CHAPTER – I

INTRODUCTION

A graph is a collection of nodes and lines that we call vertices and edges respectively. A graph can be labeled or unlabeled. In this thesis, we are interested in labeled graph with orientation. In many labeled graphs, the labels are used for identification only. Labeling can be used not only to identify vertices and edges, but also to signify some additional properties, depending on the particular type of labeling. The kind of labeling we are interested, can serve in dual purposes.

This work deals with directed graph labeling. All the graphs considered here are finite and directed unless otherwise specified.

A **graph labeling** is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices or edges then the labeling is called **vertex or edge labeling**. Graph labelings were first introduced in the late 1960’s. In the recent years, dozens of graph labeling techniques have been studied in over 1000 papers.

Labeled graph serve as useful models for a broad range of applications such as coding theory [17], radar, astronomy [3], communications network addressing [16], solar ranging [18], broadcast frequency assignments [12], X-ray crystallography, circuit design, data base management and models for constraint programming over finite domains [35, 42, 43, 45].
By a \((p, q)\) graph \(G\), we mean a graph \(G = (V, E)\) with \(|V| = p\) and \(|E| = q\). A **vertex labeling** of graph \(G\) is an assignment \(f\) of labels to the vertices of \(G\) that induces for each edge \(uv\), a label depending on the vertex labels \(f(u)\) and \(f(v)\).

Most graph labeling methods trace their origin from the definition introduced by Rosa [38] in 1967. Rosa called a function \(f\) a **\(\beta\)-valuation** of a graph \(G\) with \(q\) edges if \(f\) is an injection from the vertices of \(G\) to the set \(\{0, 1, 2, ..., q\}\) such that, when each edge \(xy\) is assigned the label \(|f(x) - f(y)|\), the resulting edge labels are distinct. Golomb [14] subsequently called such labeling as **graceful**.

Graceful labeling originated as means of attacking the conjecture of Ringel [37] that \(K_{2n+1}\) can be decomposed into \(2n + 1\) subgraphs that are all isomorphic to a given tree with \(n\) edges.

Most graphs that have some sort of regularity of structure are graceful. Sheppard [39] has shown that there are exactly \(q!\) gracefully labeled graphs with \(q\) edges. Rosa [38] has identified three reasons when a graph fails to be graceful.

1. \(G\) has “too many vertices” and “not enough edges”
2. \(G\) has “too many edges”
3. \(G\) has “wrong parity”

In 1982, Acharya [1] proved that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph.
In 2008, Acharya, Rao and Arumugam [2] proved that every tree can be embedded as an induced subgraph of a graceful tree.

The Ringel – Kotzig conjectured [24] that all trees are graceful has been the focus of many papers. Graham and Sloane [15] defined a graph $G$ with $q$ edges to be harmonious if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $\left( f( x ) + f( y ) \right) \mod q$, the resulting edge labels are distinct.

Numerous variations and generalizations of graceful labelings such as $\alpha$ - labeling, $k$ and $(k, d)$ - graceful labeling, $k$ – equitable labeling, $\gamma$ - labeling, skolem graceful and graceful like labeling are discussed in [13].

In 1990, Hartsfield and Ringel [20] introduced antimagic graphs. A graph with $q$ edges is called antimagic, if its edges can be labeled with 1, 2, ..., $q$ such that the sum of the labels of the edges incident to each vertex are distinct. Sonntag [44] has extended the notion of antimagic labeling to hypergraphs.

In 1985, Lo[32] introduced the edge - graceful graph which is a dual notion of graceful labeling. A graph $G \left( V, E \right)$ is said to be edge - graceful if there exists a bijection $f$ from $E$ to $\{1, 2, ..., q\}$ such that the induced mapping $f^+$ from $V$ to $\{1, 2, ..., p – 1\}$ given by $f^+ \left( x \right) = \left( \sum f \left( xy \right) \right) \mod p$ taken over all edges $xy$ is a bijection.

Every edge - graceful graph is found to be antimagic and Lo [32] found a necessary condition for a graph with $p$ vertices and $q$ edges to be
edge-graceful is that \( q (q + 1) \equiv 0 \text{ or } P \pmod{p} \). Lee [29] noted that this necessary condition extends to any multigraph with \( p \) vertices and \( q \) edges.

Lee [29] conjectured that any connected simple \((p, q)\)–graph with \( q (q + 1) \equiv P (p+1) \pmod{p} \) vertices is edge-graceful. He also conjectured that this condition is sufficient for the edge-gracefulness of connected graphs.

Further Lee [28] has conjectured that all trees of odd order are edge-graceful. Small [41] has proved that spiders for which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [27] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low and Mitchem [10] have shown that regular spiders of odd order are edge-graceful.

Lee, Seah and Wang [30] gave a complete characterization of edge-graceful \( P_n^k \)-graphs. Shiu, Lam and Cheng [40] proved that the composition of the path \( P_3 \) and any null graph of odd order is edge-graceful.

Many physical situations require directed graphs. The street map of a city with one-way streets, flow networks with valves in the pipes and electrical networks are represented by directed graphs. Directed graphs are employed in abstract representations of computer programs, where the
vertices stand for the program instructions and the edges specify the execution sequence. The directed graph is an invaluable tool in the study of sequential machines. Directed graphs in the form of signal – flow graphs are used for system analysis in control theory.

Harary, Norman and cartwright [19,21] written a book on the theory of digraphs. A 100 – page monograph was written by Moon [36] on tournaments (complete asymmetric digraphs) alone. For applications of directed graphs in operations research, Kaufmann’s book [26] is a good source.

Chen and Wing [11] give some properties and interesting applications of acyclic digraphs. Minimal decyclization of a digraph was the subject of the doctoral thesis by Lempel [31].

In the early 1980s, Bloom and Hsu [5, 6, 7] extended graceful labelings to directed graphs by defining a graceful labeling on a directed graph \( D(V, E) \) as a one - to - one map \( \theta \) from \( V \) to \( \{0, 1, 2, ..., |E|\} \) such that \( \theta(y) - \theta(x) \mod (|E|+1) \) is distinct for every edge \( xy \) in \( E \). Graceful labelings of directed graphs also arose in the characterization of finite neofields by Hsu and keedwell [22, 23]. Graceful labelings of directed graphs was the subject of Marr’s 2007 Ph.D. dissertation [33]. In [33] and [34] Marr presents results of graceful labelings of directed paths, stars, wheels and umbrellas.

The concept of magic, antimagic and conservative labelings have been extended to directed graphs [25]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference
sets, sequenceable groups, generalized complete mappings, near-complete mappings and neo fields are discussed in [4, 8, 9].

Motivated by the notion of edge-gentle graphs and the applications of directed graphs, we focused our attention towards directed edge-gracefulness and the author defined in this thesis the definition namely directed edge-graceful labeling.

The new findings are categorized into nine chapters. We now present chapter wise summary.

Chapter I briefly introduces the thesis.

Chapter II provides fundamental concepts of graph theory and definitions of graphs which are needed for the rest of this thesis.

Chapter III introduces the directed edge-graceful labeling and investigates the directed edge-gracefulness of some graphs such as fan, wheel and cycle. We also proved that $nC_3$ snake and Butterfly $B_n$ are directed edge-graceful.

Chapter IV deals with the directed edge-graceful labeling schemes of path and star related trees. The graphs considered here are star, path, \( \langle K_{1,n} : K_{1,n} \rangle \), festoon, \( \langle K_{1,n} : m \rangle \) graphs and y-tree.

Chapter V focuses on the directed edge-graceful labeling of some trees such as the tree $JE_{m,n}$, $S(m,n)$, sparklers, $D_{m,n} @ K_{1,t}$ and twig.

Chapter VI investigates the directed edge-graceful labeling of a special tree $T_{i,n,m}$ in the following cases.

(i) \( t, n \) and \( m \) are odd

(ii) \( t, n \) are even and \( m \) is odd
(iii) \( n \) and \( m \) are even and \( t \) is odd

(iv) \( t,m \) are even and \( n \) is odd.

Also, directed edge - graceful labeling of \( m \)-ary trees,

\[ FC(1_{m},K_{n},K_{l,m}) \] and \( m : K_{1,n} \) are established.

**Chapter VII** studies the directed edge - graceful labeling of some cycle related graphs. The graphs considered here are \( C_{n} \cup K_{m}, C_{m} @ K_{1,n}, D_{m,n}, \) flag, tortoise and friendship graphs.

**Chapter VIII** deals with the directed edge - graceful labeling of disjoint union of graphs and multigraphs. The graphs considered here are \( P_{3} \cup K_{1,2n+1}, P_{3} \cup K_{1,2n+1}, K_{1,2n} \cup K_{1,2m+1}, \) snail, \( H_{2}(C_{2n+1}), H_{m}(K_{1,n}) \) and \( C_{2n+1} \cup K_{1,2m+1} \) graphs.

**Chapter IX** introduces the edge - graceful number and found it for some family of graphs such as \( K_{1,n}, B_{n,m} \) and \( K_{1,n} : K_{1,m} \).

Finally, the thesis ends with Bibliography.