I. Introduction

The fluctuating or ‘Brownian’ motion of a quantum particle coupled to an environment serves as a model for investigation of observable macroscopic effects in open quantum systems. The problem is usually handled using the influence functional method introduced by Feynman and Vernon [1], where the object of interest is the reduced dynamics of the system evolving under the influence of the environment, which is quantified by the influence functional. In the model studied by Caldeira and Leggett [2], the coordinate of the particle was coupled linearly to an infinite set of harmonic oscillators constituting the environment, and it was also assumed that the system and the environment were initially separable. However, in general, there are physical problems modeled by the quantum Brownian motion which involve some form of nonlinearity in the interaction between the system and its environment, for example, as pointed out by Hu, Paz and Zhang [3], in the strong-field conditions in the early Universe when one cannot exercise any control over the strength of the coupling. Hu et al. [3] developed techniques to obtain the influence functional perturbatively for the case of nonlinear system-environment coupling, but with the assumption of separable initial conditions.

There may be difficulties associated with the separable (product) initial state, which assumes a sudden artificial switch-on of the interaction between the system and the environment at time $t = 0$, and thus influences the subsequent short-time behavior of the system.
The treatment of the quantum Brownian motion with linear coupling has been generalized to the physically reasonable initial condition of a mixed state of the system and its environment by Hakim and Ambegaokar [4], Smith and Caldeira [5], and Grabert, Schramm and Ingold [6], and by us [7] for the case of a system in a Stern-Gerlach potential. There is no treatment available in the literature for the quantum Brownian motion with nonlinear system-environment couplings and generalized nonseparable initial conditions, which is the aim of the present work.

In this chapter [10] we use the influence functional to get the propagator for the problem when the particle is under the influence of a harmonic potential, and also an anharmonic potential, the so-called washboard potential [8, 9], which models the motion of a heavy charged particle in the interior or at the surface of a metal. From the propagator for the particle in a harmonic potential we obtain the master equation and the Wigner equation for a general nonseparable initial condition, and then for the specific case of a ‘thermal’ initial condition, introduced by Hakim and Ambegaokar [4], where the off-diagonal elements of the density matrix of the total system are suppressed and thus the transients due to the switching on of the system-environment interaction at $t = 0$ are avoided. We also establish a fluctuation-dissipation theorem connecting the dissipative and the fluctuating influences of the stochastic environment.

The present chapter is organized as follows. In Section II we present the influence functional for nonlinear system-environment couplings. In particular, for interactions quadratic in the position of the environmental oscillators coupled with an arbitrary linear function $f(x)$ of the system, the influence functional is obtained up to second order in the coupling constant. In Section III we use the influence functional with the simple form of coupling linear in the system coordinate, i.e., with $f(x) = x$, to obtain the propagator of the particle in a harmonic potential (Section IIIA), and in an additional anharmonic ‘washboard’ potential (Section IIIB). In Section IVA we use the propagator in Section IIIA to obtain the master equation and from it the Wigner equation for the quantum Brownian oscillator.
The inhomogeneities in the master equation under generalized initial conditions disappear for the case of the 'thermal' initial conditions in Section IVB. In Section V we establish a generalized fluctuation-dissipation theorem for our quantum Brownian oscillator. Finally in Section VI we summarize our results.

II. Influence Functional

Our model for the quantum Brownian motion consists of the usual system of a quantum particle of mass $M$ moving in a potential $V(x)$ and coupled to an environment of a set of harmonic oscillators. Here

$$S[x, \{q_n\}] = S_S[x] + S_E[\{q_n\}] + S_{SE}[x, \{q_n\}] \quad (V.1)$$

where

$$S_S[x] = \int_0^t ds \left[ \frac{1}{2} M \dot{x}^2 - V(x) \right] \quad (V.2)$$

is the system action,

$$S_E[\{q_n\}] = \int_0^t ds \sum_{n=1}^N \left[ \frac{1}{2} m_n \ddot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 q_n^2 \right] \quad (V.3)$$

is the environment action and

$$S_{SE}[x, \{q_n\}] = \int_0^t ds \sum_n \left[ - \lambda C_n f(x) q_n^k \right]$$

$$= \int_0^t ds \sum_n v_n(x) q_n^k \quad (V.4)$$

is the action for the system-environment interaction. As in Hu et al. [3] we have introduced a new dimensionless coupling constant $\lambda$ for use as a small parameter in the perturbative expansions and $v_n(x) = -\lambda C_n f(x)$ plays the role of a vertex function. Here $x$ denotes the
position of the particle, \( m_n, \omega_n \) and \( q_n \) are the mass, frequency and position of the \( n \)th environmental oscillator and \( k \) is an integer, the system-environment coupling is nonlinear with the environmental nonlinearity resting in the power \( k \) of \( q_n \) and the system effect being described by a general function \( f(x) \).

Now following Grabert et al. [6] we write down the reduced density matrix of the system as

\[
\rho(x_f, x', t) = \int dx dx' d\vec{x} d\vec{x}' \lambda_0(x, \vec{x}, x', \vec{x}') Z^{-1} \int Dx Dx' D\vec{x} \times \exp \left[ \frac{i}{\hbar} \left( S_S[x] - S_S[x'] + iS_{SE}^{EQ}[\vec{x}] \right) \right] \tilde{F}[x, x', \vec{x}]. \tag{V.5}
\]

Here \( Z^{-1} \) is a normalizing constant, the factor \( \lambda_0 \) comes from the initial condition and is called the preparation function. It describes the deviation of the initial (nonequilibrium) state from the equilibrium distribution. \( \tilde{F} \) is the influence functional produced by the environment and is given by

\[
\tilde{F}[x, x', \vec{x}] = \prod_n \int dq_n dq_n' dq_n'' Z_n^{-1} \int Dq_n Dq_n' Dq_n'' \times \exp \left[ \frac{i}{\hbar} \left( S_E[\{q_n\}] + S_{SE}[x, \{q_n\}] - S_E[\{q_n'\}] - S_{SE}[x', \{q_n'\}] \right) \right]
\]

\[
- \frac{1}{\hbar} \left( S_{SE}^{EQ}[\{\vec{q}_n\}] + S_{SE}^{EQ}[\vec{x}, \{\vec{q}_n\}] \right), \tag{V.6}
\]

assuming that initially the interacting system is in a thermal equilibrium state at a temperature \( T = (k_B \beta)^{-1} \), with

\[
q_n(t) = q'_n(t) = q_{nf}, \quad q_n(0) = \vec{q}_n(h\beta) = q_{ni}, \quad \vec{q}_n(0) = q'_n(0) = q_{ni'}, \quad \vec{x}(0) = \vec{x}', \quad \vec{x}(h\beta) = \vec{x}, \tag{V.7}
\]

where the superscript \( EQ \) in (V.6) stands for the Euclidean action which comes from the Euclidean functional integral representation of the the correlation that exists between the
system and the environment initially, and \( Z_{R_n}^{-1} \) is a normalization constant that ensures that \( \tilde{F} = 1 \) in the case of vanishing interactions. Thus it can be seen that the functional integration is over closed paths of the environment. Here

\[
S^{EQ}[\bar{x}, \{\bar{q}_n\}] = S_S^{EQ}[\bar{x}] + S_E^{EQ}[\{\bar{q}_n\}] + S_{SE}^{EQ}[\bar{x}, \{\bar{q}_n\}],
\]

(V.8)

with

\[
S_S^{EQ}[\bar{x}] = \int_0^{\hbar \beta} d\tau \left[ \frac{1}{2} \dot{\bar{x}}^2 + V(\bar{x}) \right],
\]

(V.9)

\[
S_E^{EQ}[\{\bar{q}_n\}] = \int_0^{\hbar \beta} d\tau \sum_{n=1}^N \left[ \frac{1}{2} m_n \dot{\bar{q}}_n^2 + \frac{1}{2} m_n \omega_n^2 \bar{q}_n^2 \right],
\]

(V.10)

and

\[
S_{SE}^{EQ}[\bar{x}, \{\bar{q}_n\}] = \int_0^{\hbar \beta} d\tau \sum_n \left[ \lambda C_n f(\bar{x}) \bar{q}_n^k \right]
\]

\[
= \int_0^{\hbar \beta} d\tau \sum_n \bar{v}_n(\bar{x}) \bar{q}_n^k.
\]

(V.11)

Now the influence functional

\[
\tilde{F}[x, x', \bar{x}] = \prod_n \tilde{F}_n[x, x', \bar{x}]
\]

\[
= \prod_n \exp \left[ \frac{i}{\hbar} \delta A_n[x, x', \bar{x}] \right]
\]

\[
= \exp \left[ \frac{i}{\hbar} \delta A[x, x', \bar{x}] \right],
\]

(V.12)

where \( \delta A = \sum_n \delta A_n \) is the total influence action.

As a consequence of the nonlinearity in the coupling, the influence functional cannot be obtained exactly. However, utilizing the small coupling constant \( \lambda \) we can proceed perturbatively. Following Hu et al. [3] we start with the influence functional of the linear model with
the system coordinates acting as the source terms and obtain the desired influence functional for the nonlinear case perturbatively from this by means of functional differentiation with respect to the source terms.

The influence functional of an environment where the environment oscillators are linearly coupled to the coordinate of the system is [3]

\[
\tilde{F}_n^{(1)}[J, J', J] = \int_{-\infty}^{\infty} dq_n dq'_n dq_{n_f} Z_R^{-1} q_{n_f}^{q_f} q_{n_i}^{q_{n_i}} q_{n_i}^{q_{n_i}} Dq_n Dq'_n D\bar{q}_n \times \exp \left[ \frac{i}{\hbar} \left\{ S_E[q_n] + \int_0^t ds J(S) q_n(S) - S_E[q'_n] - \int_0^t ds J'(S) q_n'(S) \right\} \right. \\
\left. \frac{1}{\hbar} \left\{ S_{E_E}^{EQ}[\bar{q}_n] + \int_0^t d\tau [-\bar{J}(\tau) \bar{q}_n(\tau)] \right\} \right] \\
= \left\langle \exp \left[ \frac{i}{\hbar} \left\{ \int_0^t ds J(S) q_n(S) - \int_0^t ds J'(S) q_n'(S) \right\} \right. \\
\left. \frac{1}{\hbar} \left\{ \int_0^t d\tau [-\bar{J}(\tau) \bar{q}_n(\tau)] \right\} \right] \right\rangle_0. \quad (V.13)
\]

Here \( \langle f(q_n, q'_n, \bar{q}_n) \rangle_0 \), i.e., average of any function of environment variables is given by

\[
\langle f(q_n, q'_n, \bar{q}_n) \rangle_0 = Z_R^{-1} q_{n_f}^{q_f} q_{n_i}^{q_{n_i}} q_{n_i}^{q_{n_i}} \times \exp \left[ \frac{i}{\hbar} \left\{ S_E[q_n] - S_E[q'_n] + i S_{E_E}^{EQ}[\bar{q}_n] \right\} \right] f(q_n, q'_n, \bar{q}_n) \\
= f \left( \frac{\hbar}{i \delta J(S)} - \frac{\hbar}{i \delta J'(S)} \right) \tilde{F}_n^{(1)}[J, J', J] \bigg|_{\{J, J', J=0\}}. \quad (V.14)
\]

Using this definition the influence functional can be expressed as

\[
\tilde{F}_n[x, x', \bar{x}] = \left\langle \exp \left[ \frac{i}{\hbar} \left\{ S_{SE}[x, q_n] - S_{SE}[x', q'_n] + i S_{SE}^{EQ}[\bar{x}, \bar{q}_n] \right\} \right] \right\rangle_0
\]

97
\[
\begin{align*}
\mathcal{F}_n(x, x', \bar{x}) &= \exp \left[ \frac{i}{\hbar} \left\{ S_{SE} \left[ x, \frac{\hbar}{i} \frac{\delta}{\delta J} \right] - S_{SE} \left[ x', -\frac{\hbar}{i} \frac{\delta}{\delta J'} \right] \ight. \right. \\
&\quad + \left. \left. i S_{SE}^{EQ} \left[ \bar{x}, -\frac{\delta}{\delta J} \right] \right\} \right] \mathcal{F}_n^{(1)}[J, J', \bar{J}] \bigg|_{(J, J', \bar{J} = 0)}. \tag{V.15}
\end{align*}
\]

Expanding the above exponential and keeping up to second order terms in \(\lambda\) we have the influence functional as

\[
\mathcal{F}_n(x, x', \bar{x}) = \exp \left[ \frac{i}{\hbar} \delta A_n[x, x', \bar{x}] \right], \tag{V.16}
\]

with

\[
\delta A_n[x, x', \bar{x}] = \left\{ \begin{array}{l}
\langle S_{SE}[x, q_n] \rangle_0 - \langle S_{SE}[x', q_n'] \rangle_0 + i \left\langle S_{SE}^{EQ}[\bar{x}, \bar{q}_n] \right\rangle_0 \\
+ \frac{i}{2\hbar} \left\{ \left( \langle S_{SE}[x, q_n] \rangle_0 \right)^2 - \left( \langle S_{SE}[x, q_n] \rangle_0 \right)^2 \right\} \\
+ \frac{i}{2\hbar} \left\{ \left( \langle S_{SE}[x', q_n'] \rangle_0 \right)^2 - \left( \langle S_{SE}[x', q_n'] \rangle_0 \right)^2 \right\} \\
- \frac{i}{\hbar} \left\{ \langle S_{SE}[x, q_n] S_{SE}[x', q_n'] \rangle_0 - \langle S_{SE}[x, q_n] \rangle_0 \langle S_{SE}[x', q_n'] \rangle_0 \right\} \\
- \frac{i}{2\hbar} \left\{ \left\langle S_{SE}^{EQ}[^{\bar{x}}, \bar{q}_n] \right\rangle_0 - \left\langle S_{SE}^{EQ}[^{\bar{x}}, \bar{q}_n] \right\rangle_0^2 \right\} \\
- \frac{1}{\hbar} \left\{ \langle S_{SE}[x, q_n] S_{SE}^{EQ}[\bar{x}, \bar{q}_n] \rangle_0 - \langle S_{SE}[x, q_n] \rangle_0 \langle S_{SE}^{EQ}[\bar{x}, \bar{q}_n] \rangle_0 \right\} \\
+ \frac{1}{\hbar} \left\{ \langle S_{SE}[x', q_n'] S_{SE}^{EQ}[\bar{x}, \bar{q}_n] \rangle_0 \right. \\
\left. - \langle S_{SE}[x', q_n'] \rangle_0 \langle S_{SE}^{EQ}[\bar{x}, \bar{q}_n] \rangle_0 \right\}.
\right. \tag{V.17}
\]

All the terms in Eq. (V.17) are calculated in terms of the unperturbed influence functional of Eq. (V.13) which can be solved exactly to give [6]
\[
\tilde{F}^{(1)}_n[J, J', \bar{J}] = \exp \left\{ \frac{1}{2\hbar} \int_0^t \int_0^{\sigma} d\tau d\sigma \ k_n^{(2)}(\tau - \sigma) \bar{J}(\tau) \bar{J}(\sigma) \right.
\]
\[
+ \frac{i}{\hbar} \int_0^t d\tau \int_0^{s_1} ds \ K_n^{*}(s - i\tau) \bar{J}(\tau) \{J(s) - J'(s)\}
\]
\[
- \frac{i}{\hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 [J(s_1) - J'(s_1)] \eta_n^{(1)}(s_1 - s_2) [J(s_2) + J'(s_2)]
\]
\[
- \frac{1}{\hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 [J(s_1) - J'(s_1)] \nu_n^{(1)}(s_1 - s_2)
\]
\[
\times [J(s_2) - J'(s_2)] \}
\]

(V.18)

Here

\[
\eta_n^{(1)}(u) = -\frac{1}{2m_n\omega_n} \sin(\omega_n u), \quad (V.19)
\]

\[
\nu_n^{(1)}(u) = \frac{1}{2m_n\omega_n} \mathcal{Z} \cos(\omega_n u), \quad (V.20)
\]

\[
\mathcal{Z} = \coth \left( \frac{\hbar\omega\beta}{2} \right), \quad \beta = \frac{1}{k_BT}, \quad (V.21)
\]

\[
k_n^{(2)}(\tau - \sigma) = \frac{1}{2m_n\omega_n} \cosh \left[ \omega_n \left( \frac{1}{2} \hbar\beta - (\tau - \sigma) \right) \right], \quad (V.22)
\]

\[
K_n(s - i\tau) = \frac{1}{2m_n\omega_n} \cosh \left[ \omega_n \left( \hbar\beta/2 - i(s - i\tau) \right) \right], \quad (V.23)
\]

Now using the form of the unperturbed influence functional \( \tilde{F}^{(1)}_n[J, J', \bar{J}] \) given by Eq. (V.18) we obtain
\[ \langle q_n(s) \rangle_0 = \langle q'_n(s) \rangle_0 = \langle \bar{q}_n(\tau) \rangle_0 = 0. \]  \hspace{1cm} (V.24)

The various two-point functions are

\[ \langle q_n(s_1)q_n(s_2) \rangle_0 = -i\hbar \left[ -\eta_n^{(1)}(s_1 - s_2)\text{sgn}(s_1 - s_2) + i\nu_n^{(1)}(s_1 - s_2) \right], \]  \hspace{1cm} (V.25)

\[ \langle q'_n(s_1)q'_n(s_2) \rangle_0 = -i\hbar \left[ \eta_n^{(1)}(s_1 - s_2)\text{sgn}(s_1 - s_2) + i\nu_n^{(1)}(s_1 - s_2) \right], \]  \hspace{1cm} (V.26)

\[ \langle q_n(s_1)q'_n(s_2) \rangle_0 = -i\hbar \left[ \eta_n^{(1)}(s_1 - s_2) + i\nu_n^{(1)}(s_1 - s_2) \right], \]  \hspace{1cm} (V.27)

\[ \langle \bar{q}_n(\tau_1)\bar{q}_n(\tau_2) \rangle_0 = \hbar k_n^{(2)}(\tau_1 - \tau_2), \]  \hspace{1cm} (V.28)

\[ \langle q_n(s_1)\bar{q}_n(\tau_1) \rangle_0 = \hbar K_n^*(s_1 - i\tau_1), \]  \hspace{1cm} (V.29)

\[ \langle q'_n(s_1)\bar{q}_n(\tau_1) \rangle_0 = \hbar K_n^*(s_1 - i\tau_1). \]  \hspace{1cm} (V.30)

Here \( \text{sgn} \) is the sign function given by \( \text{sgn}(u) = \frac{u}{|u|} \). Now Feynman diagrams are introduced to illustrate the perturbative calculations. Equations (V.25) to (V.30) represent the bath propagators. These are given in Fig.1. Here the vertices (coupling terms) are labeled as \( v, v' \) and \( \bar{v} \) and depicted by a solid circle, an open circle and a circle with line in it respectively.

The average \( (...)_0 \) is obtained by closing the bath propagator lines upon themselves.

We now consider the case where the coupling is quadratic in the environmental variables, i.e., when \( k = 2 \) in Eqs. (V.4) and (V.11). With the help of the above equations we compute the various terms in Eq. (V.17) and get the influence action \( \delta A_n \) as

\[
\delta A_n[x, x', \bar{x}] = \int_0^t ds \left\{ -\delta V_n(x) \right\} - \int_0^t ds \left\{ -\delta V_n(x') \right\} + \int_0^t d\tau \left\{ i\delta \bar{V}_n(\bar{x}) \right\}
\]
Figure 1: Propagators for the bath variables $qq, q'q', qq', \bar{q} \bar{q}, q\bar{q}$ and $q'\bar{q}$ are depicted in (a), (b), (c), (d), (e) and (f), respectively. The vertices $v, v'$ and $\bar{v}$ are denoted by a solid circle, an open circle and a circle with line in it, respectively.
\[-i \int_0^{\tau} d\tau \int_0^\sigma d\sigma \frac{\lambda^2}{2} f(\bar{\tau}(\sigma)) \left[ 2\hbar C_n^2 k_n^{(2)^2}(\tau - \sigma) \right] f(\bar{\tau}(\sigma))
\]
\[+ \int_0^t ds \int_0^s d\tau \frac{\lambda^2}{2} \left[ 4\hbar C_n^2 K_n^2(s - i\tau) \right] f(\bar{\tau}(\tau)) \left[ f(x(s)) - f(x'(s)) \right]
\]
\[-\lambda^2 \int_0^t ds_1 \int_0^{s_1} ds_2 \left[ f(x(s_1)) - f(x'(s_1)) \right] \eta_n^{(2)}(s_1 - s_2) \left[ f(x(s_2)) + f(x'(s_2)) \right]
\]
\[+ i\lambda^2 \int_0^t ds_1 \int_0^{s_1} ds_2 \left[ f(x(s_1)) - f(x'(s_1)) \right] \nu_n^{(2)}(s_1 - s_2)
\]
\[\times \left[ f(x(s_2)) - f(x'(s_2)) \right]. \quad (V.31)
\]

Here

\[\delta V_n(x) = \hbar \frac{\lambda C_n}{2m_n \omega_n} Z f(x), \quad (V.32)\]

\[\delta V_n(x') = \hbar \frac{\lambda C_n}{2m_n \omega_n} Z f(x'), \quad (V.33)\]

\[\delta \bar{V}_n(\bar{x}) = \hbar \frac{\lambda C_n}{2m_n \omega_n} Z f(\bar{x}), \quad (V.34)\]

\[k_n^{(2)^2}(\tau - \sigma) = \frac{1}{(2m_n \omega_n)^2} \cosh^2 \left[ \omega_n \left( \frac{1}{2} \hbar \beta - (\tau - \sigma) \right) \right], \quad (V.35)\]

\[K_n^2(s - i\tau) = \frac{1}{4m_n^2 \omega_n^2} \left[ Z^2 \cosh^2[\omega_n(\tau - is)] + \sinh^2[\omega_n(\tau - is)] \right.
\]
\[- Z \sinh[2\omega_n(\tau - is)] \right] \quad (V.36)\]

\[\eta_n^{(2)}(s) = -\hbar \frac{2C_n^2}{4m_n^2 \omega_n^2} Z \sin(2\omega_n s), \quad (V.37)\]
In Fig. 2 we give the diagramatic representation of the expressions obtained in Eq. (V.31). Here we note that the first three terms on the right hand side of Eq. (V.31) introduce a renormalization of the potential and are depicted by the diagrams (a), (b) and (c) in Fig. 2.

The next term in Eq. (V.31) is depicted by the diagram (d) in Fig. 2. The next term in Eq. (V.31) can be split into two terms
\[
\frac{\lambda^2}{2} \int_0^t ds \int_0^{h_\beta} d\tau \left[ 4\hbar C_n^2 K_n^2(s - i\tau) \right] f(\bar{x}(\tau)) f(x(s)),
\]  
(V.39)

and

\[
- \frac{\lambda^2}{2} \int_0^t ds \int_0^{h_\beta} d\tau \left[ 4\hbar C_n^2 K_n^2(s - i\tau) \right] f(\bar{x}(\tau)) f(x'(s)).
\]  
(V.40)

These are given by diagrams (e) and (f) in Fig. 2. The last two expressions in Eq. (V.31) come from the addition of the following three terms

\[
\int_0^t ds_1 \int_0^t ds_2 \frac{\lambda^2}{2} f(x(s_1)) \left\{ - \eta_n^{(2)}(s_1 - s_2) \text{sgn}(s_1 - s_2) + i\nu_n^{(2)}(s_1 - s_2) \right\} f(x(s_2)),
\]  
(V.41)

\[
\int_0^t ds_1 \int_0^t ds_2 \frac{\lambda^2}{2} f(x'(s_1)) \left\{ \eta_n^{(2)}(s_1 - s_2) \text{sgn}(s_1 - s_2) + i\nu_n^{(2)}(s_1 - s_2) \right\} f(x'(s_2)),
\]  
(V.42)

and

\[
\int_0^t ds_1 \int_0^t ds_2 \lambda^2 f(x(s_1)) \left\{ - \eta_n^{(2)}(s_1 - s_2) - i\nu_n^{(2)}(s_1 - s_2) \right\} f(x'(s_2)).
\]  
(V.43)

Equations (V.41), (V.42) and (V.43) are given by the diagrams (g), (h) and (i) in Fig. 2.

Taking the continuum limit, the influence functional becomes

\[
\tilde{F}[x, x', \bar{x}] = \exp \left\{ - \frac{i}{\hbar} \int_0^t ds \delta V(x) + \frac{i}{\hbar} \int_0^t ds \delta V(x') - \frac{1}{\hbar} \int_0^{h_\beta} d\tau \delta \bar{V}(\bar{x}) + \frac{1}{2\hbar} \int_0^{h_\beta} d\tau \int_0^{h_\beta} d\sigma k^{(2)}(\tau - \sigma) f(\bar{x}(\tau)) f(\bar{x}(\sigma)) + \frac{i}{\hbar} \int_0^t ds \int_0^{h_\beta} d\tau K^{(2)}(s - i\tau) f(\bar{x}(\tau)) \left[ f(x(s)) - f(x'(s)) \right] \right\}
\]
\[-\frac{i}{\hbar} \int_0^t ds \int_0^s du \left[ f(x(s)) - f(x'(s)) \right] \eta^{(2)}(s - u) \left[ f(x(u)) + f(x'(u)) \right] \]

\[-\frac{1}{\hbar} \int_0^t ds \int_0^s du \left[ f(x(s)) - f(x'(s)) \right] \nu^{(2)}(s - u) \]

\[x \left[ f(x(u)) - f(x'(u)) \right] \right\}. \tag{V.44} \]

Here

\[\delta V(x) = \sum_n \delta V_n(x) = \sum_n \hbar^2 \frac{\lambda C_n}{2 m_n \omega_n} \mathcal{Z} f(x) = \int_0^\infty \frac{d\omega}{\pi} \rho_D(\omega) \mathcal{Z} f(x), \tag{V.45} \]

\[\delta V(x') = \sum_n \delta V_n(x') = \sum_n \hbar^2 \frac{\lambda C_n}{2 m_n \omega_n} \mathcal{Z} f(x') = \int_0^\infty \frac{d\omega}{\pi} \rho_D(\omega) \mathcal{Z} f(x'), \tag{V.46} \]

\[\delta V(\bar{x}) = \sum_n \delta V_n(\bar{x}) = \sum_n \hbar^2 \frac{\lambda C_n}{2 m_n \omega_n} \mathcal{Z} f(\bar{x}) = \int_0^\infty \frac{d\omega}{\pi} \rho_D(\omega) \mathcal{Z} f(\bar{x}), \tag{V.47} \]

with

\[\rho_D(\omega) = \sum_n \delta(\omega - \omega_n) \frac{\pi \hbar \lambda C(\omega)}{2 m \omega}. \tag{V.48} \]

Also, in Eq. (V.44),

\[k^{(2)}(\tau - \sigma) = \sum_n 2\hbar^2 \lambda^2 C_n^2 k_n^{(2)} (\tau - \sigma) \]

105
Here is the spectral density of the environment oscillators. Now using
\[ \cosh^2 \left[ \omega \left( \frac{1}{2} \hbar \beta - (\tau - \sigma) \right) \right] = \frac{1}{2} \cosh \left[ 2 \omega \left( \frac{1}{2} \hbar \beta - (\tau - \sigma) \right) \right] + \frac{1}{2}, \]  
and
\[ \cosh \left[ 2\omega \left( \frac{1}{2} \hbar \beta - \tau \right) \right] = \frac{\sinh [\omega \hbar \beta]}{\hbar \beta} \sum_{k=\infty}^{\infty} \frac{4\omega}{4\omega^2 + \nu_k^2} e^{i\nu_k \tau}, \]  
where \( 0 \leq \tau \leq \hbar \beta \) and \( \nu_k = 2\pi k/\hbar \beta \) we have
\[ k^{(2)}(\tau - \sigma) = \frac{M}{\hbar \beta} \sum_{k=\infty}^{\infty} \zeta_k^{(2)} e^{i\nu_k (\tau - \sigma)} + \int_0^\infty \frac{d\omega}{\pi} I(\omega)(\mathcal{Z}^2 - 1), \]  
where
\[ \zeta_k^{(2)} = \frac{8}{M} \int_0^\infty \frac{d\omega}{\pi} I(\omega) \mathcal{Z} \frac{\omega}{4\omega^2 + \nu_k^2}, \]  
and
\[ K^{*{(2)}}(s-i\tau) = \sum_n 2\hbar \lambda^2 C_n^2 K^{*{(2)}}_n(s-i\tau) \]
\[ = \sum_n 2\hbar \frac{\lambda^2 C_n^2}{(2m \omega_n)^2} \left[ \mathcal{Z}^2 \cosh^2[\omega_n(\tau - is)] + \sinh^2[\omega_n(\tau - is)] \right]. \]
\[
-Z \sinh[2\omega_n(\tau - is)]
= \int_0^\infty \frac{d\omega}{\pi} 2I(\omega) \left\{ \left[ \frac{1}{2}(Z^2 + 1) \cos(2\omega s) + iZ \sin(2\omega s) \right] \cosh(2\omega \tau) 
- i \left[ \frac{1}{2}(Z^2 + 1) \sin(2\omega s) - iZ \cos(2\omega s) \right] \sinh(2\omega \tau) 
+ \frac{1}{2}(Z^2 - 1) \right\}.
\]

(V.55)

Now using

\[
cosh[2\omega \tau] = \frac{\sinh[2\omega \hbar \beta]}{\hbar \beta} \sum_{k=-\infty}^{\infty} \frac{2\omega}{[4\omega^2 + \nu_k^2]} e^{i\nu_k \tau} 
- \frac{1 - \cosh[2\omega \hbar \beta]}{\hbar \beta} \sum_{k=-\infty}^{\infty} \frac{\nu_k}{[4\omega^2 + \nu_k^2]} e^{i\nu_k \tau},
\]

(V.56)

and

\[
\sinh[2\omega \tau] = \frac{1}{\hbar \beta} \{ \cosh[2\omega \hbar \beta] - 1 \} \sum_{k=-\infty}^{\infty} \frac{2\omega}{[4\omega^2 + \nu_k^2]} e^{i\nu_k \tau} 
+ \frac{1}{\hbar \beta} \sinh[2\omega \hbar \beta] \sum_{k=-\infty}^{\infty} \frac{\nu_k}{[4\omega^2 + \nu_k^2]} e^{i\nu_k \tau},
\]

(V.57)

we get

\[
K^{(2)}(s - i\tau) = \frac{M}{\hbar \beta} \sum_{k=-\infty}^{\infty} [g_k(s) + \bar{h_k}(s)] e^{i\nu_k \tau} 
+ \int_0^\infty \frac{d\omega}{\pi} I(\omega)(Z^2 - 1).
\]

(V.58)

Here

\[
g_k(s) = \frac{8}{M} \int_0^\infty \frac{d\omega}{\pi} I(\omega) Z \frac{\omega}{[4\omega^2 + \nu_k^2]} \cos(2\omega s),
\]

(V.59)

107
and

\[ h_k(s) = \frac{4}{M} \int_0^\infty \frac{d\omega}{\pi} I(\omega) Z \frac{\nu_k}{[4\omega^2 + \nu_k^2]} \sin(2\omega s). \]  \hspace{1cm} (V.60)

The other two terms in Eq. (V.44) are defined as

\[ \eta^{(2)}(s) = \sum_n \lambda^2 \eta_n^{(2)}(s) \]
\[ = - \int_0^\infty \frac{d\omega}{\pi} 2I(\omega) Z \sin(2\omega s), \]  \hspace{1cm} (V.61)

\[ \nu^{(2)}(s) = \sum_n \lambda^2 \nu_n^{(2)}(s) \]
\[ = \int_0^\infty \frac{d\omega}{\pi} I(\omega) \left[ (Z^2 - 1) + (Z^2 + 1) \cos(2\omega s) \right]. \]  \hspace{1cm} (V.62)

For the case of separable initial conditions, Eq. (V.44) reduces satisfactorily to the influence functional obtained by Hu et al. [3].

### III. The Propagator

In the following we use the influence functional obtained in Sec. II, with \( f(x) = x \), i.e., with couplings linear in the system coordinate, to get the propagator for the particle in a harmonic potential and an additional anharmonic potential. The propagator in the path-integral representation is given by

\[ J(x_f, x_f', t, x_i, x_i', \bar{x}, \bar{x}') = \frac{1}{Z} \int D\bar{x} Dx Dx' D\bar{x}' \]
\[ \times \exp \left[ \frac{i}{\hbar} \left( S_S[x] - S_S[x'] + iS_S^{EQ}[\bar{x}] \right) \right] \tilde{F}[x, x', \bar{x}], \]  \hspace{1cm} (V.63)

where \( S_S[x] \) is the action (V.2) describing the system and \( S_S^{EQ}[\bar{x}] \) is its Euclidean counterpart.
A. Harmonic potential

We consider the quantum particle in a harmonic potential,

$$V(x) = V_h(x, t) \equiv \frac{1}{2} M \omega_0^2 x^2 - x F(t) \quad (V.64)$$

is the harmonic potential and $F(t)$ is an external time-dependent force, acting on the system. As in Grabert, Schramm and Ingold [6] we assume that this force does not influence the initial state, i.e.,

$$F(t) = 0, \quad t \leq 0. \quad (V.65)$$

Now the propagator $J$ in (V.63) can be written as

$$J(x_f, x'_f, t, x_i, x'_i, \bar{x}, \bar{x}') = \frac{1}{Z} \int Dx Dx' D\bar{x} D\bar{x}' \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left[ \frac{1}{2} M \dot{x}^2 - V(x) + \frac{1}{2} M \dot{x'}^2 - V(x') \right] - \frac{i}{\hbar} \int_0^t ds \delta V(x) + \frac{i}{\hbar} \int_0^t ds \delta V(x') - \frac{1}{\hbar} \int_0^t d\tau \delta \bar{V}(\bar{x}) \right. \right.$$  

$$+ \frac{1}{2 \hbar} \int_0^t d\tau \int d\sigma \kappa^{(2)}(\tau - \sigma) \bar{x}(\tau) \bar{x}(\sigma)$$  

$$+ \frac{i}{\hbar} \int_0^t ds \int d\tau \kappa^{*^{(2)}}(s - i \tau) \bar{x}(s) x(s) - x'(s)]$$  

$$- \frac{i}{\hbar} \int_0^t ds \int du [x(s) - x'(s)] \eta^{(2)}(s - u) [x(u) + x'(u)]$$  

$$- \frac{1}{\hbar} \int_0^t ds \int du [x(s) - x'(s)] \nu^{(2)}(s - u) [x(u) - x'(u)] \right\}, \quad (V.66)$$

where $Z$ is a normalization constant. All the terms appearing in $J$ have been described in the previous section. It can be seen that the functional integrals in Eq. (V.66) (with $f(x) = 109$)
and \( f(\vec{x}) = \vec{x} \) are Gaussian and hence can be worked out using the fact that significant contribution to the effective action comes from the minimal action paths. Proceeding in a manner similar to Grabert, Schramm and Ingold [6] we get the propagator as

\[
J = \frac{1}{Z} \exp \left\{ \frac{i}{\hbar} \sum (r_f, q_f, t, r_i, q_i, \vec{r}, \vec{q}) \right\}, \tag{V.67}
\]

where

\[
\frac{i}{\hbar} \sum (r_f, q_f, t, r_i, q_i, \vec{r}, \vec{q}) = - (\alpha_1 \vec{r}^2 + \alpha_2 \vec{q}^2) + i\alpha_3 (q_f r_f + q_i r_i)
+ i\alpha_4 q_i r_f + i\alpha_5 q_f r_i + i\alpha_6 q_i \vec{r}
- \alpha_7 q_i \vec{q} + i\alpha_8 q_f \vec{r} - \alpha_9 q_f \vec{q}
- (\alpha_{10} q_i^2 + \alpha_{11} q_f q_i + \alpha_{12} q_f^2)
+ i\alpha_{13} q_i + i\alpha_{14} q_f. \tag{V.68}
\]

with

\[
q(s) = x(s) - x'(s), \quad r(s) = \frac{x(s) + x'(s)}{2}, \tag{V.69}
\]

\[
q(0) = q_i = x_i - x'_i, \quad r(0) = r_i = \frac{x_i + x'_i}{2}, \tag{V.70}
\]

\[
q(t) = q_f = x_f - x'_f, \quad r(t) = r_f = \frac{x_f + x'_f}{2}, \tag{V.71}
\]

\[
\vec{q} = \vec{x} - \vec{x}', \quad \vec{r} = \frac{\vec{x} + \vec{x}'}{2}. \tag{V.72}
\]

The various coefficients on the RHS of Eq. (V.68) are

\[
\alpha_1 = \frac{M}{2\hbar}, \quad \alpha_2 = \frac{M}{2\hbar} \Omega^{(2)}, \quad \alpha_3 = \frac{M}{\hbar} \dot{G}^+_+(t), \tag{V.73}
\]

\[
\alpha_4 = - \frac{M}{\hbar} \frac{1}{G^+_+(t)}, \quad \alpha_5 = - \frac{M}{\hbar} \frac{1}{G^-_+(t)}, \quad \alpha_6 = \frac{M}{\hbar} C_1^{(2)+}(t),
\]

\[
\alpha_7 = - \frac{M}{\hbar} C_2^{(2)+}(t), \quad \alpha_8 = \frac{M}{\hbar} C_1^{(2)-}(t), \quad \alpha_9 = - \frac{M}{\hbar} C_2^{(2)-}(t),
\]

\[
\alpha_{10} = \frac{M}{2\hbar} R^{(2)++}(t), \quad \alpha_{11} = \frac{M}{\hbar} R^{(2)+-}(t), \quad \alpha_{12} = \frac{M}{2\hbar} R^{(2)-+}(t),
\]

\[
\alpha_{13} = \frac{1}{\hbar} \int_0^t ds \frac{G^+_+(t-s)}{G^+_+(t)} F_1(s), \quad \alpha_{14} = \frac{1}{\hbar} \int_0^t ds \frac{G^-_-(s)}{G^-_-(t)} F_1(s). \tag{V.73}
\]
with

$$\Lambda = \frac{1}{\hbar \beta} \sum_{k=-\infty}^{\infty} u_k^{(2)}, \tag{V.74}$$

$$u_k^{(2)} = \left( \nu_k^2 + \omega_0^2 - \zeta_k^{(2)} \right)^{-1}, \tag{V.75}$$

$$\nu_k = \frac{2\pi k}{\hbar \beta}, \tag{V.76}$$

$$\Omega^{(2)} = \frac{1}{\hbar \beta} \sum_{k=-\infty}^{\infty} u_k^{(2)} \left( \omega_0^2 - \zeta_k^{(2)} \right), \tag{V.77}$$

$$C_{m}^{(2)+}(t) = \int_0^t ds \frac{G_+(t-s)}{G_+(t)} C_{m}^{(2)}(s), \quad m = 1, 2, \tag{V.78}$$

$$C_{m}^{(2)-}(t) = \int_0^t ds \frac{G_-(s)}{G_-(t)} C_{m}^{(2)}(s), \quad m = 1, 2, \tag{V.79}$$

$$C_1^{(2)}(s) = \frac{1}{\hbar \beta} \sum_{k=-\infty}^{\infty} u_k^{(2)} g_k(s), \tag{V.80}$$

$$C_2^{(2)}(s) = \frac{1}{\hbar \beta} \sum_{k=-\infty}^{\infty} \nu_k u_k^{(2)} h_k(s), \tag{V.81}$$

$$R^{(2)+-}(t) = \int_0^t ds \int_0^t du \, R^{(2)}(s,u) \frac{G_+(t-s)}{G_+(t)} \frac{G_-(u)}{G_-(t)}, \tag{V.82}$$

$$R^{(2)}(s,u) = R^{(2)'}(s,u) + \frac{\nu^{(2)}(s-u)}{M}, \tag{V.83}$$

$$R^{(2)'}(s,u) = -\Lambda C_1^{(2)}(s) C_1^{(2)}(u)$$

$$+ \frac{1}{\hbar \beta} \sum_{k=-\infty}^{\infty} u_k^{(2)} \left[ g_k(s)g_k(u) - h_k(s)h_k(u) \right], \tag{V.84}$$

\(G_+(t)\) is a solution of the equation

$$\ddot{r} + \frac{2}{M} \int_0^s du \ \eta^{(2)}(s-u) r(u) + \omega_0^2 r = 0, \tag{V.85}$$

satisfying \(G_+(0) = 0\) and \(\dot{G}_+(0) = 1,\)

$$G_-(s) = \frac{G_+(t-s) \dot{G}_+(t) - G_+(t) \dot{G}_+(t-s)}{G_+(t)G_+(t) - G_+^2(t)} \tag{V.86}$$
and

\[
F_1(s) = \left[ F(s) - \int_0^\infty \frac{d\omega}{\pi} \rho_D(\omega) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right].
\] (V.87)

The normalization constant \(Z\) is found to be

\[
Z(t) = \left( \frac{2\pi \hbar}{M} \right)^{3/2} \wedge^{1/2} |G_+(t)|.
\] (V.88)

Eqs. (V.67) to (V.88) determine the propagator. It is to be noticed that it has the same form as the corresponding linear coupling case [6] but now with rather complicated coefficients and the coefficients have an additional temperature-dependent term \(\coth(\hbar \omega/2k_BT)\) which appears in the nonlinear coupling problem even for separable initial conditions [3].

B. Additional anharmonic potential

We now compute the propagator for the quantum particle in a potential

\[
V(x) = V_h(x) + V_a(x),
\] (V.89)

where \(V_h(x)\) is the harmonic potential (V.64), and

\[
V_a(x) = -V_0 \cos(k_0x)
\] (V.90)

is the anharmonic potential. The potential

\[
V_w(x) \equiv -V_0 \cos(k_0x) - xF
\] (V.91)

is called the washboard potential, with \(F\) being a time-independent external force. The washboard potential gives an idealized description of the motion of a heavy charged particle in the interior or at the surface of a metal [8], where the underlying crystal provides the periodic potential (the first term on the RHS of (V.91)) with lattice constant \(2\pi/k_0\), and there is a potential drop \(2\pi F/k_0\) per period (the second term on the RHS of (V.91)) because of friction with the conduction electrons. The washboard potential is often used to model a single Josephson tunnel junction.
The propagator $J$ as given in Eq. (V.63) has terms like

$$\exp \left\{ - \frac{i}{\hbar} \int_0^t ds \, V_a(x) \right\}, \quad \exp \left\{ \frac{i}{\hbar} \int_0^t ds \, V_a(x') \right\}, \quad \exp \left\{ - \frac{1}{\hbar} \int_0^\beta d\tau \, V_a(x(\tau)) \right\}. $$

Because of the nonlinearities that enter due to $V_a(x)$, a direct evaluation of the functional integrations in $J$ is not possible. However, proceeding in a manner suggested by Fisher and Zwerger [8] we write

$$\exp \left\{ - \frac{i}{\hbar} \int_0^t ds \, V_a(x) \right\} = \sum_{n_1=0}^{\infty} \left( \frac{iV_0}{2\hbar} \right)^{n_1} \int_{\{e_i\}} \prod_{i=1}^{n_1} \int ds_1 \int ds_2 \ldots \int ds_{n_1}$$

$$\times \exp \left\{ - \frac{i}{\hbar} \int_0^t ds \, \rho(s) x(s) \right\}, \quad (V.92)$$

where we have introduced $n_1$ variables or 'charges' $e_i$ with a 'charge density'

$$\rho(s) = \hbar k_0 \sum_{i=1}^{n_1} e_i \delta(s - s_i), \quad e_i = \pm 1. \quad (V.93)$$

Similarly, expanding the other two terms with two new sets of 'charges' $\sigma_k$ and $\lambda_j$, we have

$$\exp \left\{ \frac{i}{\hbar} \int_0^t ds \, V_a(x'(s)) \right\} = \sum_{n_2=0}^{\infty} \left( - \frac{iV_0}{2\hbar} \right)^{n_2} \int_{\{\sigma_k\}} \prod_{k=1}^{n_2} \int ds'_1 \int ds'_2 \ldots \int ds'_{n_2}$$

$$\times \exp \left\{ \frac{i}{\hbar} \int_0^t ds \, \rho'(s) x'(s) \right\}, \quad (V.94)$$

where

$$\rho'(s) = \hbar k_0 \sum_{k=1}^{n_2} \sigma_k \delta(s - s'_k), \quad \sigma_k = \pm 1, \quad (V.95)$$

and

$$\exp \left\{ - \frac{1}{\hbar} \int_0^\beta d\tau \, V_a(\bar{x}(\tau)) \right\} = \exp \left\{ \frac{V_0}{\hbar} \int_0^\beta d\tau \, \cos(k_0 \bar{x}(\tau)) \right\}$$

$$= \sum_{n_3=0}^{\infty} \left( \frac{V_0}{2\hbar} \right)^{n_3} \int_{\{\lambda_j\}} \prod_{j=1}^{n_3} \int \tau_1 \int \tau_2 \ldots \int \tau_{n_3}$$

$$\times \exp \left\{ - \frac{1}{\hbar} \int_0^\beta d\tau \, \bar{\rho}(\tau) \bar{x}(\tau) \right\}, \quad (V.96)$$

113
\[\overline{\rho}(\tau) = \frac{i\hbar k_0}{\omega_0} \sum_{j=1}^{n_3} \lambda_j \delta(\tau - \tau_j), \quad \lambda_j = \pm 1. \tag{V.97}\]

We now use Eq. (V.63) to have the propagator as

\[
J(x_f, x'_f, t, x_i, x'_i, \overline{x}, \overline{x'}) = \sum_{n_1, n_2, n_3=0}^{\infty} \left( \frac{iV_0}{2\hbar} \right)^{n_1} \left( \frac{-iV_0}{2\hbar} \right)^{n_2} \left( \frac{V_0}{2\hbar} \right)^{n_3} \\
\times \sum_{(c_1, \sigma_1, \lambda_1)} \int ds_1 \int ds_2 \ldots \int ds'_1 \int ds'_2 \ldots \int ds'_{n_2} \int d\tau_1 \int d\tau_2 \ldots \int d\tau_{n_3} \\
\times J_1(x_f, x'_f, t, x_i, x'_i, \overline{x}, \overline{x'}, \rho, \rho', \overline{\rho}). \tag{V.98}\]

Here we assume that the criterion of uniform convergence as established by Chen et al. [9] is satisfied in that there exists a \(V_0\), say \(\overline{V}\), such that for \(V_0 < \overline{V}\) the above series has a uniform convergence. In the above equation \(J_1\) contains the functional integrals,

\[
J_1(x_f, x'_f, t, x_i, x'_i, \overline{x}, \overline{x'}, \rho, \rho', \overline{\rho}) = \frac{1}{Z} \int DxDx'D\overline{x} \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left[ \frac{1}{2} M \dot{x}^2 - \frac{1}{2} M \omega_0^2 \overline{x}^2 \right] \right. \\
- \frac{i}{\hbar} \int_0^t ds \left[ \frac{1}{2} M \dot{x}'^2 - \frac{1}{2} M \omega_0^2 \overline{x}'^2 \right] \\
- \frac{1}{\hbar} \int_0^{\hbar} d\tau \left[ \frac{1}{2} M \dot{x}^2 + \frac{1}{2} M \omega_0^2 \overline{x}^2 \right] \\
+ \frac{i}{\hbar} \int_0^t ds \left[ F_1(s)[x(s) - x'(s)] - \frac{1}{\hbar} \int_0^t ds \rho(s)x(s) \right] \\
+ \frac{i}{\hbar} \int_0^t ds \rho'(s)x'(s) - \frac{1}{\hbar} \int_0^{\hbar} d\tau \overline{\rho}(\tau)\overline{x}(\tau) \\
+ \frac{1}{2\hbar} \int_0^{\hbar} d\tau \int_0^{\hbar} d\sigma k^{(2)}(\tau - \sigma)\overline{x}(\tau)\overline{x}(\sigma) \\
+ \frac{i}{\hbar} \int_0^t ds \int_0^{\hbar} d\tau K^{(2)}(s - i\tau)\overline{x}(\tau)[x(s) - x'(s)] \\
- \frac{i}{\hbar} \int_0^t ds \int_0^s du [x(s) - x'(s)]\eta^{(2)}(s - u)[x(u) + x'(u)] \\
\right. \\
114
\[
-\frac{1}{h} \int_0^t ds \int_0^s du [x(s) - x'(s)] \nu^{(2)}(s - u) \\
\times [x(u) - x'(u)]. \tag{V.99}
\]

The advantage of using the representations in Eqs. (V.92), (V.94) and (V.96) is that the functional integrals in Eq. (V.99) are Gaussian and can be evaluated exactly. Proceeding in the same manner as before, we have

\[
J_1(q_f, r_f, t, q_i, r_i, \bar{q}, \bar{r}) = \frac{M}{2\pi h G_+(t) \left(\frac{2\pi h \Lambda}{M}\right)^{1/2}} \\
\times \exp \left\{ \frac{1}{2\hbar M} B^2 \right\} \exp\{-P\} \\
\times \exp \left\{ -\frac{M}{\hbar} \left[ \frac{1}{2\Lambda} \bar{r}^2 + \frac{\Omega^{(2)}}{2} \bar{q} \right] \right\} - iM \frac{1}{\hbar^2 \beta \Lambda} \bar{r} \rho_0 \\
+ iM \frac{\bar{q}}{\hbar^2 \beta} \rho_7 + iM \frac{1}{\hbar} (q_f r_f + q_i r_i) \frac{\dot{G}_+(t)}{G_+(t)} \\
- \frac{iM}{\hbar} \left( q_f r_i \frac{1}{G_-(t)} + q_i r_f \frac{1}{G_+(t)} \right) \\
+ iM \frac{\bar{r}}{\hbar} \left( q_i C_1^{(2)}+(t) + q_f C_1^{(2)}-(t) \right) \\
+ iM \frac{\bar{q}}{\hbar} \left( q_i C_2^{(2)}+(t) + q_f C_2^{(2)}-(t) \right) \\
- \frac{M}{2\hbar} \left[ q_i^2 R^{(2)}++(t) + 2q_i q_f R^{(2)}++-(t) + q_f^2 R^{(2)}--(t) \right] \\
+ \frac{i}{\hbar} \int_0^t ds \left[ q_i \frac{G_+(t-s)}{G_+(t)} + q_f \frac{G_-(s)}{G_-(t)} \right] F_1(s) \\
- \frac{1}{2} \rho_2(s) + \rho_4(s) - i[-\rho_3(s) + \rho_5(s)] \\
+ \frac{M}{\hbar} q_i \int_0^t ds \int_0^t du \ R^{(2)}(s, u)[f_1(u) - f_2(u)] \frac{G_+(t-s)}{G_+(t)} \\
+ \frac{M}{\hbar} q_f \int_0^t ds \int_0^t du \ R^{(2)}(s, u)[f_1(u) - f_2(u)] \frac{G_-(s)}{G_-(t)}
\]

115
\[-\frac{i}{\hbar} r_f \int_0^t ds \rho_1(s) \frac{G_+(s)}{G_+(t)} - \frac{i}{\hbar} r_i \int_0^t ds \rho_1(s) \frac{G_-(t-s)}{G_-(t)}\]

\[-\frac{M}{\hbar} \bar{q} \int_0^t ds [f_1(s) - f_2(s)] C_2^{(2)}(s)\]

\[-\frac{i}{\hbar} \bar{r} \int_0^t ds \int_0^s du \rho_1(s) G_+(s-u) C_1^{(2)}(u)\]

\[+\frac{i}{\hbar} \bar{r} \int_0^t ds \int_0^t du \rho_1(s) \frac{G_+(s)}{G_+(t)} G_+(t-u) C_1^{(2)}(u)\]  \hspace{1em} (V.100)

Here \(q(s), \tau(s)\) and \(\bar{q}, \bar{\tau}\) are as defined in Eqs. (V.69) and (V.72), respectively, and

\[\rho_1(s) = \rho(s) - \rho'(s),\]  \hspace{1em} (V.101)

\[\rho_2(s) = \rho(s) + \rho'(s),\]  \hspace{1em} (V.102)

\[\rho_3(s) = \frac{k_0}{\hbar \beta^2} \sum_{k=\infty}^{\infty} u_k^{(2)^2} \left( \sum_{j=1}^{n_3} \lambda_j \cos(\nu_k \tau_j) \right) g_k(s),\]  \hspace{1em} (V.103)

\[\rho_4(s) = \frac{k_0}{\beta} \sum_{k=\infty}^{\infty} u_k^{(2)^2} \left( \sum_{j=1}^{n_3} \lambda_j \sin(\nu_k \tau_j) \right) h_k(s),\]  \hspace{1em} (V.104)

\[\rho_5(s) = \frac{k_0}{\beta} \sum_{k=\infty}^{\infty} u_k^{(2)^2} \left( \sum_{j=1}^{n_3} \lambda_j \cos(\nu_k \tau_j) \right) g_k(s),\]  \hspace{1em} (V.105)

\[\rho_6 = \frac{\hbar k_0}{M} \sum_{k=\infty}^{\infty} u_k^{(2)^2} \left( \sum_{j=1}^{n_3} \lambda_j \cos(\nu_k \tau_j) \right),\]  \hspace{1em} (V.106)

\[\rho_7 = \frac{\hbar k_0}{M} \sum_{k=\infty}^{\infty} u_k^{(2)^2} \nu_k \left( \sum_{j=1}^{n_3} \lambda_j \sin(\nu_k \tau_j) \right),\]  \hspace{1em} (V.107)

\[f_1(s) = \frac{1}{M} \int_0^{t-s} du G_+(t-s-u) \rho_1(u),\]  \hspace{1em} (V.108)

\[f_2(s) = \frac{1}{M} \frac{G_+(t-s)}{G_+(t)} \int_0^t du G_+(t-u) \rho_1(u).\]  \hspace{1em} (V.109)
\( G_+ (t) \) and \( G_- (s) \) are as given by Eqs. (V.85) and (V.86), respectively. Also,

\[
B = \left[ \frac{M}{\hbar \beta} \rho_6 + \frac{\dot{G}_+ (t)}{G_+ (t)} \int_0^t ds \rho_1 (s) G_+ (s) \right.
\]

\[+ C_1^{(2)+} (t) \int_0^t ds \rho_1 (s) G_+ (s) + \int_0^t ds \rho_1 (s) \frac{G_- (t - s)}{G_- (t)}
\]

\[+ \int_0^t ds \int_0^s du \rho_1 (s) G_+ (s - u) C_1^{(2)} (u)
\]

\[- \int_0^t ds \int_0^t du \rho_1 (s) \frac{G_+ (s)}{G_+ (t)} G_+ (t - u) C_1^{(2)} (u) \right],
\]

(V.110)

\[
P = \frac{1}{\hbar M} \left( \int_0^t ds \rho_1 (s) G_+ (s) \right)^2 \left[ - \frac{\Omega^{(2)}}{2} + C_2^{(2)+} (t) - \frac{1}{2} R^{(2)++} (t) \right]
\]

\[+ \frac{1}{\hbar^2 \beta} \rho_7 \int_0^t ds \rho_1 (s) G_+ (s) \]

\[- \frac{1}{\hbar M} \left( \int_0^t ds \rho_1 (s) G_+ (s) \right) \int_0^t du \frac{G_+ (t - u)}{G_+ (t)} \left[ F_1 (u) - \frac{1}{2} \rho_2 (u) \right.
\]

\[+ \rho_4 (u) - i[\rho_3 (u) + \rho_5 (u)] \]

\[- \frac{1}{\hbar} \left( \int_0^t ds \rho_1 (s) G_+ (s) \right) \int_0^t du \int_0^t dv R^{(2)} (u, v) [f_1 (v) - f_2 (v)] \frac{G_+ (t - u)}{G_+ (t)}
\]

\[+ \frac{1}{\hbar} \left( \int_0^t ds \rho_1 (s) G_+ (s) \right) \int_0^t du \left[ f_1 (u) - f_2 (u) \right] C_2^{(2)} (u). \]

(V.111)

All the other terms are as given before. Equation (V.98) along with Eqs. (V.100) to (V.111) give the required propagator. The extra terms in Eq. (V.100) compared to Eq. (V.68) come primarily due to the additional inhomogeneities, in the equations of motion arising from the charge densities (V.93), (V.95) and (V.97) for both the imaginary time paths (because of nonseparable initial states) and the real time paths. It can be seen that dropping the anharmonicity we recover the propagator given by Eqs. (V.67) and (V.68) in Section IIIA.
This completes the derivation of the propagator for a particle in a harmonic plus an anharmonic potential where the system-environment coupling is nonlinear and initial conditions are quite general. The influence functional is obtained up to the second order of perturbation. The treatment for a quantum particle in the washboard potential was given by Chen et al. [9] using generalized initial conditions but for linear system-environment couplings. They assumed an Ohmic spectral density of the reservoir for which the generalized initial state happens to be equivalent to the product (separable) initial state. Our treatment is more general in that we do not make any assumptions on the reservoir spectral density, and the various terms are worked out explicitly.

IV. Applications

In this section we use the propagator (V.67) obtained in Section IIIA to derive the master equation for the dynamics of the reduced density matrix of the system in a harmonic potential. We first derive the master equation for a general nonseparable initial condition and then consider the specific ‘thermal’ initial condition [4]. We also derive the corresponding Wigner equations.

A. General nonseparable initial condition

1. The master equation

The time variation of the reduced density matrix of the oscillator is given as

$$\frac{\partial}{\partial t} \rho(q_f, r_f, t) = \int dq_i dr_i d\bar{q} d\bar{r} \frac{\partial}{\partial t} J(q_f, r_f, t, q_i, r_i, \bar{q}, \bar{r}) \lambda_0(q_i, r_i, \bar{q}, \bar{r}).$$  \hspace{1cm} (V.112)

where $\lambda_0$ is the preparation function. We now use Eqs. (V.67), (V.68) for $J$ and the simplified method of Paz [11] to get the master equation. The trick is to express $\partial J/\partial t$ in terms of derivatives of the final coordinates so that the differential can be pulled out of the integral in Eq. (V.112). From the form of $J$ given by Eqs. (V.67), (V.68) we have
Here $\alpha_0 = 1/Z$ with $Z$ given by Eq. (V.88). Now

\[
\frac{i}{\hbar} J \frac{\partial \Sigma}{\partial t} = \left\{ - (\dot{\alpha}_1 \overline{p}^2 + \dot{\alpha}_2 \overline{q}^2) + i \dot{\alpha}_3 (q_f r_f + q_i r_i) + i \dot{\alpha}_4 q_i r_f \\
+ i \dot{\alpha}_5 q_f r_i + i \dot{\alpha}_6 q_i \overline{p} - \dot{\alpha}_7 q_i \overline{q} + i \dot{\alpha}_8 q_f \overline{q} - \dot{\alpha}_9 q_f \overline{q} \\
- (\dot{\alpha}_{10} q_i^2 + \dot{\alpha}_{11} q_f q_f + \dot{\alpha}_{12} q_f^2) + i \dot{\alpha}_{13} q_i + i \dot{\alpha}_{14} q_f \right\} J. \tag{V.114}
\]

Here

\[
q_i J = -\frac{i}{\alpha_4} (\partial_{r_f} - i \alpha_3 q_f) J, \tag{V.115}
\]

\[
(\partial_{r_f} = \partial / \partial r_f)
\]

\[
r_i J = -\frac{i}{\alpha_5} \partial_{q_f} J - \frac{1}{\alpha_5} \left[ \alpha_3 r_f + \alpha_8 \overline{p} + i \alpha_9 \overline{q} + \frac{\alpha_{11}}{\alpha_4} (\partial_{r_f} - i \alpha_3 q_f) + 2 i \alpha_{12} q_f + \alpha_{14} \right] J, \tag{V.116}
\]

\[
q_i^2 J = \frac{\alpha_3^2}{\alpha_4^2} q_f^2 J + \frac{2i \alpha_3}{\alpha_4^2} q_f \partial_{r_f} J - \frac{1}{\alpha_4^2} \partial_{r_f}^2 J, \tag{V.117}
\]

\[
r_i q_i J = -\frac{1}{\alpha_4 \alpha_5} \partial_{r_f q_f} J + i \frac{\alpha_3}{\alpha_4 \alpha_5} J + i \frac{\alpha_3}{\alpha_4 \alpha_5} r_f (\partial_{r_f} - i \alpha_3 q_f) J \\
- 2 \frac{\alpha_{12}}{\alpha_4 \alpha_5} q_f (\partial_{r_f} - i \alpha_3 q_f) J + i \frac{\alpha_{14}}{\alpha_4 \alpha_5} (\partial_{r_f} - i \alpha_3 q_f) J \\
+ i \frac{\alpha_3}{\alpha_4 \alpha_5} q_f \partial_{q_f} J - i \frac{\alpha_{11} \alpha_3^2}{\alpha_4^2 \alpha_5} q_f^2 J + \frac{2 \alpha_3 \alpha_{11}}{\alpha_4^2 \alpha_5} q_f \partial_{r_f} J \\
+ \frac{i \alpha_{11}}{\alpha_4 \alpha_5} \partial_{r_f}^2 J + \frac{i \alpha_8}{\alpha_4 \alpha_5} \overline{p} (\partial_{r_f} - i \alpha_3 q_f) J \\
- \frac{i \alpha_9}{\alpha_4 \alpha_5} \overline{q} (\partial_{r_f} - i \alpha_3 q_f) J. \tag{V.118}
\]

Now using Eqs. (V.113) to (V.118) in Eq. (V.112) we get the master equation as
\[
\frac{\partial}{\partial t} \rho(x_f, x'_f, t) = i \left[ \frac{\hbar}{2M} (\partial_{x_f}^2 - \partial_{x_f'}^2) - \frac{\theta(t)}{\hbar} (x_f - x'_f) \right] \\
- \frac{\omega^2(t)}{2\hbar} (x_f^2 - x'_f^2) \rho(x_f, x'_f, t) \\
- \frac{1}{\hbar} \Gamma(t)(x_f - x'_f)(\partial_{x_f} - \partial_{x'_f}) \rho(x_f, x'_f, t) \\
- \frac{1}{\hbar} D_{pp}(t)(x_f - x'_f)^2 \rho(x_f, x'_f, t) \\
- \frac{i}{\hbar} \left[ D_{xp}(t) + D_{px}(t) \right] (x_f - x'_f)(\partial_{x_f} + \partial_{x'_f}) \rho(x_f, x'_f, t) \\
- \frac{1}{\hbar} D_{xx}(t)(\partial_{x_f} + \partial_{x'_f})^2 \rho(x_f, x'_f, t) \\
+ \frac{i}{\hbar} \tilde{C}_1(t)(x_f - x'_f) \rho_1(x_f, x'_f, t) \\
- \frac{1}{\hbar} \tilde{C}_2(t)(x_f - x'_f) \rho_2(x_f, x'_f, t). \tag{V.119}
\]

Here we have reverted back to the original coordinates using Eqs. (V.69) to (V.71), and \( p \) is the momentum of the particle. The various coefficients in Eq. (V.119) are

\[
\theta(t) = \hbar \left[ \frac{\dot{\alpha}_5 \alpha_{14}}{\alpha_5} - \dot{\alpha}_{14} \right], \tag{V.120}
\]

\[
\omega^2(t) = \hbar \left[ \frac{\alpha_3 \dot{\alpha}_5}{\alpha_5} - \dot{\alpha}_3 \right], \tag{V.121}
\]

\[
\Gamma(t) = \frac{\hbar}{2\alpha_5} \left[ \frac{\dot{\alpha}_3 \alpha_3}{\alpha_4} - \dot{\alpha}_5 \right] \tag{V.122}
\]

is the dissipation term,

\[
D_{pp}(t) = \hbar^2 \left[ \dot{\alpha}_{12} + \frac{2\dot{\alpha}_3 \alpha_3 \alpha_{12}}{\alpha_4 \alpha_5} - \frac{\dot{\alpha}_3 \alpha_{11} \alpha_3^2}{\alpha_4^2 \alpha_5} \\
+ \frac{\dot{\alpha}_5 \alpha_{11} \alpha_3}{\alpha_4 \alpha_5} - \frac{2\dot{\alpha}_3 \alpha_{12}}{\alpha_5} + \frac{\alpha_3^2 \dot{\alpha}_{10}}{\alpha_5^2} - \frac{\dot{\alpha}_{11} \alpha_3}{\alpha_4} \right] \tag{V.123}
\]

causes decoherence in \( x \),

\[
D_{xp}(t) + D_{px}(t) = \hbar \left[ \frac{2\dot{\alpha}_3 \alpha_{12}}{\alpha_4 \alpha_5} - \frac{2\dot{\alpha}_3 \alpha_3 \alpha_{11}}{\alpha_4^2 \alpha_5} \\
+ \frac{\dot{\alpha}_5 \alpha_{11}}{\alpha_4 \alpha_5} + \frac{2\dot{\alpha}_{10} \alpha_3}{\alpha_5^2} - \dot{\alpha}_{11} \right] \tag{V.124}
\]

120
is the anomalous diffusion term,

\[ D_{xx}(t) = \hbar \left[ \frac{\dot{\alpha}_3 \alpha_{11}}{\alpha_4^2 \alpha_5} - \frac{\dot{\alpha}_{10}}{\alpha_4^2} \right] \quad (V.125) \]

generates decoherence in \( p \),

\[ \tilde{C}_1(t) = \hbar \left[ \dot{\alpha}_8 - \frac{\dot{\alpha}_5 \alpha_8}{\alpha_5} \right] \quad (V.126) \]

does inhomogeneity in master equation,

\[ \tilde{C}_2(t) = \hbar \left[ \dot{\alpha}_9 - \frac{\dot{\alpha}_5 \alpha_9}{\alpha_5} \right] \quad (V.127) \]

does inhomogeneity in master equation,

\[ \rho_1(x_f, x'_f, t) = \int dx_i dx'_i dq dq' \tilde{r} J_{\lambda_0}(x_i, x'_i, \tilde{q}, \tilde{r}), \quad (V.128) \]

and

\[ \rho_2(x_f, x'_f, t) = \int dx_i dx'_i dq dq' \tilde{q} J_{\lambda_0}(x_i, x'_i, \tilde{q}, \tilde{r}). \quad (V.129) \]

The above coefficients of the master equation can be written in a more compact form as follows:

\[ \Gamma(t) = -\frac{\hbar}{2} \frac{\dot{W}(t)}{W(t)}, \quad (V.130) \]

where

\[ W(t) = G_+(t) \tilde{G}_+(t) - \tilde{G}_+^2(t), \quad (V.131) \]

\[ \omega^2(t) = -\frac{M}{G_+(t)} \left[ \frac{2}{\hbar} \frac{\dot{G}_+(t)}{G_+(t)} \Gamma(t) + \tilde{G}_+(t) \right] \]

\[ = \frac{\dot{G}_+(t) \tilde{G}_+(t) - \tilde{G}_+^2(t)}{W(t)}, \quad (V.132) \]

\[ \theta(t) = -\frac{2}{\hbar} \Gamma(t) \int_0^t ds \frac{G_-(S)}{G_-(t)} F_1(s) - F_1(t) \]

\[ - \int_0^t ds \left[ \tilde{G}_+(t-s) - \frac{G_+(t-s)}{G_+(t)} \tilde{G}_+(t) \right] F_1(s), \quad (V.133) \]
\[ C_1(t) = \hbar \left[ \frac{\partial}{\partial t} + \frac{2}{\hbar} \Gamma(t) \partial_{\nu} + \frac{1}{M} \omega^2(t) \right] \left( G_+(t) \alpha_6(t) \right), \] (V.134)

\[ C_2(t) = \hbar \left[ \frac{\partial}{\partial t} + \frac{2}{\hbar} \Gamma(t) \partial_{\nu} + \frac{1}{M} \omega^2(t) \right] \left( G_+(t) \alpha_7(t) \right), \] (V.135)

\[ D_{xx}(t) = 0, \] (V.136)

\[ D_{pp}(t) = \hbar M \left[ \frac{\partial^2}{\partial \nu^2} + \frac{2}{\hbar} \Gamma(t) \partial_{\nu} + \frac{1}{M} \omega^2(t) \right] \tilde{U}^{(2)}(t, t')|_{t=t'}, \] (V.137)

\[ D_{xp}(t) + D_{px}(t) = \hbar \left[ \frac{\partial^2}{\partial \nu^2} + \frac{2}{\hbar} \Gamma(t) \partial_{\nu} + \frac{1}{M} \omega^2(t) \right] U^{(2)}(t, t')|_{t=t'}, \] (V.138)

where

\[ U^{(2)}(t, t') = \int_0^t ds \int_0^{t'} du \ G_+(t - s) R^{(2)}(s, u) G_+(t' - u), \] (V.139)

and

\[ \tilde{U}^{(2)}(t, t') = \frac{\partial}{\partial t} U^{(2)}(t, t'). \] (V.140)

It can be seen that the above equations have structures similar to the ones for the linear coupling case [13] though the coefficients here are quite complicated. The last two expressions on the RHS of Eq. (V.119) make the master equation inhomogeneous. This implies that for generalized initial conditions, in the case of nonlinear system-environment couplings, it is not possible to obtain an exact Liouville operator \( L \) where \( L \) satisfies the equation

\[ \frac{\partial \rho}{\partial t} = L \rho. \] (V.140)

This is a feature of nonseparable initial conditions even for the case of linear system-environment couplings [12].

2. The Wigner equation

The Wigner equation is obtained from the master equation by writing [14]

\[ \frac{\partial}{\partial t} W(p, x, t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dy \ e^{i p y} \left\langle x - \frac{1}{2} y \right| \frac{\partial}{\partial t} \rho \left| x + \frac{1}{2} y \right\rangle. \] (V.141)
Equation (V.119) is expressed in operator form as

\[
\frac{\partial \rho}{\partial t} = -i \frac{p^2}{2M, \rho} - i \frac{\theta(t)}{\hbar} [x, \rho] - i \frac{\omega^2(t)}{2\hbar} [x^2, \rho]
\]

\[+ \frac{\Gamma(t)}{2i\hbar} [\{p, x\}^+, \rho] + \frac{\Delta}{\hbar^3} [[x, \rho]\rho] - [p, \rho x]
\]

\[+ \frac{D_{xx}(t)}{\hbar^3} [p, [p, \rho]] - \frac{D_{pp}(t)}{\hbar^2} [x, [x, \rho]]
\]

\[+ \frac{1}{2\hbar^2} [D_{xp}(t) + D_{px}(t)] [x, [p, \rho]] + [p, [x, \rho]]
\]

\[+ \frac{i}{\hbar} \tilde{C}_1(t) [x, \rho_1] - \frac{1}{\hbar} \tilde{C}_2(t) [x, \rho_2]. \]  

(V.142)

Here \([a, b]\) stands for the commutator \((ab - ba)\), and \(\{a, b\}_+\) for \(ab + ba\), and \(p, x, \rho, \rho_1, \rho_2\) are operators. Also

\[
\Delta = -i\hbar \Gamma(t). \]  

(V.143)

Now Eq. (V.142) is substituted into Eq. (V.141) to give the Wigner equation

\[
\frac{\partial W}{\partial t} = - \frac{1}{M} \frac{\partial}{\partial x} pW + \omega^2(t) \frac{\partial}{\partial p} xW + \theta(t) \frac{\partial}{\partial p} W
\]

\[+ D_{pp}(t) \frac{\partial^2}{\partial p^2} W - \frac{1}{\hbar} D_{xx}(t) \frac{\partial^2}{\partial x^2} W + \left[D_{xp}(t) + D_{px}(t)\right] \frac{\partial^2}{\partial x \partial p} W
\]

\[+ \frac{2}{\hbar} \Gamma(t) \frac{\partial}{\partial p} pW - \tilde{C}_1(t) \frac{\partial}{\partial p} W_1 - i \tilde{C}_2(t) \frac{\partial}{\partial p} W_2, \]  

(V.144)

where

\[W_1(p, x, t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dy \ e^{\frac{i}{\hbar} py} \left< x - \frac{1}{2} y \right| x + \frac{1}{2} y \right>, \]  

(V.145)

and

\[W_2(p, x, t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dy \ e^{\frac{i}{\hbar} py} \left< x - \frac{1}{2} y \right| x + \frac{1}{2} y \right>. \]  

(V.146)
The Wigner equation may be employed for calculation of various correlation functions in a quasiclassical manner. The equation obtained here for the nonlinear couplings and nonseparable initial conditions has wider applicability than the one obtained earlier by Romero and Paz [13] for the linear coupling case.

B. Thermal initial condition

We now consider the simple case of a thermal initial condition [4], for which the off-diagonal elements of operators in position space of the particle are suppressed in thermal equilibrium.

1. The master equation

The thermal initial condition has the preparation function [12]

\[
\lambda_0(q_i, r_i, \bar{q}, \bar{r}) = f(q_i, r_i) \delta(\bar{q} - q_i) \delta(\bar{r} - r_i)
\]  \hspace{1cm} (V.147)

in (V.112). Thus, in (V.119), we now have

\[
\rho_1(q_f, r_f, t) = \int dq_i dr_i \: r_i J(q_f, r_f, t, q_i, r_i, q_i, r_i) f(q_i, r_i),
\]  \hspace{1cm} (V.148)

(cf. Eq. (V.128)) and

\[
\rho_2(q_f, r_f, t) = \int dq_i dr_i \: q_i J(q_f, r_f, t, q_i, r_i, q_i, r_i) f(q_i, r_i),
\]  \hspace{1cm} (V.149)

(cf. Eq. (V.129)). Using (V.147) the reduced density matrix of the quantum Brownian oscillator becomes

\[
\rho(q_f, r_f, t) = \int dq_i dr_i \: J(q_f, r_f, t, q_i, r_i, q_i, r_i) f(q_i, r_i).
\]  \hspace{1cm} (V.150)

Now from Eqs. (V.67) we have

\[
J(q_f, r_f, t, q_i, r_i, \bar{q}_i, \bar{r}_i) = \alpha_0 \exp \left\{ \frac{i}{\hbar} \sum (q_f, r_f, t, q_i, r_i, \bar{q}_i, \bar{r}_i) \right\},
\]  \hspace{1cm} (V.151)

where \( \alpha_0 = 1/Z \) with \( Z \) given by Eq. (V.88), and from (V.68),
\[ \frac{i}{\hbar} \sum (q_f, r_f, t, q_i, r_i, q_i, r_i) = - \left( \alpha_1 q_i^2 + \alpha_2 q_i^2 \right) + i\alpha_3 (q_f r_f + q_i r_i) + i\alpha_4 q_i r_f \\
+ i\alpha_5 q_f r_i + i\alpha_6 q_i r_i - \alpha_7 q_i^2 + i\alpha_8 q_f r_i \\
- \alpha_9 q_f q_i - \left( \alpha_{10} q_i^2 + \alpha_{11} q_f q_f + \alpha_{12} q_f^2 \right) \\
+ i\alpha_{13} q_i + i\alpha_{14} q_f. \]  

(V.152)

Using

\[ q_i J = -\frac{i}{\alpha_4} (\partial_{r_f} - i\alpha_3 q_f) J, \]  

(V.153)

\[ r_i J = \frac{i}{\alpha_5 + \alpha_8} \frac{\partial q_f J - 1}{\alpha_5 + \alpha_8} \left[ \alpha_3 r_f + \frac{(\alpha_9 + \alpha_{11})}{\alpha_4} (\partial_{r_f} - i\alpha_3 q_f) \right] \\
+ 2i\alpha_{12} q_f + \alpha_{14} \right] J, \]  

(V.154)

we have

\[ \rho_1(q_f, r_f, t) = - \frac{i}{\alpha_5 + \alpha_8} \partial_{q_f} \rho(q_f, r_f, t) \\
- \frac{1}{\alpha_5 + \alpha_8} \alpha_3 r_f \rho(q_f, r_f, t) \\
- \frac{\alpha_9 + \alpha_{11}}{\alpha_4 (\alpha_5 + \alpha_8)} (\partial_{r_f} - i\alpha_3 q_f) \rho(q_f, r_f, t) \\
- \frac{2i\alpha_{12}}{\alpha_5 + \alpha_8} q_f \rho(q_f, r_f, t) \\
- \frac{\alpha_{14}}{\alpha_5 + \alpha_8} \rho(q_f, r_f, t), \]  

(V.155)

and

\[ \rho_2(q_f, r_f, t) = \frac{-i}{\alpha_4} (\partial_{r_f} - i\alpha_3 q_f) \rho(q_f, r_f, t). \]  

(V.156)

Using Eqs. (V.155) and (V.156) in Eq. (V.119), and with \( D_{xx}(t) = 0 \) (as in Eq. (V.136)), we obtain the master equation for the case of thermal initial conditions as

\[ \frac{\partial}{\partial t} \rho(x_f, x'_f, t) = i \left[ \frac{\hbar}{2M} \left( \partial^2_{x_f} - \partial^2_{x'_f} \right) - \frac{\vec{\theta}(t)}{\hbar} (x_f - x'_f) \right] \]
This has the form of an exact master equation, i.e., there are no inhomogeneities, and in this case an exact Liouville operator $L$ exists. This is in agreement with the findings of Karrlein and Grabert [12] for thermal initial conditions in the linear coupling case. The inhomogeneities in the master equation emerge only for the general nonseparable initial conditions.

2. The Wigner equation

Proceeding as before, we obtain the Wigner equation from the master equation (V.157) as

$$\frac{\partial W}{\partial t} = - \frac{1}{M} \frac{\partial}{\partial x} pW + \frac{\sigma^2}{2\hbar} \frac{\partial}{\partial p} xW + \tilde{\theta}(t) \frac{\partial}{\partial p} W$$

$$+ \tilde{D}_{pp}(t) \frac{\partial^2}{\partial p^2} W + \left[ \tilde{D}_{xp}(t) + \tilde{D}_{px}(t) \right] \frac{\partial^2}{\partial x \partial p} W$$

$$+ \frac{2}{\hbar} \tilde{\Gamma}(t) \frac{\partial}{\partial p} pW.$$  (V.158)

The coefficients on the RHS of (V.158) are

$$\tilde{\theta}(t) = \theta(t) + \frac{\alpha_1 \alpha_4}{\alpha_5 + \alpha_8} \tilde{C}_1(t),$$  (V.159)

$$\tilde{\omega}^2(t) = \omega^2(t) + \frac{\alpha_3}{\alpha_5 + \alpha_8} \tilde{C}_1(t),$$  (V.160)

$$\tilde{\Gamma}(t) = \Gamma(t) - \frac{\tilde{C}_1(t)}{2(\alpha_5 + \alpha_8)},$$  (V.161)

$$\tilde{D}_{pp}(t) = D_{pp}(t) + \hbar \left[ \frac{\alpha_3(\alpha_9 + \alpha_{11})}{\alpha_4(\alpha_5 + \alpha_8)} \tilde{C}_1(t) \right.$$  

$$\left. - 2 \frac{\alpha_{12}}{\alpha_5 + \alpha_8} \tilde{C}_1(t) - \frac{\alpha_3}{\alpha_4} \tilde{C}_2(t) \right],$$  (V.162)
Equation (V.158) has the form of the generalized Fokker-Planck equation.

Thus we see that nonlinearity in the environment up to second-order perturbation does not introduce any nonlinear behavior in the system for either separable initial conditions [3] or nonseparable initial conditions.

V. Fluctuation-Dissipation Theorem

The real and imaginary parts of the coordinate autocorrelation function of the quantum particle are not independent and should be related by a generalized fluctuation-dissipation theorem. In this section we establish a fluctuation-dissipation theorem using the propagator in Eq. (V.67) for the quantum Brownian particle in a harmonic potential. Proceeding as in Grabert et al. [6]

\[
\langle x \rangle_t = \int dr_f dq_i dr_i r_f \tilde{J}(0, r_f, t, q_i, r_i) \nonumber
\]
\[
= \frac{1}{Z} \int dr_f dq_i dr_i r_f \exp \left\{ -i \frac{M}{\hbar G_+(t)} q_i \left( r_f - \frac{1}{M} \int_0^t ds G_+(t - s) F_1(s) \right) \right\} \nonumber
\]
\[
\times \exp \left\{ -\frac{1}{2} a_i M(t) a_i \right\}. \tag{V.164}
\]

Here \( \tilde{J}(q_f, r_f, t, q_i, r_i) \) is defined as

\[
\tilde{J}(q_f, r_f, t, q_i, r_i) = \int dq d\bar{r} J(q_f, r_f, t, q_i, r_i, \bar{q}, \bar{r}) \tilde{\lambda}(q_i, r_i, \bar{q}, \bar{r}) \tag{V.165}
\]
\[
= J(q_f, r_f, t, q_i, r_i, q_i, r_i),
\]

where

\[
\tilde{\lambda}(q_i, r_i, \bar{q}, \bar{r}) = \delta(\bar{q} - q_i) \delta(\bar{r} - r_i), \tag{V.166}
\]

127
\[ a_i = \left( \begin{array}{c} q_i \\ r_i \end{array} \right), \quad \text{(V.167)} \]

\[ M(t) = \frac{M}{\hbar} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad \text{(V.168)} \]

with

\[ M_{11} = \left[ \Omega^{(2)} - 2C_2^{(2)+} (t) + R^{(2)++} (t) \right], \quad \text{(V.169)} \]

\[ M_{12} = M_{21} = -i \left[ \frac{\dot{G}_+(t)}{G_+(t)} + C_1^{(2)+} (t) \right], \quad \text{(V.170)} \]

\[ M_{22} = \frac{1}{\Lambda}. \quad \text{(V.171)} \]

Doing the integrations in Eq. (V.164) we get

\[ \langle x \rangle_t = \int_0^t ds \, \chi(t-s) F_1(s), \quad \text{(V.172)} \]

where

\[ \chi(t) = \frac{1}{M} G_+(t) \quad \text{(V.173)} \]

is the response function. The coordinate autocorrelation function \( C(t) \) is

\[ C(t) = \langle x(t)x \rangle = \int dr_f dq_i dr_i r_f(r_i + \frac{1}{2}q_i) \tilde{J}(0, r_f, t, q_i, r_i). \quad \text{(V.174)} \]

Here \( \tilde{J} \) is given by Eq. (V.165) with \( F_1(s) = 0 \). Using Eqs. (V.67), (V.68), (V.165) we have

\[ C(t) = \frac{1}{Z} \int dr_f dq_i dr_i r_f \left( r_i + \frac{1}{2}q_i \right) \times \exp \left\{ - \frac{i}{\hbar} \frac{M}{G_+(t)} q_i r_f \right\} \exp \left\{ - \frac{1}{2}a_i M(t) a_i \right\}. \quad \text{(V.175)} \]
Doing the integrations we get the coordinate autocorrelation function $C(t)$ as

$$C(t) = S(t) + iA(t),$$  \hspace{1cm} (V.176)

where $S(t)$ is the symmetrized correlation given by

$$S(t) = \frac{1}{2} \langle x(t)x + xx(t) \rangle = \frac{\hbar}{M} \wedge \left[ \hat{G}_+(t) + G_+(t)C_1^{(2)+}(t) \right],$$  \hspace{1cm} (V.177)

and $A(t)$ is the antisymmetrized correlation given by

$$A(t) = \frac{1}{2i} \langle x(t)x - xx(t) \rangle = -\frac{\hbar}{2} \chi(t), \quad t \geq 0,$$  \hspace{1cm} (V.178)

with $\chi(t)$ given by Eq. (V.173), these have the same form as the linear coupling case, and

$$C_1^{(2)+}(t) = \int_0^t ds \frac{G_+(t-s)}{G_+(t)} C_1^{(2)}(s),$$

(Eq. (V.78) with $m = 1$). Substituting $u_k^{(2)}$ from Eq. (V.75), $\zeta_k^{(2)}$ from Eq. (V.54), $\eta^{(2)}(s)$ from Eq. (V.61) and using

$$\hat{G}_+(z) \equiv \mathcal{L}\{G_+(t)\} = \frac{1}{z^2 + \frac{2}{M} \eta^{(2)}(z) + \omega_0^2},$$  \hspace{1cm} (V.179)

from Eq. (V.85), and

$$\hat{\eta}^{(2)}(z) \equiv \mathcal{L}\{\eta^{(2)}(s)\} = -\mathcal{L} \left\{ \int_0^\infty \frac{d\omega}{\pi} 2I(\omega) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \sin(2\omega s) \right\}$$

$$= -\int_0^\infty \frac{d\omega}{\pi} 4I(\omega) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \frac{\omega}{\left[z^2 + 4\omega^2\right]},$$  \hspace{1cm} (V.180)

thereby showing that

$$\frac{2}{M} \hat{\eta}^{(2)}(\nu_k) = -\zeta_k^{(2)},$$  \hspace{1cm} (V.181)
we get

\[ u_k^{(2)} = \hat{G}_+ (|\nu_k|). \]  \hspace{1cm} (V.182)

From the above we can see that

\[ G_+(t)C_1^{(2)+}(t) = \frac{1}{\hbar \beta} \sum_{k=-\infty}^{\infty} \hat{G}_+ (|\nu_k|) \int_0^t ds G_+(t-s)g_k(s), \]  \hspace{1cm} (V.183)

and

\[ \hat{S}(z) \equiv \mathcal{L}\{S(t)\} = \frac{\hbar \Lambda}{M} z \hat{G}_+(z) + \frac{1}{\beta M} \sum_{k=-\infty}^{\infty} \hat{G}_+ (|\nu_k|) \hat{G}_+(z) \tilde{g}_k(z). \]  \hspace{1cm} (V.184)

Here \( \mathcal{L} \) stands for the Laplace transform. Using Eq. (V.59) for \( g_k(s) \), we have

\[ g_k(s) = \frac{8}{M} \int_0^\infty \frac{d\omega}{\pi} I(\omega) \coth \left( \frac{\hbar \omega}{2kBT} \right) \frac{\omega}{[4\omega^2 + \nu_k^2]} \cos(2\omega s) \]

\[ = \overline{\gamma}^{(2)}(s) - \overline{\zeta}^{(2)}_k(s), \]  \hspace{1cm} (V.185)

where

\[ \overline{\gamma}^{(2)}(s) = \frac{2}{M} \int_0^\infty \frac{d\omega}{\pi} I(\omega) \frac{\hbar \omega}{2kBT} \cos(2\omega s), \]  \hspace{1cm} (V.186)

and

\[ \overline{\zeta}^{(2)}_k(s) = \frac{2}{M} \int_0^\infty \frac{d\omega}{\pi} I(\omega) \coth \left( \frac{\hbar \omega}{2kBT} \right) \frac{\nu_k^2}{\omega[4\omega^2 + \nu_k^2]} \cos(2\omega s). \]  \hspace{1cm} (V.187)

Inverting Eq. (V.186) we get

\[ I(\omega) = \frac{2}{\coth \left( \frac{\hbar \omega}{2kBT} \right)} M \omega \int_0^\infty ds \overline{\gamma}^{(2)}(s) \cos(2\omega s). \]  \hspace{1cm} (V.188)

Substituting Eq. (V.188) in (V.187) we get

130
\[ \zeta_k^{(2)}(s) = \frac{|\nu_k|}{2} \int_0^\infty du \, \gamma^{(2)}(u) \left[ e^{-|\nu_k(s+u)|} + e^{-|\nu_k(s-u)|} \right], \quad (V.189) \]

where we have used

\[ \int_0^\infty \frac{d\omega}{[4\omega^2 + \nu_k^2]} \cos[2\omega(u \pm s)] = \frac{\pi}{4|\nu_k|} e^{-|\nu_k(s\pm u)|}. \quad (V.190) \]

From (V.189) we get

\[ \zeta_k^{(2)}(z) \equiv \mathcal{L}\{\zeta_k^{(2)}(s)\} = \int_0^\infty ds \, e^{-zs} \zeta_k^{(2)}(s) = \frac{1}{(z^2 - \nu_k^2)} \left\{ z|\nu_k| \gamma^{(2)}(|\nu_k|) - \nu_k^2 \gamma^{(2)}(z) \right\}, \quad (V.191) \]

where we have used

\[ \int_0^\infty ds \, e^{-zs} e^{-|\nu_k(s+u)|} = \frac{e^{-|\nu_k|u}}{(z + |\nu_k|)}, \quad (V.192) \]

and

\[ \int_0^\infty ds \, e^{-zs} e^{-|\nu_k(s-u)|} = \frac{e^{-zu}}{(z + |\nu_k|)} - \frac{e^{-zu}}{(z - |\nu_k|)} + \frac{e^{-|\nu_k|u}}{(z - |\nu_k|)}. \quad (V.193) \]

Thus we have

\[ \hat{S}(z) = \frac{1}{\beta M} \sum_{k=-\infty}^\infty \hat{G}_+(|\nu_k|) \hat{G}_+(z) \left\{ z + \gamma^{(2)}(z) ight. \left. - \frac{z}{(z^2 - \nu_k^2)} |\nu_k| \gamma^{(2)}(|\nu_k|) + \frac{\nu_k^2}{(z^2 - \nu_k^2)} \gamma^{(2)}(z) \right\}. \quad (V.194) \]

Now using Eq. (V.179) we have

\[ \hat{n}^{(2)}(z) = \frac{M}{2 \hat{G}_+(z)} - \frac{M}{2} (z^2 + \omega_0^2), \quad (V.195) \]

131
Also, using Eqs. (V.61) and (V.186), we have

\[ \hat{\eta}^{(2)}(z) = \frac{M}{2} \left\{ z \hat{\gamma}^{(2)}(z) - \gamma^{(2)}(0) \right\}, \]  

(V.196)

where

\[ \gamma^{(2)}(0) = \frac{1}{M} \int_0^\infty \frac{d\omega}{\pi} \frac{2I(\omega)}{\omega} \coth \left( \frac{\hbar \omega}{2k_B T} \right). \]  

(V.197)

Combining Eqs. (V.195) and (V.196), we have

\[ \hat{\gamma}^{(2)}(z) = \frac{1}{z \hat{G}_+(z)} - \frac{(z^2 + \omega_0^2)}{z} + \gamma^{(2)}(0). \]  

(V.198)

Using Eq. (V.198) in Eq. (V.194) we get

\[ \tilde{S}(z) = \frac{1}{\beta M} \sum_{k=0}^\infty \frac{z}{(\nu_k^2 - z^2)} \left[ \hat{G}_+(z) - \hat{G}_+(|\nu_k|) \right]. \]  

(V.199)

This can then be cast in the form

\[ \tilde{S}(\omega) = \hbar \coth \left( \frac{\hbar \omega}{2k_B T} \right) \tilde{\chi}'(\omega), \]  

(V.200)

which is the usual statement of the fluctuation-dissipation theorem, where \( \tilde{S}(\omega) \) is the Fourier transform of \( S(t) \):

\[ \tilde{S}(\omega) = \tilde{S}(i\omega) + \tilde{S}(-i\omega), \]  

(V.201)

and

\[ \tilde{\chi}''(\omega) = \frac{i}{2} \left[ \tilde{\chi}(i\omega) - \tilde{\chi}(-i\omega) \right], \]  

(V.202)

with

\[ \tilde{\chi}(\omega) = \int_{-\infty}^\infty dt \chi(t) e^{i\omega t} = \tilde{\chi}(-i\omega), \]  

(V.203)
where $\chi(t)$ is the response function (V.173). Here we have used

$$
\sum_{k=-\infty}^{\infty} \frac{\omega}{[\omega^2 + \nu_k^2]} = \frac{\hbar}{2k_B T} \coth \left( \frac{\hbar \omega}{2k_B T} \right).
$$

(V.204)

It is thus seen that for our case of couplings nonlinear in the environment coordinates and treated up to second order of perturbation, the form of the fluctuation-dissipation theorem is preserved for separable initial conditions [3] as well as general nonseparable initial conditions. The proportionality of $\tilde{S}(\omega)$ and $\chi''(\omega)$ illustrates the close connection between fluctuation and dissipation mechanisms acting on the quantum Brownian oscillator.

VI. Summary

In this chapter we have investigated the quantum Brownian motion (QBM) with couplings nonlinear (quadratic) in the environment coordinates, treating it up to second order of perturbation for general nonseparable initial conditions [10]. We have thus extended the work of Hu et al. [3] and Brun [15] who set out the basic foundations for handling nonlinear QBM with separable initial conditions.

We have constructed the influence functional for nonlinear interactions up to second order of perturbation with generalized initial conditions. We then used the influence functional, restricting the nonlinearity to the environment, to get the propagator for the particle in a harmonic potential as well as for the particle in an additional anharmonic potential, called the washboard potential used to describe the ideal motion of a heavy charged particle in a metal. For the harmonic potential case we have obtained the propagator that is similar to the one in the corresponding linear coupling case [6] even though the coefficients are more complicated – among other things having an additional temperature-dependent factor in them. For the case of the particle in the washboard potential we have been able to work out the propagator and get all the terms explicitly. This is a step forward from the previous treatment by Chen et al. [9] for the case of linear system-environment couplings and an Ohmic spectrum of the reservoir.
From the propagator for the particle in a harmonic potential we have obtained the master equation and the Wigner equation. Both of these equations exhibited inhomogeneities which imply that it is not possible to construct an exact Liouville operator for generalized initial conditions for either the linear coupling case of Karrlein and Grabert [12], Romero and Paz [13], or when there is nonlinearity in the environment coordinate in the system-environment interaction. We have then considered the specific case of a simple initial condition, called the thermal initial condition, where an exact master equation and a Wigner equation resembling the generalized Fokker-Planck equation are obtained. Thus under such simpler initial conditions, an exact Liouville operator exists for the linear [12] as well as the nonlinear coupling case.

We have also used the propagator for the quantum Brownian particle in a harmonic potential to establish a generalized fluctuation-dissipation theorem. Even though the coefficients in our propagator are more complicated than the corresponding linear coupling case, the form of the fluctuation-dissipation relation is found to be the same as that in the linear coupling case, for both separable [3] and nonseparable initial conditions, confirming that the results are physically consistent, and the same physical mechanism is responsible both for the fluctuations of the position of the quantum oscillator and for its damping. The results presented here are applicable to all the physical problems modeled by the quantum Brownian motion with initially correlated and nonlinearly coupled environment.
References


