CHAPTER – VII

DIRECTED EDGE - GRACEFUL LABELING OF CYCLE RELATED GRAPHS

Introduction 7.1

In this chapter, we study the directed edge - graceful labeling of cycle related graphs such as,

\[ C_n \circ K_m \]
\[ C \circ K \]
\[ C \circ K \]

Flag \( Fl_{2n} \)

D

Tortoise

Friendship graph

Definition 7.2

The Corona \( G_1 \circ G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph \( G \) obtained by taking one copy of \( G_1 \) (which has \( p_1 \) points) and \( p_1 \) copies of \( G_2 \), and then joining the \( i^{th} \) point of \( G_1 \) to every point in the \( i^{th} \) copy of \( G_2 \).

Theorem 7.3

The graph \( C_n \circ \overline{K}_m \) (\( m \geq 2, n \geq 3 \)) is directed edge - graceful for all \( n \) odd and \( m \) even.
Proof

Let $V = \{v_1, v_2, ..., v_n, v_{11}, v_{12}, ..., v_{1m}, v_{21}, v_{22}, ..., v_{2m}, ..., v_{n1}, v_{n2}, ..., v_{nm}\}$ be the set of vertices. Now, we orient the edges of $C_n \odot K_m$ such that the arc set $A$ is given by

$$A = \{v_{2i+1}, v_{2i}\}, 1 \leq i \leq \frac{n-1}{2} \cup \{v_{2i+1}, v_{2i}\}, 1 \leq i \leq \frac{n-1}{2} \cup \{(v_n, v_1)\} \cup \{(v_i, v_{ij})\}, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

The edges and their orientation of $C_n \odot K_m$ are as in Fig. 7.1:

![Fig. 7.1: $C_n \odot K_m$ with orientation](image)

We now label the arcs of $A$ as follows:

$$f((v_{2i+1}, v_{2i})) = i \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f((v_{2i+1}, v_{2i})) = n (m+1) - 1 + i \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(v, v_{n1}) = n(m + 1)$$
The computed values of $f^+ (v_i)$, $f^+ (v_{ij})$ and $f^- (v_i)$, $f^- (v_{ij})$ are given below:

$$f^+ (v_1) = n(m + 1)$$

$$f^- (v_1) = - (nm + n) \frac{m}{2} + 1 \frac{n - 1}{2}$$

$$f^+ (v_2) = (nm + n) \frac{n - 1}{2} - 1 + 2i \quad 1 \leq i \leq \frac{n - 1}{2}$$

$$f^- (v_2) = - \frac{m}{2} (nm + n) \quad 1 \leq i \leq \frac{n - 1}{2}$$

$$f^+ (v_{2,i+1}) = 0 \quad 1 \leq i \leq \frac{n - 1}{2}$$

$$f^- (v_{2,i+1}) = - (nm + n) \frac{m}{2} + 1 \frac{n - 1}{2} + 2i \quad 1 \leq i \leq \frac{n - 1}{2}$$

$$f^+ (v_{(2j-1)i}) = \frac{n - 1}{2} + \frac{m}{2} (i - 1) + j \quad 1 \leq i \leq n, 1 \leq j \leq m$$

$$f^- (v_{(2j-1)i}) = 0$$

$$f^+ (v_{(2j)i}) = n(m + 1)$$

$$f^- (v_{(2j)i}) = - \frac{n - 1}{2}$$

$$f^- (v_{(2j)i}) = 0, -$$
Then the induced vertex labels are:

Case (i) \( \frac{n-1}{2} \) is even

\[
g(v_{i-1, j}) = \frac{n-1}{2} + \frac{m}{2}(i-1) + j \quad 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2}
\]

\[
g(v_{i, j}) = n \left( m + 1 - \frac{n-1}{2} \right) - \frac{m}{2}(i-1) - j \quad 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2}
\]

\[
g(v_{2i-1}) = \frac{n-1}{2} + 2 - 2i \quad 1 \leq i \leq \frac{n+3}{4}
\]

\[
g(v_{2i}) = n(m+1) - \frac{n-1}{2} - 1 + 2i \quad 1 \leq i \leq \frac{n-1}{4}
\]

\[
g(v_{n-1, i-1}) = \frac{2i}{2} - 1 \quad 1 \leq i \leq \frac{n-1}{4}
\]

\[
g(v_{n+1, i-1}) = n(m+1) - 2i \quad 1 \leq i \leq \frac{n-1}{4}
\]

Case (ii) \( \frac{n-1}{2} \) is odd.

\[
g(v_{i, (2j-1)}) = \frac{n-1}{2} + \frac{m}{2}(i-1) + j \quad 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2}
\]

\[
g(v_{i, (2j)}) = n \left( m + 1 - \frac{n-1}{2} \right) - \frac{m}{2}(i-1) - j \quad 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2}
\]

\[
g(v_{2i-1}) = \frac{n-1}{2} + 2 - 2i \quad 1 \leq i \leq \frac{n+1}{4}
\]

\[
g(v_{2i}) = n(m+1) - \frac{n-1}{2} - 1 + 2i \quad 1 \leq i \leq \frac{n-3}{4}
\]
Clearly, it follows that all the vertex labels are distinct and ranges between 0 to \( p - 1 \). Thus, \( g \) is a bijection. Hence, the graph \( C_n \circ K_m \) (\( m \geq 2, n \geq 3 \)) is directed edge - graceful for all \( n \) odd and \( m \) even. The directed edge - graceful labeling of \( C_5 \circ K_8 \), \( C_7 \circ K_8 \), \( C_9 \circ K_6 \) and \( C_{11} \circ K_4 \) are given in Fig. 7.2, Fig. 7.3, Fig. 7.4 and Fig. 7.5 respectively.

\[
g(v) = 2i - 2 \quad \text{for} \quad 1 \leq i \leq \frac{n+1}{4}
\]

\[
g(v) = n(m + 1) + 1 - 2i \quad \text{for} \quad 1 \leq i \leq \frac{n+1}{4}
\]
Fig. 7.3: $C_7 \odot \overline{K}_8$ with directed edge - graceful labeling

Fig. 7.4: $C_9 \odot \overline{K}_6$ with directed edge - graceful labeling
Definition 7.4

The graph $C_m @ K_{1,n}$ is obtained by joining the vertex of a cycle $C_m$ to the centre of a star $K_{1,n}$.

Theorem 7.5

The graph $C_{2m} @ K_{1,2n+1}$ ($m \geq 2, n \geq 1$) is directed edge - graceful for all $m$ and $n$.

Proof

Let $V = \{v_1, v_2, \ldots, v_{2m}, u_1, u_2, \ldots, v_{2n+1}\}$ be the set of vertices. Now, we orient the edges of $C_{2m} @ K_{1,2n+1}$ such that the arc set $A$ is given by
\[ A = \{(v_{2i-1}, v_{2i}), \, 1 \leq i \leq m - 1\} \cup \{v_1, v_{2m}\} \cup \{(v_{2i-1}, v_{2i}), \, 1 \leq i \leq m\} \cup \{(v_1, u_j), \, 1 \leq j \leq 2n + 1\}. \]

The edges and their orientation of \(C_{2m} \@ K_{1,2n+1}\) are as in Fig. 7.6:

![Diagram](image)

**Fig. 7.6:** \(C_{2m} \@ K_{1,2n+1}\) with orientation

We now label the arcs of \(A\) as follows:

\[
\begin{align*}
  f((v_{2i+1}, v_{2i})) &= i & 1 \leq i \leq m - 1 \\
  f((v_1, v_{2m})) &= m \\
  f((v_{2i-1}, v_{2i})) &= m + 2n + 1 + i & 1 \leq i \leq m \\
  f((v_1, u_{2j-1})) &= m + j & 1 \leq j \leq n + 1 \\
  f((v_1, u_{2j})) &= m + 2n + 2 - j & 1 \leq j \leq n 
\end{align*}
\]

The computed values of \(f^+(v_i), \, f^+(u_j)\) and \(f^-(v_i), \, f^-(u_j)\) are given below:

\[
\begin{align*}
  f^+(v_{2i}) &= m + 2n + 1 + 2i & 1 \leq i \leq m \\
  f^-(v_{2i}) &= 0 & 1 \leq i \leq m
\end{align*}
\]
\[ f^+(v_{2i+1}) = 0 \quad 1 \leq i \leq m - 1 \]
\[ f^-(v_{2i+1}) = -(m + 2n + 2 + 2i) \quad 1 \leq i \leq m - 1 \]
\[ f^+(v_1) = 0 \]
\[ f^-(v_1) = -(m + n + 1) [2(n + 1) + 1] \]
\[ f^+(u_{2j-1}) = m + j \quad 1 \leq j \leq n + 1 \]
\[ f^-(u_{2j-1}) = 0 \quad 1 \leq j \leq n + 1 \]
\[ f^+(u_{2j}) = m + 2n + 2 - j \quad 1 \leq j \leq n \]
\[ f^-(u_{2j}) = 0 \quad 1 \leq j \leq n \]

Then the induced vertex labels are:

**Case (i) m is odd**
\[ g(u_{2j-1}) = m + j \quad 1 \leq j \leq n + 1 \]
\[ g(u_{2j}) = m + 2n + 2 - j \quad 1 \leq j \leq n \]
\[ g(v_{2i-1}) = m + 1 - 2i \quad 1 \leq i \leq \frac{m+1}{2} \]
\[ g(v_{2i}) = m + 2n + 1 + 2i \quad 1 \leq i \leq \frac{m-1}{2} \]
\[ g(v_{m+2i}) = 2i - 1 \quad 1 \leq i \leq \frac{m+1}{2} \]
\[ g(v_{m+2i}) = 2m + 2n + 1 - 2i \quad 1 \leq i \leq \frac{m-1}{2} \]

**Case (ii) m is even**
\[ g(u_{2j-1}) = m + j \quad 1 \leq j \leq n + 1 \]
\[ g(u_{2j}) = m + 2n + 2 - j \quad 1 \leq j \leq n \]
\[ g(v_{2i-1}) = m + 1 - 2i \quad 1 \leq i \leq \frac{m}{2} \]
\[ g(v_{2i}) = m + 2n + 1 + 2i \quad 1 \leq i \leq \frac{m - 2}{2} \]
\[ g(v_{m-2+2i}) = 2i - 2 \quad 1 \leq i \leq \frac{m+2}{2} \]
\[ g(v_{m-1+2i}) = 2m + 2n + 2 - 2i \quad 1 \leq i \leq \frac{m}{2} \]

Clearly, it follows that all the vertex labels are distinct and ranges between 0 to \( p - 1 \). Thus, \( g \) is a bijection. Hence, the graph \( C_{2m} \oplus K_{1,2n+1} \) \((m \geq 2, n \geq 1)\) is directed edge-graceful for all \( m \) and \( n \). The directed edge-graceful labeling of \( C \oplus K_{10,13} \), \( C \oplus K_{12,9} \), \( C \oplus K_{14,7} \) and \( C \oplus K_{16,9} \) are given in Fig. 7.7, Fig. 7.8, Fig. 7.9 and Fig. 7.10 respectively.

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**Fig. 7.7:** \( C_{10} \oplus K_{1,13} \) with directed edge-graceful labeling
Fig. 7.8: $C_{12} @ K_{1,9}$ with directed edge - graceful labeling

Fig. 7.9: $C_{14} @ K_{1,7}$ with directed edge - graceful labeling
Fig. 7.10: $C_{16} @ K_{1,9}$ with directed edge - graceful labeling

**Theorem 7.6**

The graph $C_{2m+1} @ K_{1,2n}$ ($m \geq 1, n \geq 1$) is directed edge - graceful for all $m$ and $n$.

**Proof**

Let $V = \{v_1, v_2, ..., v_{2m+1}, u_1, u_2, ..., u_{2n}\}$ be the set of vertices. Now, we orient the edges of $C_{2m+1} @ K_{1,2n}$ such that the arc set $A$ is given by

$$A = \{(v_{2i-1}, v_{2i}), 1 \leq i \leq m\} \cup \{(u_{2i+1}, v_{2i}), 1 \leq i \leq m\}$$

$$\cup \{(v_{2m+1}, v_1)\} \cup \{(v_{m+1}, u_j), 1 \leq j \leq 2n\}.$$
The edges and their orientation of $C_{2m+1} @ K_{1,2n}$ are as in Fig. 7.11:

![Diagram of $C_{2m+1} @ K_{1,2n}$]

**Fig. 7.11: $C_{2m+1} @ K_{1,2n}$** with orientation

We now label the arcs of $A$ as follows:

\[
\begin{align*}
    f((v_{2i-1}, v_{2i})) & = m + 2n + i & 1 \leq i \leq m \\
    f((v_{2i+1}, v_{2i})) & = i & 1 \leq i \leq m \\
    f((v_{m+1}, v_1)) & = 2m + 2n + 1 & f((v_{m+1}, u_1)) = m + 1 \\
    f((v_m, u_2j)) & = m + 1 + j & 1 \leq j \leq n \\
    f((v_m, u_2j+1)) & = m + 2n + 1 - j & 1 \leq j \leq n - 1
\end{align*}
\]

The computed values of $f^+(v_i)$, $f^+(u_j)$ and $f^-(v_i)$, $f^-(u_j)$ are given below:

\[
\begin{align*}
    f^+(v_1) & = 2m + 2n + 1 ; \\
    f^+(v_{2i}) & = m + 2n + 2i & 1 \leq i \leq \frac{m}{2} \\ 
    f^-(v_{2i}) & = 0 & 1 \leq i \leq \frac{m}{2} \\ 
    f^+(v_{2i}) & = m + 2n + 2i & 1 \leq i \leq \frac{m-1}{2} \\ 
\end{align*}
\]
\[ f^-(v_{2i}) = 0 \quad 1 \leq i \leq \frac{m-1}{2} \text{ if } m \text{ is odd} \]

\[ f^+(v_{m+1}) = 2m + 2n + 1 \quad \text{if } m \text{ is odd} \]

\[ f^-(v_{m+1}) = -n - 1 \quad 2m + 2n + 2 + 2m + 1 + n \text{ if } m \text{ is odd} \]

\[ f^+(v_{m+1}) = 0 \quad \text{if } m \text{ is even} \]

\[ f^-(v_{m+1}) = -n (2m + 2n + 3) + 2m + 1 \quad \text{if } m \text{ is even} \]

\[ f^+(v_{m+2i}) = 2m + 2n + 2i \quad 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \]

\[ f^-(v_{m+2i}) = 0 \quad 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \]

\[ f^+(v_{m+2i}) = 0 \quad 1 \leq i \leq \frac{m+1}{2} \text{ if } m \text{ is odd} \]

\[ f^-(v_{m+2i}) = -(2m + 2n + 2i) \quad 1 \leq i \leq \frac{m+1}{2} \text{ if } m \text{ is odd} \]

\[ f^+(v_{m+1+2i}) = 0 \quad 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \]

\[ f^-(v_{m+1+2i}) = -(2m + 2n + 1 + 2i) \quad 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \]

\[ f^+(v_{m+1+2i}) = 2m + 2n + 1 + 2i \quad 1 \leq i \leq \frac{m-1}{2} \text{ if } m \text{ is odd} \]

\[ f^-(v_{m+1+2i}) = 0 \quad 1 \leq i \leq \frac{m-1}{2} \text{ if } m \text{ is odd} \]

\[ f^+(u_1) = m + 1 \quad ; \quad f^-(u_1) = 0 \]

\[ f^+(u_{2j}) = m + 1 + j \quad 1 \leq j \leq n \]

\[ f^-(u_{2j}) = 0 \quad 1 \leq j \leq n \]
Then the induced vertex labels are:

**Case (i)  \( m \) is odd**

\[
g(u_1) = m + 1
\]

\[
g(u_2) = m + 1 + j \quad 1 \leq j \leq n
\]

\[
g(u_{2j+1}) = m + 2n + 1 - j \quad 1 \leq j \leq n - 1
\]

\[
g(v_{2i-1}) = m + 2 - 2i \quad 1 \leq i \leq \frac{m+1}{2}
\]

\[
g(v_{2i}) = m + 2n + 2i \quad 1 \leq i \leq \frac{m-1}{2}
\]

\[
(v_{m+1}) = 0
\]

\[
(v_{m+2i}) = 2m + 2n + 2 - 2i \quad 1 \leq i \leq \frac{m+1}{2}
\]

\[
g(v_{m+1+2i}) = 2i \quad 1 \leq i \leq \frac{m-1}{2}
\]

**Case (ii)  \( m \) is even**

\[
g(u_1) = m + 1
\]

\[
g(u_2) = m + 1 + j \quad 1 \leq j \leq n
\]

\[
g(u_{2j+1}) = m + 2n + 1 - j \quad 1 \leq j \leq n - 1
\]

\[
g(v_{2i-1}) = m + 2 - 2i \quad 1 \leq i \leq \frac{m}{2}
\]

\[
g(v_{2i}) = m + 2n + 2i \quad 1 \leq i \leq \frac{m}{2}
\]

\[
g(v_{m+1}) = 0
\]
\[ g \left( v_{m+2i} \right) = 2i - 1 \quad 1 \leq i \leq \frac{m}{2} \]

\[ g \left( v_{m+1+2i} \right) = 2m + 2n + 1 - 2i \quad 1 \leq i \leq \frac{m}{2} \]

Clearly, it follows that all the vertex labels are distinct and ranges between 0 to \( p - 1 \). Thus, \( g \) is a bijection. Hence, the graph \( C_{2m+1} \circ K_{1,2n} \) \((m \geq 1, n \geq 1)\) is directed edge - graceful for all \( m \) and \( n \). The directed edge - graceful labeling of \( C_9 \circ K_{1,10} \), \( C_{11} \circ K_{1,8} \), \( C_{13} \circ K_{1,8} \) and \( C_{15} \circ K_{1,12} \) are given in Fig. 7.12, Fig. 7.13, Fig. 7.14 and Fig. 7.15 respectively.

**Fig. 7.12:** \( C_9 \circ K_{1,10} \) with directed edge - graceful labeling

**Fig. 7.13:** \( C_{11} \circ K_{1,8} \) with directed edge - graceful labeling
Fig. 7.14: $C_{13} @ K_{1,8}$ with directed edge - graceful labeling

Fig. 7.15: $C_{15} @ K_{1,12}$ with directed edge – graceful labeling

Definition 7.7

The flag $Fl_n$ is obtained by joining an edge to the vertex of a cycle $C_n$.

Theorem 7.8

The flag $Fl_{2n}$ ($n \geq 2$) is directed edge - graceful for all $n$.

Proof

Let $V[Fl_{2n}] = \{v, v_1, v_2, ..., v_{2n}\}$ be the set of vertices. Now, we orient the edges of $Fl_{2n}$ such that the arc set $A$ is given by
\[ A = \{(v_1, v)\} \cup (v_{2i-1}, v_{2i}), 1 \leq i \leq n \} \cup \{(v_{2i+1}, v_{2i}), 1 \leq i \leq n - 1 \} \cup \{(v_1, v_{2n})\}. \]

The edges and their orientation of \( F_{2n} \) are as in Fig. 7.16:

![Fig. 7.16: \( F_{2n} \) with orientation](image)

We now label the arcs of \( A \) as follows:

\[
\begin{align*}
  f((v_1, v)) &= 1 \\
  f((v_{2i-1}, v_{2i})) &= 2i & 1 \leq i \leq n \\
  f((v_{2i+1}, v_{2i})) &= 2i + 1 & 1 \leq i \leq n \\
  f((v_1, v_{2n})) &= 2n + 1
\end{align*}
\]

The computed values of \( f^+(v), f^+(v_i) \) and \( f^-(v), f^-(v_i) \) are given below:

\[
\begin{align*}
  f^+(v) &= 1 ; \quad f^-(v) = 0 \\
  f^+(v_1) &= 0 ; \quad f^-(v_1) = -(2n + 4) \\
  f^+(v_{2i}) &= 4i + 1 & 1 \leq i \leq n \\
  f^-(v_{2i}) &= 0 & 1 \leq i \leq n \\
  f^+(v_{2i+1}) &= -(4i + 3) & 1 \leq i \leq n - 1 \\
  f^-(v_{2i+1}) &= 0 & 1 \leq i \leq n - 1
\end{align*}
\]
Then the induced vertex labels are:

**Case (i) :** \( n \) even

\[
g(v) = 1 \quad ; \quad g(v_1) = 2(n-1)
\]

\[
g(v_{2n}) = 2n
\]

\[
g(v_{2i}) = 4i + 1 \quad 1 \leq i \leq \frac{n}{2} - 1
\]

\[
g(v_n) = 0
\]

\[
g(v_{2i+1}) = 2n - 4i - 2 \quad 1 \leq i \leq \frac{n}{2} - 1
\]

\[
\bar{g}(v_{n+2i+1}) = 2n + 3 - 4i \quad 1 \leq i \leq \frac{n}{2}
\]

\[
g(v_{n+2i}) = 4i \quad 1 \leq i \leq \frac{n}{2} - 1
\]

**Case (ii) :** \( n \) odd

\[
g(v) = 1 \quad ; \quad g(v_1) = 2(n-1)
\]

\[
g(v_{2n}) = 2n
\]

\[
g(v_{2i}) = 4i + 1 \quad 1 \leq i \leq \frac{n-1}{2}
\]

\[
g(v_n) = 0
\]

\[
g(v_{2i+1}) = 2n - 4i - 2 \quad 1 \leq i \leq \frac{n-3}{2}
\]

\[
\bar{g}(v_{n+2i+1}) = 4i - 2 \quad 1 \leq i \leq \frac{n-1}{2}
\]

\[
g(v_{n+2i}) = 2n + 1 - 4i \quad 1 \leq i \leq \frac{n-1}{2}
\]
Clearly, it follows that all the vertex labels are distinct and ranges between 0 to $p - 1$. Thus, $g$ is a bijection. Hence, the flag $Fl_{2n}$ ($n \geq 2$) is directed edge-graceful for all $n$. The directed edge-graceful labeling of $Fl_{10}$, $Fl_{12}$, $Fl_{14}$ and $Fl_{16}$ are given in Fig. 7.17, Fig. 7.18, Fig. 7.19 and Fig. 7.20 respectively.

Fig. 7.17: $Fl_{10}$ with directed edge-graceful labeling

Fig. 7.18: $Fl_{12}$ with directed edge-graceful labeling
Definition 7.9

A graph **dragon** is formed by joining the end point of the path $P_m (m \geq 2)$ to a cycle $C_n (n \geq 3)$. It is denoted by $D_{m,n}$. 

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**Fig. 7.19:** $F_{14}$ with directed edge-graceful labeling

**Fig. 7.20:** $F_{16}$ with directed edge-graceful labeling
Theorem 7.10

The graph $D_{m,n}$ ($m, n \geq 2$) is directed edge-graceful for all $m$ and $n$ even or $m$ and $n$ odd.

Proof

Let $V = \{u_i, 1 \leq i \leq m + n - 1\}$ be the set of vertices. Now, we orient the edges of $D_{m,n}$ such that the arc set $A$ is given by

$$
A = \{ (u_{2i-1}, u_{2i}) \mid 1 \leq i \leq \frac{m + n - 2}{2} \} \cup \{ (u_{2i-1}, u_{2i+1}) \mid 1 \leq i \leq \frac{m + n - 2}{2} \} \cup \{ (u_{m+n-1}, u_1) \}
$$

The edges and their orientation of $D_{m,n}$ are as in Fig. 7.21:

![Fig. 7.21: $D_{m,n}$ with orientation](image)

We now label the arcs of $A$ as follows:

For $1 \leq i \leq \frac{m + n - 2}{2}$

$$
f((u_{2i+1}, u_{2i})) = i
$$
For $1 \leq i \leq m + n - 2$

$$f\left((u_{2i-1}, u_{2i})\right) = \frac{m + n - 2 + i}{2}$$

$$f\left((u_{m+n-1}, u_m)\right) = m + n - 1$$

The computed values of $f^+\left(u_i\right)$ and $f^-\left(u_i\right)$ are given below:

**Case (i): $m$ and $n$ are even**

$$f^+\left(u_1\right) = 0$$

$$f^+\left(u_{2i}\right) = \frac{m + n - 2 + 2i}{2} \quad ; \quad 1 \leq i \leq \frac{m - 2}{2}$$

$$f^-\left(u_{2i}\right) = 0 \quad ; \quad 1 \leq i \leq \frac{m - 2}{2}$$

$$f^+\left(u_{2i+1}\right) = 0 \quad ; \quad 1 \leq i \leq \frac{m - 2}{2}$$

$$f^-\left(u_{2i+1}\right) = -\frac{m + n}{2} + 2i \quad ; \quad 1 \leq i \leq \frac{m - 2}{2}$$

$$f^+\left(u_m\right) = \frac{5m + 3n - 4}{2}$$

$$f^+\left(u_{m+2i}\right) = \frac{3m + n - 1 + 2i}{2} \quad ; \quad 1 \leq i \leq \frac{n - 1}{2}$$

$$f^-\left(u_{m+2i}\right) = 0 \quad ; \quad 1 \leq i \leq \frac{n - 1}{2}$$

$$f^+\left(u_{m+2i+1}\right) = 0 \quad ; \quad 1 \leq i \leq \frac{n - 1}{2}$$

$$f^-\left(u_{m+2i+1}\right) = \frac{3m + n - 4 + 2i}{2} \quad ; \quad 1 \leq i \leq \frac{n - 1}{2}$$
Case (ii): \( m \) and \( n \) are odd

\[
\begin{align*}
f^+(u_1) &= 0, \\
f^+(u_{2i}) &= \frac{m+n-2}{2} + 2i, \quad 1 \leq i \leq \frac{m-1}{2} \\
f^-(u_{2i}) &= 0, \quad 1 \leq i \leq \frac{m-1}{2} \\
f^+(u_{2i+1}) &= 0, \quad 1 \leq i \leq \frac{m-3}{2} \\
f^-(u_{2i+1}) &= -\frac{m+n}{2} + 2i, \quad 1 \leq i \leq \frac{m-3}{2} \\
f^+(u_m) &= m + n - 1, \quad f^-(u_m) = -\frac{3m+n-2}{2} \\
f^+(u_{m+2i}) &= 0, \quad 1 \leq i \leq \frac{n-1}{2} \\
f^-(u_{m+2i}) &= -\frac{3m+n-2}{2} + 2i, \quad 1 \leq i \leq \frac{n-1}{2} \\
f^+(u_{m+2i-1}) &= \frac{3m+n-4}{2} + 2i, \quad 1 \leq i \leq \frac{n-1}{2} \\
f^-(u_{m+2i-1}) &= 0, \quad 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

Then the induced vertex labels are:

Case (i): \( \frac{m+n}{2} \) is even

\[
\begin{align*}
g(u_{2i-1}) &= \frac{m+n}{2} - 2i, \quad 1 \leq i \leq \frac{m+n}{4} \\
\end{align*}
\]
\[ g(u_{2i}) = \frac{m+n-2}{2} + 2i \quad 1 \leq i \leq \frac{m+n}{4} - 1 \]
\[ 2i - \frac{m+n}{2} \leq i \leq \frac{m+n-2}{2} \]

Case (ii): \( \frac{m+n}{2} \) is odd

\[ g(u_{2i-1}) = \frac{m+n+2}{4} - 2i \quad 1 \leq i \leq m+n + 2 \]
\[ \frac{m+n}{2} \leq i \leq \frac{m+n+2}{2} \]

\[ g(u_{2i}) = \frac{m+n-2}{2} + 2i \quad 1 \leq i \leq m+n-2 \]
\[ 2i - \frac{m+n}{2} \leq i \leq \frac{m+n-2}{2} + 1 \leq i \leq m+n \]

Clearly, if follows that all the vertex labels are distinct and ranges between 0 to \( p - 1 \). Thus, \( g \) is a bijection. Hence, the graph \( D_{m,n} \) (\( m, n \geq 2 \)) is directed edge - graceful for all \( m \) and \( n \) even or \( m \) and \( n \) odd. The directed edge - graceful labeling of \( D_{15,11}, D_{17,7}, D_{14,12} \) and \( D_{14,14} \) are given in Fig. 7.22, Fig. 7.23, Fig. 7.24 and Fig. 7.25 respectively.

**Fig. 7.22:** \( D_{15,11} \) with directed edge - graceful labeling
Fig. 7.23: $D_{17,7}$ with directed edge - graceful labeling

Fig. 7.24: $D_{14,12}$ with directed edge - graceful labeling

Fig. 7.25: $D_{14,14}$ with directed edge - graceful labeling

Definition 7.11

A tortoise $(To)_n$ is obtained from a path $v_1, v_2, ..., v_n$ by attaching an edge between $v_i$ and $v_{n-i+1}$ for $i = 1$ to $n - \frac{n}{2}$ and $n \geq 3$. 
Theorem 7.12

The tortoise graph \((To)_{2n+1}\) \((n \geq 2)\) is directed edge-graceful for all \(n\).

Proof

Let \(V (To)_{2n+1} = \{v_1, v_2, ..., v_{2n+1}\}\) be the set of vertices. Now, we orient the edges of \((To)_{2n+1}\) such that the arc set \(A\) is given by

\[A = \{(v_i, v_{i+1}), 1 \leq i \leq n\} \cup \{(v_{i+1}, v_i), n + 1 \leq i \leq 2n\}
\]

\[\cup \{(v_i, v_{2n+2-i}), 1 \leq i \leq n\}.
\]

The edges and their orientation of \((To)_{2n+1}\) are as in Fig. 7.26:

![Graph](image)

**Fig. 7.26: \((To)_{2n+1}\) with orientation**

We now label the arcs of \(A\) as follows:

\[f((v_i, v_{i+1})) = i \quad 1 \leq i \leq 2n\]

\[f((v_{n+1-i}, v_{n+1+i})) = 2n + i \quad 1 \leq i \leq n\]

The computed values of \(f^+(v_i)\), \(f^+(v)\) and \(f^-(v_i)\), \(f^- (v)\) are given below:

\[f^+(v_i) = i - 1 \quad 1 \leq i \leq n\]
\[ f^-(v_i) = -(3n + 1) \quad 1 \leq i \leq n \]
\[ f^+(v_{n+1}) = 2n + 1 \quad ; \quad f^-(v_{n+1}) = 0 \]
\[ f^+(v_{n+i}) = 3n + 1 + 2i \quad 1 \leq i \leq n - 1 \]
\[ f^-(v_{n+i}) = -(n + i) \quad 1 \leq i \leq n - 1 \]
\[ f^+(v_{2n+1}) = 3n \quad ; \quad f^-(v_{2n+1}) = -2n \]

Then the induced vertex labels are:
\[ g(v_i) = n + i \quad 1 \leq i \leq n \]
\[ g(v_{n+1}) = 0 \]
\[ g(v_{n+i}) = i - 1 \quad 2 \leq i \leq n + 1 \]

Clearly, it follows that all the vertex labels are distinct and ranges between 0 to \( p - 1 \). Thus, \( g \) is a bijection. Hence, the tortoise graph \( (To)_{2n+1} \ (n \geq 2) \) is directed edge - graceful for all \( n \). The directed edge - graceful labeling of \( (To)_{11} \), \( (To)_{13} \), \( (To)_{15} \) and \( (To)_{17} \) are given in Fig. 7.27, Fig. 7.28, Fig. 7.29 and Fig. 7.30 respectively.

Fig. 7.27: \((To)_{11}\) with directed edge - graceful labeling
Fig. 7.28: \((To)_{13}\) with directed edge - graceful labeling

Fig. 7.29: \((To)_{15}\) with directed edge - graceful labeling

Fig. 7.30: \((To)_{17}\) with directed edge - graceful labeling
Definition 7.13

The friendship graph $C_3^{(n)}$ is the one – point union of $n$ copies of the cycle $C_3$

Theorem 7.14

The friendship graph $C_3^n (n \geq 3)$ is directed edge - graceful for all $n$.

Proof

Let $V\left(C_3^n\right) = \{v, v_1, v_2, \ldots, v_{2n}\}$ be the set vertices. Now, we orient the edges of $C_3^n$ such that the arc set $A$ is given by

Case (i): If $n$ is odd

$$A = \left\{ (v, v_i), 1 \leq i \leq 2n \right\} \cup (v_{4i-3}, v_{4i-2}), 1 \leq i \leq \frac{n+1}{2} \cup (v_{4i}, v_{4i-1}), 1 \leq i \leq \frac{n-1}{2}. \right\}

Case (ii): If $n$ is even

$$A = \left\{ (v, v_i), 1 \leq i \leq 2n \right\} \cup (v_{4i-3}, v_{4i-2}), 1 \leq i \leq \frac{n}{2} \cup (v_{4i}, v_{4i-1}), 1 \leq i \leq \frac{n-1}{2}. \right\}

The edges and their orientation of $C_3^n$ are as in Fig. 7.31:

![Fig. 7.31: $C_3^n$ with orientation](image_url)
We now label the arcs of $A$ as follows:

**Case (i): $n$ odd**

For $1 \leq i \leq 2n$

\[
\begin{align*}
  f((v, v_i)) &= \frac{i}{2} \quad \text{if } i = 1 \\
  2n - \frac{i - 1}{2} \quad &\text{if } i \equiv 1 \pmod{4} \\
  2n - \frac{i - 2}{2} \quad &\text{if } i \equiv 2 \pmod{4} \\
  f((v_{4i-3}, v_{4i-2})) &= 2n + 2i - 1 \quad 1 \leq i \leq \frac{n+1}{2} \\
  f((v_{4i}, v_{4i+1})) &= 2i + 2n \quad 1 \leq i \leq \frac{n-1}{2}
\end{align*}
\]

**Case (ii): $n$ even**

For $1 \leq i \leq 2n$

\[
\begin{align*}
  f((v, v_i)) &= \frac{i}{2} \quad \text{if } i = 1 \\
  2n - \frac{i - 1}{2} \quad &\text{if } i \equiv 1 \pmod{4} \\
  2n - \frac{i - 2}{2} \quad &\text{if } i \equiv 2 \pmod{4}
\end{align*}
\]
\[ f \left( (v_{4i-3}, v_{4i-2}) \right) = 2n + 2i - 1 \quad 1 \leq i \leq \frac{n}{2} \]

\[ f \left( (v_{4i}, v_{4i-1}) \right) = 2i + 2n \quad 1 \leq i \leq \frac{n}{2} \]

The computed values of \( f^+ (v) \), \( f^+ (v_i) \) and \( f^- (v) \), \( f^- (v_i) \) are given below:

\[
\begin{align*}
f^+ (v_i) &= \begin{cases} 
1 & i = 1 \\
4n + 1 & i = 2 \\
2n + 1 + i & i \equiv 3 \pmod{4} \\
5n & i \equiv 2 \pmod{4} \\
2n - \frac{i - 1}{2} & i \equiv 1 \pmod{4} \\
2n - \frac{i - 2}{2} & i \equiv 0 \pmod{4} \\
- \left(2n + 1\right) & i = 1 \\
0 & i = 2 \\
0 & i \equiv 3 \pmod{4} \\
0 & i \equiv 2 \pmod{4} \\
- 2n + \frac{i + 1}{2} & i \equiv 1 \pmod{4} \\
- 2n + \frac{i}{2} & i \equiv 0 \pmod{4}
\end{cases}
\]

\[
f^- (v_i) = \begin{cases} 
0 & i = 1 \\
0 & i = 2 \\
0 & i \equiv 3 \pmod{4} \\
0 & i \equiv 2 \pmod{4} \\
- 2n + \frac{i + 1}{2} & i \equiv 1 \pmod{4} \\
- 2n + \frac{i}{2} & i \equiv 0 \pmod{4}
\end{cases}
\]

\[
f^+ (v) = 0
\]

\[
f^- (v) = - \left( n \left( 2n + 1 \right) \right) \sum_{i=1}^{2n} f \left( (v, v_i) \right)
\]
Then the induced vertex labels are:

For $1 \leq i \leq 2n$

$$g(v_i) = \begin{cases} 
  i & i = 1 \\
  2n & i = 2 \\
  \frac{i}{i-1} & i \equiv 3 \pmod{4} \\
  \frac{2n+1-i}{2n} & i \equiv 2 \pmod{4} \\
  i & i \equiv 1 \pmod{4} \\
  i & i \equiv 0 \pmod{4} 
\end{cases}$$

Clearly, it follows that all the vertex labels are distinct and ranges between 0 to $p - 1$. Thus, $g$ is a bijection. Hence, the graph $C_3^n \ (n \geq 3)$ is directed edge - graceful for all $n$. The directed edge - graceful labeling of $C_3^6, C_3^7, C_3^8$ and $C_3^9$ are given in Fig. 7.32, Fig. 7.33, Fig. 7.34 and Fig. 7.35 respectively.

![Diagram](http://www.novapdf.com/)

**Fig. 7.32: $C_3^6$ with directed edge - graceful labeling**
Fig. 7.33: $C_7^7$ with directed edge - graceful labeling

Fig. 7.34: $C_8^8$ with directed edge - graceful labeling
Fig. 7.35: $C^9_3$ with directed edge - graceful labeling