petryshyn, (1962(i))] has considered the gradient method for equations of the form $A u = f$, where $A$ is an unbounded self-adjoint positive-bounded below operator. Petryshyn [Petryshyn, (1962(ii))] further generalized the method to unbounded $k$-symmetric and k.p.d operators.

Petryshyn [Petryshyn, (1965)] generalized the method to operators which are densely defined linear and unbounded in the real Hilbert space $H$ and which are neither symmetric nor $k$-symmetric. Petryshyn has also considered the iterative method which is applicable to the class of operators forming acute angles studied by Sobolevesky [Petryshyn, (1965)].

Sen [Sen, (1968)] has extended his method [Sen, (1965)] for solving bounded linear operator equations in $H$ to an unbounded linear operator equation in a separable Hilbert space. He has applied his method in solving torsion problems for rods of rectangular or trapezoidal cross-section [Sen, (1970)].

Altman has written a number of papers [Altman, 1957 (ii), (iii), (iv), (v), (vi)]; (1961(i),(ii),(iii),(iv),(v)) offering various methods of solving linear functional equations.
A very powerful theorem which forms the basis of several
iterative procedures in almost every branch of Applied Mathematics
is the fixed point theorem. Given an operator equation \( u = Au \)
this theorem asserts under some specific conditions satisfied
by \( A \) the existence of at least one solution \( u \) of the
equation \( u = Au \). Fixed point theorems can be broadly divided
into two classes, (i) topological fixed point theorems and
(2) algebraic or constructive fixed point theorems. Theorems
of Topological type are strictly existence theorems i.e. they
establish conditions under which a fixed point exists but they
do not provide a method for finding it. On the other hand theorems
of constructive type give an iterative method of finding a fixed
point.

The prototype of most constructive fixed point theorems is
the contraction mapping theorem or principle due to Banach
\[ \text{Lusternik & Sobolev, (1965)} \] Theorem. In a complete metric
space \( X \), given an operator \( A \) which takes the elements
of the space \( X \) again into the elements of this space. Further,
for all \( x \) and \( y \) in \( X \), let \( \rho(A(x), A(y)) \leq \alpha \rho(x, y) \)
with $0 < \alpha < 1$ and not depending on $x$ and $y$. Then, there is a unique point $x_0$ such that, $A(x_0) = x_0$. The point $x_0$ is called a fixed point of $A$. A useful generalization of the previous theorem \cite{kolmogorov1954} states as follows:

If $T$ is an operator mapping a complete metric space $X$ into itself and if $T^n$ is a contractive mapping for some $n$ (where $n$ is a positive integer), then $T$ has a unique fixed point.

If in particular $T$ satisfies the condition $\rho(Tx, Ty) \leq \alpha \{ \rho(x, Tx) + \rho(y, Ty) \}$ for all $x, y \in X$ then $T$ is said to be a nonexpansive mapping \cite{ortega1970}. A fixed point theorem for nonexpansive mapping was given by Browder and Petryshyn \cite{browder1966} for uniformly convex Banach spaces and for more general domains by Kirk \cite{kirk1965}, Belluce and Kirk, \cite{belluce1966}. In a paper \cite{kirk1975} Kirk has studied the existence of fixed points for nonexpansive mappings satisfying certain boundary conditions.

Kannan \cite{kannan1969} has proved a result as follows:

If $T$ is a map of a complete metric space $E$ into itself and if $\rho(Tx, Ty) \leq \alpha \{ \rho(x, Tx) + \rho(y, Ty) \}$ where $x, y \in E$ and $0 < \alpha < 1/2$. Then $T$ has a unique fixed point in $E$. 
Kannan has proved many other fixed point theorems / Bharucha-Reid, (1976).

The Schauder's fixed point theorem which is a generalization of Brouwer's fixed point theorem / Bharucha-Reid, (1976) is a topological fixed point theorem. On the other hand Krasnosel'skii's fixed point theorem / Krasnosel'skii, (1955) is an interesting combination of Schauder theorem and contraction mapping principle. Hardy-Rogers, Chi Song Wong have written a number of papers on fixed point theorems / Hardy-Rogers, (1973); Chi Song Wong, (1974); (1976). An incredibly large number of papers on fixed point theorems proving the existence of solutions of equations has been written. We do not enlist them here because our interest is more in finding the solution than in simply proving its existence.

Iterative methods of solving operator equations in general:

One of the most fundamental methods of solving linear or nonlinear equation \( P(x) = 0 \) is Newton's method. The principle underlying Newton's method in the case of a nonlinear equation is that of reducing it to a sequence of linearized local approximating equation. Fine / Fine, (1916) seems to be the first to prove the theorem in \( n \)-dimensional space. Ostrowski
Ostrowski, (1936) unaware of Pine's discovery presented independently new convergence theorems and error estimates.

Kantorovich, (1939) was the first to give theorems for equation in a space with norm in a partially ordered space. But more fundamental theorems came in 1948 Kantorovich, (1948(i),(ii)) in the setting of a Banach space. Mysovskii Kantorovich & Akilov, (1959) is a notable contributor to Newton's method. Among others who have significant contributions to Newton's method, mention may be made of Rall, (1961(i)); (1961(ii)); (1966); (1974); (1976) Dennis, (1968); (1969); (1970) Collatz, (1966(ii)) Schroder, (1956); (1957), Rheinboldt and Ortega, (1971) to name a few. Moore, (1968); Anselone & Moore, (1964) has considered Newton's method in the context of solving approximately nonlinear integral equations. Sen, (1966); Sen, (1973(i)); (1973(ii)); (1978(i)) has also investigated Newton's method. In papers Sen, (1973(i)); (1973(ii)), he has sought to unify and generalize a list of Newton-like methods. Rheinboldt and Ortega, (1970) in their excellent monograph has listed more than three hundred references of Newton's method.

There are several other iterative methods for solving functional equations both linear and nonlinear like methods of
steepest descent \( \text{Vainberg, (1960)} \), method of over-relaxation \( \text{Collatz, (1966(1))} \), method of regular falsi \( \text{Collatz, 1966(i)} \), method of nonlinear Gauss-Seidel, non-linear successive over-relaxation methods \( \text{Rheinboldt & Ortega, (1970)} \), continuation methods \( \text{Rheinboldt & Ortega (1970); Rheinboldt, (1975); Dennis, (1971)} \), Broyden's methods \( \text{Byrne & Hall, (1975)} \), contractor theory by Altmann \( \text{Byrne & Hall (1973)} \). Collatz has made an intensive study of monotonically decomposable operators, i.e. operators which can be expressed as the sum of an isotone operator and an antitone operator. His school has successfully applied the theory to solve many non-linear p.d.e. equations \( \text{ordinary & partial differential equations (1974)} \) and non-linear integral equations \( \text{Hall, (1971)} \).

Krasnosel'ski and others \( \text{Krasnosel'ski, et.al, (1969)} \) in their monograph on approximate solution of operator equations have given an elaborate reference on iterative methods. Other references are \( \text{Antonosiewicz, (1968); Goldstein, (1965); Keller, (1968); Lika, (1965); Ben-Israel, (1965); (1966); Browder & Petryshyn 1966(i),(ii); Rheinboldt (1974); Sen, (1971);(1978)(ii); Chakravorty (1978); Chatterjee, (1972); Neogi, (1973), (1977)} \). Neogi has written a number of papers on iterative methods of solving singular nonlinear integral equations.
Many authors like Browder \cite{Browder1963}, \cite{Browder1965}, \cite{Browder1967}, Minty \cite{Minty1962}, \cite{Minty1963}, Vainberg and Kachurovsky \cite{Vainberg1959}, Petryshyn \cite{Petryshyn1962}, \cite{Petryshyn1966}, \cite{Petryshyn1967}, \cite{Petryshyn1968}, Edmunds \cite{Edmunds1967}, have profoundly contributed towards the theory of solvability of nonlinear operator equations. We do not have any scope of discussing at length over their contributions.

Object and Scope:

The thesis as the title suggests offers some iterative methods of solving operator equations in some abstract spaces. The order followed is that, exclusively bounded linear operator equations in a supermetric space come first. Next come the bounded linear operator equations in a Banach space. Then comes the semi-bounded linear operator equation in a Hilbert space. In this content concrete application of the method supported by numerical computation appears. Next in order appears the study of nonlinear operator equation in a metric space. In the last chapter we have studied the operator equation in general.

The object of this thesis is broadly speaking to widen and extend the areas of applicabilities of some iterative methods.
for linear operator equations in different abstract spaces and to offer certain generalizations. The convergence theorems for nonlinear equations in metric spaces are intended to solve those class of bounded nonlinear operator equations for which Banach's contraction mapping principle is of no help. The type of nonlinear equations which could be treated as above is also indicated. In considering the Picard-Poincare' Neumann type of iterations for operator equations, apart from offering some convergence theorems in metric spaces our object is to build up some general theorems which can accommodate some standard methods like Newton's method or its variants in a general setting.

A wide class of iterative methods in a Banach space for bounded linear operator equations need for their convergence the value of the spectral radius of a certain operator to be less than one. In the setting of a supermetric space we have found an analogue of such a condition for the convergence of iterative method and have found more and more relaxed conditions of convergence. That some of the theorems do reduce to standard theorems in Banach spaces have been exhibited and a linear integral equation in $C(0,1)$ has been numerically solved by the method. The above is in relation to the content of chapter I, section 1. In the next section we have treated a more general operator...
The application of the semi-inner product in an iterative method in a Banach space justifies a separate chapter being devoted to it. It raises the question as to whether a semi-balance of Hilbert space type treatment could be reduced to equation in Banach spaces.

In the case of semi-bounded invertible operator equation in a Hilbert space, the operator has been converted to a bounded linear operator equation and one of the standard methods of solving bounded linear operator equations has been applied. The computation for a torsion problem has some redeeming features.

Kannan's criterion for the existence of a unique fixed point of an operator in a complete metric space has been used to obtain the solution of a class of nonlinear problems. We are unaware of any such attempt made earlier.

While discussing the operator equations in general we considered the Picard-Poincaré-Neumann type of iteration. The theorem in a partially
ordered topological linear space tries to utilize the idea of linearization of a given operator and has demonstrated that convergence of Newton's method can also be looked upon as the consequence of the theorem.

All that have been said earlier can be taken as our justification of the object of our thesis. The scope of the thesis is mainly two fold. On one hand one can generalize the theorems of bounded linear operators to more abstract spaces and on the other hand one can find the types of operators for which the methods are suitable from the standpoint of concrete applications. Section 2, chapter I, already contains a discussion to this effect.

The method for semi-bounded linear operators can be applied to p.d.e. equations having variable coefficients.

The methods of chapter IV will be useful for nonlinear integral equations. Infact, the example we have considered Chapter IV, § 1.3 does not satisfy the criterion of monotone decomposition as discussed by Collatz Rall, 1971.
A brief survey of the main results:

This thesis consists of four chapters. The first chapter contains two sections. The second and third chapters have each within it one section only. The fourth chapter contains also two sections.

In all the chapters the iterative methods of finding the approximate solutions of various types of operator equations have been studied and the application of the various methods of solving particular problems have also been studied. The principle followed in arranging the chapters has been that the methods of solving exclusively bounded linear operator equations have been given first preference and then come the methods of solving unbounded linear operator equations and then the methods of solving non-linear operator equations and the operator equations in general.

In section 1 of chapter I, we have considered the iterative methods for the approximate solution of bounded linear operator equation \( Au = f, \ f \in \mathbb{R} \) in a complete linear supermetric space \( \mathbb{R} \). Where \( A \) is a bounded linear operator mapping \( \mathbb{R} \) into \( \mathbb{R} \).
For this purpose we have considered a known bounded linear operator $P$ such that $\mathcal{D}(A) \supseteq \mathcal{R}(P)$. Here the iterative sequence $\{f_n\}$ is taken in the form

$$f_n = f_{n-1} - Ap_{n-1}, f = f_0, n = 1, 2, \ldots \quad (\cdot).$$

A number of theorems has been provided to show the convergence of the iterative sequence to the unique solution of the given equation, under different conditions, some of those are as follows:

**Theorem 1.** Vide section 1 chapter I, theorem 1.2.1. Let the following conditions be fulfilled:

(i) $\rho(Au, Av) \geq \alpha \rho(u, v)$ for all $u, v \in \mathbb{R}, \alpha > 0$;

(ii) $\rho(Au, APu) \leq \beta \rho(u, v), \beta < 1$ for all $u \in \mathbb{R}$;

(iii) $\sum_{i=0}^{\infty} P f_i$ is defined and belongs to $\mathbb{R}$; then the equation $Au = f$ has a unique solution $u^*$ given by

$$u^* = A^{-1} f = \sum_{i=0}^{\infty} P f_i$$

and the error estimates are given by

$$\rho(u^*, u_n) \leq \beta / \alpha \rho((I - AP)^{n-1} f_0, \theta) = \beta / \alpha \rho(f_{n-1}, \theta).$$
Theorem 2. Let the following conditions be fulfilled:

(i) \( \rho (A u, \theta) \geq m_1 \rho (u, \theta), \quad m_1 > 0 \) for all non-null \( u \),

(ii) \( \rho (A \varphi, u) \geq m_2 \rho (u, \theta), \quad m_2 > 1 \) for all non-null \( u \),

(iii) \( (1 - A \varphi)^n f - (1 - A \varphi)^{n+1} f \rightarrow 0 \) as \( n \rightarrow \infty \),

(iv) \( \sum_{n=0}^{\infty} \varphi_n \), exists and belongs to \( \mathbb{R} \); then the equation \( Au = f \) has the unique solution \( u^* \) given by

\[ u^* = A^{-1} f = \sum_{n=0}^{\infty} \varphi_n , \quad f_0 = f \]

and the error estimates are given by

\[ \rho (u^*, u_0) \leq 1/m_1 \rho ((1 - A \varphi)^n f, \theta) = 1/m_1 \rho (f_n, \theta) . \]

Theorem 3. Let the following conditions be fulfilled:

(i) \( \rho (A u, \theta) \geq m_1 \rho (u, \theta), \quad m_1 > 0 \), for all \( u \in \mathbb{R} \);

(ii) \( \rho ((1 - A \varphi)^n u, (1 - A \varphi)^n v) \leq \rho (u, \theta) - \rho (v, \theta) \) for all \( u, v \in \mathbb{R} \), the equality valid: only when, \( \rho (u, \theta) = \rho (v, \theta) \).
(iii) \( \sum_{l=0}^{\infty} P_j \) exists, belongs to \( \mathbb{R} \); then the above equations admits of a unique solution given by

\[ u^* = A^{-1} j = \sum_{l=0}^{\infty} P_j . \]

The error estimates are given by

\[ \rho(u^*, u_m) \leq 1/m_1 \rho((1-AP)^m j_0, \theta) = 1/m_1 \rho(j_0, \theta). \]

In case the supermetric space reduces to a Banach space, we define the \( \| \cdot \| \) as \( \| j \| = \rho(j, \theta) \).

At the end of this section, it has been shown that the equation \( Au = f \) can be solved when \( A \) is a finite matrix, which can be expressed as the sum of two matrices \( A_1 \) and \( A_2 \) such that \( A = A_1 + A_2 \) and \( \| A_1 A_2^{-1} \| < 1 \). \( P \) is taken as \( A_2^{-1} \).

The criterion of convergence is \( \| I - AP \| < 1 \) i.e. \( \| 1 - (A_1 + A_2)A_2^{-1} \| < 1 \).

Also, it has been shown that when \( A \) is an integral operator of the form \( A = I - (A_1 + A_2) \), where \( A_1 \) and \( A_2 \) are also integral operators, the equation can be solved by taking \( P = (I - A_1)^{-1} \).
Lastly a linear Fredholm integral equation of second kind i.e. the equation \( u(t) = \int_0^1 \frac{u(t) \, dt}{s - \cos 2\pi (b + t)} = \sin 2\pi b \quad \text{for} \quad 1/2 \leq b \leq 1 \)
\( = 0 \quad \text{for} \quad 0 \leq b \leq 1/2 \) \( n \in C(0,1) \) has been solved numerically under given conditions. Here no extra effort has been made to start the iteration.

In section 2 of chapter I we have generalised the above iterative process of solving equation \( Au = f \) in a complete supermetric space, to the iterative process

\[
\hat{f}_n = (I - (AP)^m) \hat{f}_{n-1}
\]

where \( \hat{f} = \hat{f}_0 \) and \( n = 1, 2, \ldots \) and \( m \) is a positive integer. Here we start the iteration with \( \hat{f} = \hat{f}_0 \). A number of theorems has been derived to show the existence of unique solution of the equation \( Au = f \), under different conditions. We state below two of such theorems.
Theorem 4. Let the following conditions be fulfilled:

(i) \( \rho(Au, Av) \geq \alpha \rho(u, v), \quad \alpha > 0 \) \quad \text{for all} \quad u, v \in \mathbb{R};

(ii) \( \rho(u, Apv) \leq \beta \rho(u, v), \quad 0 < \beta < 1 \) \quad \text{for all} \quad u \in \mathbb{R};

(iii) \( \lim_{m \to \infty} \rho^m f_0, v \) \quad \text{is defined and belongs to} \quad \mathbb{R}, \quad m \geq 1;

(iv) \( \lim_{n \to \infty} (I - (Ap)^m f_0, v) = T f_0 \), \quad \text{is defined in the space};

then the equation \( Au = f \) has a unique solution

\[
u^* = \sum_{i=0}^{\infty} \rho(\mathbb{R}^{m-1} f_0, v), \quad m \geq 1
\]

and the error estimates are given by

\[
\rho(u^*, u_n) = \beta/\alpha \rho\left(\mathbb{R}(I - (Ap)^m f_0, v)\right) = \beta/\alpha \rho\left(\mathbb{R}(f_0, v)\right).
\]

Theorem 5. Let the following conditions be fulfilled:

(i) \( \rho(Au, Av) \geq \alpha \rho(u, v), \quad \alpha > 0 \) \quad \text{for all} \quad u, v \in \mathbb{R};

(ii) \( \lim_{n \to \infty} (I - (Ap)^m f_0) = T f_0 \); 

(iii) \( \rho(Tu, v) < \rho(u, v) \) \quad \text{for non-null} \quad u \in \mathbb{R};
(iv) \( \sum_{\nu=0}^{\infty} p(\alpha \nu^{m-1}) \hat{f}_{\nu} \) is defined in the space and belongs to \( \mathbb{R} \), then the equation \( Au = f \) admits of a unique solution

\[
U^* = \sum_{\nu=0}^{\infty} p(\alpha \nu^{m-1}) \hat{f}_{\nu}, \quad m \geq 1.
\]

The error estimates are given by

\[
\rho(U^*, U_n) \leq \frac{1}{\alpha} \rho(\hat{f}_{\nu+1}, \Theta) = \frac{1}{\alpha} \rho((1-(\alpha \nu)^m)\hat{f}_{\nu}, \Theta).
\]

From the iterative sequence (9) it is clear that when \( m=1 \), then (9) reduces to the iterative sequence that has been discussed in the section 1.

Moreover it has been shown that when the space \( \mathbb{R} \) is a real Hilbert space and \( A \) is a self-adjoint positive definite bounded below operator i.e. \( (Au, u) \geq \alpha (u, u) \), where \( \alpha > 0 \), then the iterative sequence (9) has a special significance. Under the above conditions of the space and operator, there must exists a positive square root \( B \) of the operator \( A \). It can be shown that \( B \) is one-to-one and \( B^{-1} \) exists and is bounded. Hence, if in our discussion we take \( P \) as \( B^{-1} \) then from our iterative scheme in section1 we have

\[
\hat{f}_{\nu} = (I - A^{1/2})^n \hat{f}_{\nu}, \quad n \geq 1.
\]
Normally it is very difficult to find out square root of an operator. But then, we may take the help of the iterative scheme (9). According to this sequence,

\[ f_n = (1 - (\lambda^{1/2})^m) f_0 \]  \hspace{1cm} (11).

If we take \( m = 2 \), then the above sequence reduces to the form

\[ f_n = (1 - A)^n f_0, \] which is a simpler one.

In chapter II, we have considered the solution of a bounded linear operator equation \( Au = f \) in a smooth real Banach space with a basis by considering projector in a Banach space \( \mathcal{X} \). The absence of an inner-product in a Banach space has been partially met by semi-inner-product. The iterative sequence is of the form

\[ f_n = f_{n-1} - \omega_n^{(s)} P_n f_{n-1}, \] \hspace{1cm} (12)

where \( P_n f_{n-1} = \left[ \frac{f_{n-1} - a_n}{a_n} \right] a_n \) and \( \omega_n^{(s)} \) are scalars.

In this chapter we have shown that the iterative sequence \( \{ f_n \} \) is strongly convergent and converges to \( f^1 \) (say), if \( \omega_n^{(s)} \) are so chosen, that we have,

\[ \left\| f_n - \omega_n^{(s)} \left[ \frac{f_{n-1} - a_n}{a_n} \right] a_n \right\| \leq \left\| f_{n-1} \right\|, \quad n = 1, 2, \ldots \]
Again starting from $f^{(1)}$ we build up the sequence $\{f^{(i)}\}$ as follows:

$$f^{(i)} = f^{(i-1)} - \omega^{(i)} P f^{(i-1)}$$

(13)

where $f^{(1)} = \omega^{(1)}$, $i = 1, 2, \ldots$; which again strongly converges to $f^{(2)}$ (say). In general, starting from $f^{(1)} = f^{(1)}$, the sequence $\{f^{(i)}\}$ is formed, so that

$$f^{(i)} = f^{(i-1)} - \omega^{(i)} P f^{(i-1)}$$

(14)

converges to $\theta$ as $j \rightarrow \infty$ if $\lim_{j \rightarrow \infty} \omega^{(j)}$ exists for all $i$ and the limit if there be any, is non-zero.

Lastly, it is shown that the sequence $\{u^{(k)}_k\}$, $k = 0, 1, 2, \ldots$ is strongly convergent as $\eta \rightarrow \infty$. For a $\lambda \cdot \beta \cdot \alpha \cdot \delta$ operator $A$, the equation $Au = f$ admits of a unique solution $u^*$. The error estimates at different levels are also shown. Here also the initial approximation to the solution is not required to start the iteration.

In chapter III, we have dealt with an iterative method of solving linear equations with unbounded invertible operators in a
separable complex Hilbert space $H$, with inner-product $(,)$ and norm $\|\cdot\|$. 

Our problem is to solve approximately the equation $Au = f$, where $f$ is a given element of $H$ and $A$ is in general an positive-bounded-below operator. The domain $\mathcal{D}(A)$ of $A$ is dense in $H$. First, we extended the operator $A$ to a self-adjoint continuously invertible operator $A_0$ by Friedrich's method. It has been shown that domain of definition $\mathcal{D}(A_0)$ of $A_0$ is also dense in $H$ and $A_0$ is continuously invertible.

Now we have to solve approximately the equation $A_0u = f$, where $f$ is any element of $H$. Furthermore we have assumed that there exists a bounded linear self-adjoint operator $B$, mapping $H$ onto $H$ such that

(i) $B$ is a bounded, self-adjoint;
(ii) $B$ has a bounded inverse;
(iii) $BA_0$ has a bounded inverse in the norm of $H$.

The both side of the equation $A_0u = f$ is operated by the operator $B$ and the operator $BA_0$ is denoted by $A$. Thus the above equation has been converted to a bounded linear operator equation

$$Au = f'$$

where $f' = Bf$ (15).
The unique solution of the equation (15) is the required solution.

The iterative sequence that we have framed for this purpose is of the form

\[ f_n = (I - AP)f_{n-1}, \quad n = 1,2, \ldots \quad (16) \]

where \( P \) is a known bounded linear operator such that the spectral radius of \((I - AP)\) is strictly less than one. The unique solution of the equation \( Au = f \) is \( u^* = A_0^{-1}f \), which is our first method [vide chapter III, method I, Theorem 3.1].

We have also considered another method of solving the above equation. Here \( B \) is taken as a self-adjoint bounded linear operator on \( H \) into \( H \) such that,

(i) \( \| BA_1u \| \leq k \| u \| \), for all \( u \in H \), \( k > 0 \);

(ii) \( B \) is positive-definite.

Then the iterative sequence \( \{ u_n \} \), which is taken in the form

\[ u_{n+1} = (1 - PA_1)u_n + P_1 \frac{f}{f}, \quad n = 0,1,2, \ldots \quad (17) \]

starting with \( u_0 \in H \) converges to a unique solution \( u^* \), when spectral radius of \((I - PA)\) is strictly less than one [vide, chapter III, theorem 4.1].
At the end of chapter III we have incorporated an application of the second method Vide chapter III, § 5 for the solution of unbounded linear invertible operator equation in a separable Hilbert space, to find the torsion problem for a rod of rectangular cross-section. To make an explicit representation of the torsion problem for a rod of rectangular cross-section, we have to solve the equation

\[- \nabla^2 u = 2 \Theta,\]  

(18)

where \( \mathcal{G} \) is the shear modulus and \( \Theta \) is the angle of torsion per unit length of the rod in the rectangle \( \Omega: 0 \leq x \leq a, \ 0 \leq y \leq b \) under the boundary conditions: \( u(0, y) = u(a, y) = u(x, 0) = u(x, b) = 0 \).

Here \( A = -\nabla^2 \), \( H = L^2(\Omega') \) and \( \Omega': 0 \leq x' \leq 1, \ 0 \leq y' \leq 1, \) where \( x' = \frac{x}{a}, \ y' = \frac{y}{b} \). The domain \( \mathcal{D}(A) \) of \( A \) is continuously twice differentiable function, satisfying the above boundary conditions and dense in \( L^2(\Omega') \). The operator \( \mathcal{B} \) referred earlier is taken as

\[ \mathcal{B}u = \int_0^1 k(x', t) k(y', h) u(t, h) \, dt \, dh, \text{ for all } u \in L^2(0, 1), \]

where

\[ k(x', t) = \begin{cases} x'(1-t), & x' \leq t \\ t(1-x'), & x' \geq t \end{cases} \]
The operator $B$ is a bounded linear operator and maps $C^\infty$ into $L^2$. The operator is self-adjoint and positive-definite.

(iii) The operator $b$ is defined for all $u \in \mathcal{G}(A)$ and is bounded in $L^2$.

The following lemma have been proved:

(i) The operator $B$ is a bounded linear operator and maps $L^2(\Omega')$ into $L^2(\Omega')$.

(ii) The operator $B$ is self-adjoint and positive-definite.

(iii) The operator $BA$ is defined for all $u \in \mathcal{G}(A)$ and is bounded in $L^2(\Omega')$.

The operator $A$ is of the form

$$Au = BAu = \frac{1}{\alpha^2} \int_0^1 k(y',\delta)u(x',\delta)\,d\delta + \frac{1}{\beta^2} \int_0^1 k(x',t)u(t,y')\,dt.$$ 

We have also proved the following lemma:

(i) $A$ is self-adjoint.
(ii) \( A \) is compact.

(iii) \( A \) is positive-definite.

(iv) The spectrum of \( A \) consists of positive eigenvalues only.

\( A \) being compact and self-adjoint, its spectrum consists of positive eigenvalues of \( A \). Being self-adjoint, its greatest eigenvalues is given by

\[
\lambda_m = \sup_{u \neq 0} (Au, u),
\]

Thus \( \lambda_m > 0 \).

We have then proved the lemma: If \( \mathcal{P} = \frac{1}{\lambda'} \) where

\( 0 < \lambda_m < \lambda' \) then \( n (I - \mathcal{P} A) < 1 \).

To obtain an approximate solution of the problem under consideration we have built up the iterative sequence \( \{u_n\} \) as follows:

\[
u_{n+1} = f'_{n+1} + \frac{1}{\lambda'} \sum_{\ell=0}^{\infty} f'_{\ell} - f'_{n}
\]

where \( u_0 = f' \) and \( f'_{n+1} = (I - A/\lambda')f'_{n} \), \( \ell = 1, 2, \ldots \), \( f' = f'_0 \).

Then we have proved the theorem:

Theorem 6. [Chapter III, theorem 5.1]

(i) \( f'_{n} \rightarrow 0 \) as \( n \rightarrow \infty \);
(ii) The given equation has a unique solution \( u^* \), given by

\[
u^* = \mathcal{A}^{-1} \mathcal{f}' = \frac{1}{\lambda} \sum_{n=0}^{\infty} f_n'
\]

(iii) The error estimates are given by

\[
\left\| u^* - u_n \right\| = \left\| \frac{1}{\lambda} \sum_{n=0}^{\infty} f_n' - f_n' \right\|
\]

In carrying out the computation we have considered four different types of cross-sections as follows:

\[
\begin{align*}
a = 1 \quad & a = 1 \quad & a = 1 \quad & a = 1 \\
b = 1 \quad & b = 1.2 \quad & a = 1.5 \quad & a = 2
\end{align*}
\]

and the numerical values for the torque and maximum shearing stress have been calculated and compared with those values obtained by Mikhlin, for small ratios of this sides. The results are in fairly good agreement with exact values for maximum shearing stress.

In section 1, of the chapter IV, we have considered approximate iterative processes in complete metric space. We have studied the approximate solvability of the non-linear functional equation.
\[ Au = Pu \] (20)

in a complete metric space \( \mathbb{R} \), by the iterates of the form
\[ A u_{n+1} = Pu_n \], where \( u_0 \) prechosen and \( n = 0, 1, 2, \ldots \).

\( A \) is a known nonlinear operator satisfying certain conditions.
\( P \) is also a nonlinear operator. \( A \) is an onto and \( P \) is an into self-mapping of the complete metric space \( \mathbb{R} \). Kannan's criterion for the existence of a unique fixed point of an operator has been used. The approximate solution of the equation
\[ A^n u = P^m u, \; u \in \mathbb{R}, \; m \text{ and } n \text{ are positive integers}, \]
has also been considered. Where the iterative sequence is taken in the form
\[ P^n u_{i+1} = P^n u_i, \; u_i \text{ prechosen, } n \text{ and } m \text{ are positive integers}, \; n \geq m, \; i = 0, 1, 2, \ldots \]. Different conditions for the existence of the solution of the equation have been expressed with the help of a number of theorems such as Theorem 7 (Vide section 1, Chapter IV, theorem 1.2.1). Let the following conditions be fulfilled for all

(i) \[ \beta \rho (u, v) \geq \rho (Au, Av) \geq \alpha \rho (u, v), \; \beta > \alpha > 1 \];

(ii) \[ \rho (APu, Pu) \leq \gamma \rho (Au, Pu) \];

(iii) \[ 2 \beta \gamma < \alpha (\alpha - 1) \];

then the sequence \( \{ u_n \} \) defined by (1.1.2) will converge to the unique solution of the first equation. The error
estimate is given by

\[ \rho(u_n, u^*) \leq q \left( \frac{q}{1 - q} \right)^{n-1} \rho(u_0, A^{-1}P u_0). \]

Theorem 8. Vide section 1, chapter IV, theorem 1.2.2. Let the following conditions be fulfilled for all \( u, v \in \mathbb{R} \),

(i) \( \beta \rho(u, v) \geq \rho(Au, Av) \geq \alpha \rho(u, v) \), \( \beta > \alpha > 1 \);

(ii) \( \rho(ANu, Pu) \leq \gamma \rho(Au, Pu) \);

(iii) \( \alpha \gamma < \alpha(\alpha - 1) \);

(iv) \( A \) and \( P \) commutes;

then the sequence \( \{u_i\} \) defined by \( A^n u_{i+1} = P^m u_i \), \( u_0 \) prechosen, \( n \) and \( m \) positive integers and \( n \geq m \), \( i = 0, 1, 2, \ldots \) will converge to a solution \( u^* \) of the equation \( A^n u = P^m u \).

The error estimate is given by

\[ \rho(u_n, u^*) \leq q \left( \frac{q}{1 - q} \right)^{m-1} \rho(u_0, A^{-1}P u_0). \]
Theorem 9. Vide section 1, chapter IV, theorem 1.2.3. Let \( \mathbb{R} \) be a metric linear space. Let the following conditions hold good:

(i) \( \beta \rho(u,v) \geq \rho(Au,Av) \geq \alpha \rho(u,v) \), \( \alpha > 0 \) for all \( u,v \in \mathbb{R} \);

(ii) \( \rho((A^{-n}P^{m})u, \theta) \leq k \rho(u, \theta) \) for all positive integers \( n \);

(iii) \( A^{-n}P^{m} \) is continuous at its fixed point;

(iv) \( A \) and \( P \) commute;

(v) \( P \) is compact and \( P^{n} \) is closed for all finite positive integers \( n \);

(vi) \( \rho((A^{n})^{\nu}u, (P^{m})^{\nu}u) \geq \rho(A^{n}u, P^{m}u) \)

for all finite positive integers \( n, m, \nu \);

then the sequence \( \{ u_{i} \} \) defined by \( A^{n}u_{i+1} = P^{m}u_{i} \) will converge to a solution \( u^{*} \) of the equation \( A^{n}u = P^{m}u \).

At the end of this section we have investigated the conditions so that the equation,

\[
\begin{align*}
\dot{u}_{\varepsilon}^{2}(x) + 2(x + 15)u(x) - 1.5 &= \int_{0}^{1} \left| x - t \right| [u(t) - \frac{u^{5}(t)}{8}] \, dt \quad (24)
\end{align*}
\]
admits of a unique solution in the interval 0 ≤ x ≤ 1, where u(x) is continuous in 0 ≤ x ≤ 1; starting from u'_0 = 0.06 e^x, the sequence of iterates \{u'_n\} is given by

\[ u'_n(x) = \frac{1}{2(\alpha + 1)} \left[ 1.5 - u''_{n-1}(x) + \frac{1}{\gamma} \int_0^1 |x-t| u'_n(t) - \frac{u_{n-1}(t)}{\gamma} \, dt \right], \quad n = 0, 1, \ldots \]

In section 2, of the chapter IV we have considered different conditions to obtain approximate solution of the nonlinear operator equation

\[ u = Au + f \quad (22) \]

where \( f \in X \). In the first part of section 2 the space \( X \) has been taken as a supermetric space. Later on we have considered the space \( X \) to be a pseudometric space. Two convergence theorems under different conditions have been presented here.

Theorem 10. Vide, section 2, chapter IV, theorem 2.2.1. Let \( A \) be a bounded linear operator fulfilling the following condition:

\[ \rho(A^n u, A^n v) \leq q \rho(u, v), \quad 0 < q < 1, \quad n \text{ a fixed positive integer} \].

Then the sequence of iterates \( u_{n+1} = Au_n + f \)
converges uniquely to a solution of the equation.

Theorem 11. \[\text{Vide, section 2, chapter IV, theorem 2.2.2}\] Let

A be a bounded nonlinear operator fulfilling the following conditions:

(i) there exists a linear operator L mapping X into X such that,

(ii) \( \rho(Au, Av) \leq \rho(Lu, Lv) \), for all \( u, v \in X \);

(iii) \( \rho(L^m u, L^m v) \leq q \rho(u, v) \), \( 0 < q < 1 \);

(iv) \( f \) belongs to the range of \((1 - A)\);

(v) \( L \) commutes with \( A \);

then the sequence \( \{ u_n \} \), defined by (2.1.2) converges to the unique solution of the equation.

Next we have considered that the spaces under consideration are a pseudometric space \( \mathbb{R} \) and a partially ordered space \( H \). Then for a bounded linear operator \( A \), the convergence theorem is

Theorem 12. \[\text{Vide section 2, chapter IV, theorem 2.3.1}\] .

Let \( L \) be a bounded linear operator mapping \( \mathbb{R} \) into \( \mathbb{R} \).
and \( \delta \) a fixed element in \( \mathbb{R} \) such that

\[
\begin{align*}
(i) \quad \rho (Au, Av) &\leq \rho (Lu, Lv + \delta) \quad \text{for all } u, v, \delta \in \mathbb{R}; \\
(ii) \quad \text{there exists a bounded linear operator } M \text{ in } \mathbb{R} \quad \\
\quad \text{such that, } \rho (L^b Au, L^b Av + M^q \delta) \leq \rho (L^{b+1} u, L^{b+1} v + M^{q+1} \delta) \\
\quad \text{for all } u, v \in \mathbb{R}, b = 1, 2, \ldots, \quad q = 0, 1, 2, \ldots; \\
(iii) \quad \lim_{n \to \infty} \rho (L^n u, \theta) = \theta_n \quad \text{for all } u \in \mathbb{R}; \\
(iv) \quad \lim_{n \to \infty} \rho (M^n \delta, \theta) = \theta_n; \\
\end{align*}
\]

then the sequence of iterates given by \( u_{n+1} = Au_n + \delta \) converges to a solution of the given equation.

Next we have framed another theorem where a concrete form of \( L \) and \( M \) is taken, which exposes the motivation behind the construction of the above theorem.