2.1. Introduction

This chapter is concerned with the derivation of the basic equations of elasticity and consolidation for a porous elastic material containing a viscous compressible fluid within the framework of Biot's \([10, 12, 16]\) theory. The general theory of three-dimensional consolidation of soil, regarded as an isotropic porous material was first proposed by Biot \([10]\) and his later works contain generalisations of the earlier theory to anisotropy \([12]\), to viscoelasticity of a very general nature \([13]\) and to treat dynamical and wave propagation problems of porous media \([15]\). A further refinement of the theory containing unified treatment of the linear mechanics of deformation and acoustic propagation in porous media was also presented later \([16]\). The present theory is based on the assumption of infinitesimal strain, reversibility of stress and strain and linearity of stress-strain law. It is further assumed that the deformation proceeds in such a way that the inertia terms in the field equations can be neglected so that the deformation is quasi-static in nature.

The derivation begins with the description of Biot model of a porous elastic medium and the definition of the strain energy for such a medium. The constitutive equations for the isotropic case in the context of a linear theory and the thermodynamic derivation
of Darcy's law of fluid flow are presented next. The field equations governing the deformation and stress distribution are then obtained and the boundary conditions which generally occur in the specific problems for the elastic consolidating media are discussed. The analogy which exists between the theory of elasticity and consolidation and that of coupled quasi-static thermoelasticity [14] has also been pointed out. Lastly, the generalisations of the present theory to dynamics [15, 16], to viscoelasticity [13, 16] etc. are indicated.

2.2. Biot model of a poroelastic medium and its strain energy

Let us consider a deformable elastic solid having a statistical distribution of interconnected pores which are filled with a viscous compressible fluid. The pores which are sealed off are supposed to be part of the solid. The pores are distributed in the medium at random but not necessarily in direction. The solid part of the medium is called 'solid skeleton' or 'solid frame' and the fluid in the pores is called 'interpore material'. This system of solid and fluid is supposed to be an elastic system which possesses conservation properties and the deformation of the medium is supposed to be completely reversible. Such a solid-fluid system which is porous as well as elastic is called a 'poroelastic medium' or an 'elastic consolidating medium'. Example of such a poroelastic medium is a rubber sponge saturated with water or a fluid saturated soil medium underlying civil engineering structures. The theory of elasticity
and consolidation for a porous material containing a viscous compressible fluid to be developed here will henceforth be called, for convenience, the 'theory of quasi-static poroelasticity' or simply 'poroelasticity'.

Let $V_p$ be the volume of the interconnected pores contained in a volume $V_b$ of the bulk material. Then the porosity $f$ is defined by

$$f = \frac{V_p}{V_b}.$$  

It may also be defined as

$$f = \frac{S_p}{S_b},$$

where $S_p$ is the total area occupied by the pores in any cross-sectional area $S_b$ of the bulk material.

We denote the average displacement components of the solid matrix along the $x_1, x_2, x_3$ directions of the Cartesian coordinate axes by $u_1, u_2, u_3$. The average fluid displacements $U_1, U_2, U_3$ are defined in such a way that $f U_1, f U_2, f U_3$ denote the volume of the fluid displaced through a unit area of the bulk material perpendicular to the $x_1, x_2, x_3$ axes respectively. We write $\tau_{ij}$ to denote the average total stress components of the bulk material, $\sigma_{ij}$ for the average solid stress components and $\sigma^f$ for the average fluid stress. The total stress components $\tau_{ij}$ are related to $\sigma_{ij}$ and $\sigma^f$ by the equation

$$\tau_{ij} = \sigma_{ij} + \sigma^f S_{ij}, \quad (i, j = 1, 2, 3) \quad (2.2.1)$$

where $\delta_{ij}$ is the Kronecker delta. The components $\sigma_{ij}$ and $\sigma^f$ represent respectively the forces applied to the solid part and the fluid part of the faces of a unit cube of the bulk material. If $p_f$
denotes the excess fluid pressure, then
\[ \sigma^- = -f p_f \]  

(2.2.2)

We shall henceforth drop the terms 'average' and 'excess' for brevity.

We deal with a solid-fluid system which is in thermodynamic equilibrium so that the fluid is at rest and the fluid pressure \( p_f \) remains constant throughout the body. We define the strain energy of a poroelastic medium as the isothermal free energy of the solid-fluid system. If \( W \) denotes the energy per unit volume of a system in thermodynamic equilibrium, then for small deviations of the system from the equilibrium state, the variations \( \delta W \) in the strain energy density function is given by [16]

\[ \delta W = \tau_{ij} \delta \varepsilon_{ij} + \beta_f \delta \gamma_o \]  

(2.2.3)

where

\[ \varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \]  

(2.2.4)

are the strain components of the solid and the variable \( \gamma_o \) is defined by

\[ \gamma_o = -\omega_i \]  

(2.2.5)

The vector \( \omega_i \) represents the fluid flow relative to the solid but measured in terms of volume per unit area of the bulk material. The equation (2.2.5) is valid for non-homogeneous porosity \( f \) i.e. when \( f \) is in general functions of position of a point of the medium. In particular, if \( f \) is uniform then the relation (2.2.5) reduces to

\[ \gamma_o = f (u_{i,i} - U_{i,i}) = f (e - \varepsilon) \]  

(2.2.6)
where \( \varepsilon = \varepsilon_{1} \) and \( \varepsilon = \varepsilon_{2} \) are the solid and fluid dilatations respectively. The variable \( \gamma \) measures the amount of fluid which has flowed in and out of a given element attached to the solid skeleton. We call it ' the change of fluid content ' or ' the increment of fluid content ' or simply ' the fluid content '.

Since the strain energy \( W \) must be a function of the six strain components \( e_{ij} \) defined in equation (2.2.4) and the fluid content \( \gamma \), we can write,

\[
W = W(e_{ij}, \gamma), \quad (i, j = 1, 2, 3)
\]

It now follows from equation (2.2.3) that \( \delta W \) must be an exact differential. Hence

\[
\tau_{ij} = \frac{\partial W}{\partial e_{ij}}, \quad \gamma = \frac{\partial W}{\partial \gamma}, \quad (i, j = 1, 2, 3).
\]

The equations (2.2.8) are Green's formula for a poroelastic medium which give general stress-strain relations for the medium. If, in particular, \( W \) is a quadratic form in the strain components \( e_{ij} \) and the fluid content \( \gamma \), then the equations (2.2.8) lead to linear stress-strain relations.

2.3. Stress-strain relations

For an isotropic porous material, the strain energy function \( W \) is a function of the three strain invariants and \( \gamma \). If we restrict our discussion to a linear theory then the strain energy is a homogeneous quadratic function in the strain invariants and \( \gamma \). Hence we must retain only the linear and quadratic invariants in the
the first and second power of strain components and $\gamma_0$ in the expression for $W$. We write the expression for $W$ in the form

$$2W = H I_1^2 - \mu I_2 - 2 C I_1 \gamma_0 + M \gamma_0^2$$

where the first and second invariants $I_1$ and $I_2$ are given by

$$I_1 = e_{11} + e_{22} + e_{33} = e,$$

$$I_2 = e_{11} e_{33} + e_{33} e_{11} + e_{11} e_{22} - e_{22} e_{33} - e_{33} - e_{11}$$

and $H, \mu, C, M$ are elastic constants, independent of $e_{ij}$ and $\gamma_0$ but may depend upon the coordinates $x_i$ of a point in the medium.

We can now obtain the stress-strain relations from the equations (2.2.8), (2.3.1) and (2.3.2). Again, if we put

$$H = \lambda + 2\mu, \quad C = \alpha m, \quad \lambda = \lambda + \alpha^2 m,$$

the stress-strain relations can be written in the form [16]

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij} (\lambda e - \alpha m \gamma_0),$$

$$\gamma_0 = -\lambda m e + m \gamma_0.$$

Substituting the value of $\gamma_0$ as a function of $\gamma_0$ and $e$ we get from the relations (2.3.4)

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij} (\lambda e - \alpha \gamma_0),$$

$$\gamma_0 = \lambda e + \frac{1}{m} \gamma_0.$$
On the other hand if we write 
\[ \psi_o = \psi (e - \epsilon) \]
as given in equation (2.2.6) when the porosity \( f \) is uniform, 
then the equations (2.3.5) give the stress-strain, relations 
in the form 
\[ \tau_{ij} = 2\mu \varepsilon_{ij} + \delta_{ij} \left( \lambda \varepsilon - \alpha f \right), \]
\[ \psi_f = M (1 - \alpha) \psi - M f \psi. \]

Another set of stress-strain relations can be written 
in terms of the so called 'effective stresses' defined as 
\[ \tau'_{ij} = \tau_{ij} + \delta_{ij} \psi_f \]
and representing the portion of the total stress in excess of 
the local fluid pressure. From the relations (2.3.5) and (2.3.7) 
it follows that 
\[ \tau'_{ij} - (1 - \alpha) \delta_{ij} \psi_f = 2\mu \varepsilon_{ij} + \delta_{ij} \lambda \varepsilon, \]
\[ \psi_o = \lambda \varepsilon + \frac{1}{M} \psi_f. \]

In particular, if both the solid and fluid part of the 
medium are incompressible \([10, 24-26]\) then \( \alpha = 1, \ M = \infty \). In 
this case the equations (2.3.5) and (2.3.6) take the simple form 
\[ \tau_{ij} = 2\mu \varepsilon_{ij} + \delta_{ij} \left( \lambda \varepsilon - \psi_f \right), \]
\[ \psi_o = e = - \frac{f}{1-f} \epsilon. \]
and the equations (2.3.8) become independent of the fluid pressure \( P_f \) and the fluid content \( \gamma \), equals \( \varepsilon \). For incompressible matrix material with incompressible fluid there are, therefore, two elastic constants \( \lambda \) and \( \mu \) instead of four co-efficients \( \lambda, \mu, \alpha, \beta \) in the general case. The co-efficients \( \lambda, \mu \) can be expressed in terms of Poisson's ratio \( \nu' (= \lambda/2(\lambda+\mu)) \) and shear modulus \( \alpha' (= \mu) \) of the solid skeleton.

It was shown by Biot [16] that the necessary and sufficient conditions for the strain energy \( W \) to be non-negative are that

\[
\lambda > 0, \quad \mu > 0, \quad \lambda + \frac{2}{3} \mu > 0
\]  \hspace{0.5cm} (2.3.10)

The physical interpretation of the co-efficients and methods of their measurement with the help of some 'thought' experiments were discussed in details by Biot and Willis [44]. The constant \( \alpha \) can be measured and its value lies in the range \( \frac{1}{4} \leq \alpha \leq 1 \). The 'jacketed compressibility' \( k \) is given by

\[
k = \frac{1}{\lambda + \frac{2}{3} \mu}
\]  \hspace{0.5cm} (2.3.11)

while the 'unjacketed compressibility' \( \delta \) by

\[
\delta = (1 - \alpha) k
\]  \hspace{0.5cm} (2.3.12)

The co-efficient \( \mathcal{M} \) is related to a measurable co-efficient of the fluid content \( \gamma \) for the unjacketed test by

\[
\mathcal{M} = \frac{1}{\gamma + \delta - (\delta^2/k)}
\]  \hspace{0.5cm} (2.3.13)

For some types of material when the porous matrix is homogeneous and isotropic and the fluid saturates the pores completely, then the co-efficient \( \gamma \) can be expressed in terms of the porosity \( \gamma \).
and the fluid compressibility $c'$ in the form

$$\gamma = \int (c' - \delta). \quad (2.3.14)$$

These coefficients were measured by Fatt [45] for some sandstones, the fluid saturating the pores being taken as kerosene.

Introducing a new set of coefficients $A, N, \lambda, \mu$

defined by

$$N = \mu, \quad A = \lambda + \mu \left(\lambda - \frac{\lambda^2}{\mu}\right), \quad Q = \int \left(\lambda - \frac{\lambda^2}{\mu}\right) M, \quad R = \frac{\mu^2}{\lambda} M \quad (2.3.15)$$

an alternative form of stress-strain relations can be written as

$$\tau_{ij} = \sigma_{ij} + \sigma \delta_{ij}, \quad \sigma_{ij} = 2N \varepsilon_{ij} + \varepsilon_{ij} (\lambda e + \mu e), \quad \varepsilon = \lambda e + \mu e = -\frac{1}{\lambda} \frac{\partial}{\partial} \quad (2.3.16)$$

The stress-strain relations in this form were used by Biot in his original papers [12, 32]. It follows from the relations (2.3.15) that

$$\lambda = A - \frac{\lambda^2}{\mu}, \quad \mu = N, \quad M = \frac{R}{5}, \quad A = \frac{\lambda^2 + R}{\mu} \quad (2.3.17)$$

In the present thesis we have used the stress-strain relations in the two alternative forms (2.3.6) and (2.3.16) with the corresponding set of elastic coefficients as $\lambda, \mu, \lambda, M$ and $A, N, \lambda, R$.

The equations (2.3.4) to (2.3.6), (2.3.8) and (2.3.16) are the different forms of the constitutive equations of an isotropic porous elastic medium. The forms of the stress-strain relations when the porous material are in general anisotropic can be obtained in the form [12, 16].
2.4. Darcy's law

We now consider the mechanics of flow of the fluid through the porous medium. This flow is governed by Darcy's law which is derived from thermodynamics of irreversible processes and the Onsager's relations. The rate of fluid flow in a porous medium is given by the time derivative of the volume flow vector \( \dot{\vec{w}} \).

Introducing a dissipation function \( D_f \) (which is proportional to the rate of entropy production) in a quadratic form with the help of rate of volume flow as variables, we get [16]

\[
2 D_f = \eta_i \sum_{ij} \gamma_{ij} \dot{w}_i \dot{w}_j \quad (i, j = 1, 2, 3),
\]

\( \eta \) being the viscosity of the fluid and dot denotes the derivative with respect to time. The existence of such a dissipation function is a property of the thermodynamic system which we are dealing with.

Applying the thermodynamics of irreversible processes and the principle of superposition which is valid in the context of a
linear theory, the Onsager's relations are given by [16]

\[
\frac{\partial D_f}{\partial \hat{\omega}_i} = -\rho_f \phi_{,i} \tag{2.4.2}
\]

where the total fluid potential \( \phi \) satisfies the relation

\[
\phi = \frac{p_f}{\rho_f} + G \tag{2.4.3}
\]

\( \rho_f \) being the mass density of the fluid and \( G \) is the gravitational potential per unit mass. It follows from (2.4.1) and (2.4.2) that

\[
-\frac{\rho_f}{\eta_i} \phi_{,i} = \tau_{ij} \dot{\omega}_i, \quad \left( \tau_{ij} = \gamma_{ij} \right), \quad (i, j = 1, 2, 3) \tag{2.4.9}
\]

which is the generalised form of Darcy's law. The matrix \( [\tau_{ij}] \) of the symmetric co-efficients \( \tau_{ij} \) denotes the flow resistivity and its inverse

\[
[\tau_{ij}]^{-1} = [k_{ij}] \tag{2.4.5}
\]

which is also symmetric, denotes the 'permeability matrix'. Introducing the matrix \( [k_{ij}] \), the equation (2.4.4) can be written as

\[
\dot{\omega}_i = -\frac{\rho_f}{\eta_i} k_{ij} \phi_{,i}, \quad (k_{ij} = k_{ji}), \quad (i, j = 1, 2, 3). \tag{2.4.6}
\]

If the porous medium is isotropic, then,

\[
k_{11} = k_{22} = k_{33} = k,
\]

\[
k_{23} = k_{31} = k_{12} = 0 \tag{2.4.7}
\]
and the equations (2.4.6) reduce to

\[ \dot{\tau}_{ij} = -\frac{k}{\eta_i} \rho_i \phi, \quad (i = 1, 2, 3) \quad (2.4.8) \]

which with the help of the relation (2.4.3) becomes

\[ \dot{\tau}_{ij} = -\frac{k}{\eta_i} \rho_i, \quad (i = 1, 2, 3) \quad (2.4.9) \]

The constant \( k \) is called the 'co-efficient of permeability' of the isotropic porous medium. The equation (2.4.9) is the particular form of Darcy law which will be used here.

2.5. Field equations

For an isotropic poroelastic medium, the stress-strain relations (2.3.6) are

\[ \tau_{ij} = 2\mu \varepsilon_{ij} + \delta_{ij} (\lambda e - \alpha \rho_f), \]

\[ \rho_f = \frac{M}{f} (\frac{f}{f} - x) e - M f \varepsilon \quad (2.5.1) \]

If we neglect the gravity and other body forces, the total stress field \( \tau_{ij} \) satisfies the equilibrium equations

\[ \tau_{ij,j} = 0 \quad (2.5.2) \]

and the Darcy's law of fluid flow given by the equation (2.4.9) is reduced to the form

\[ \dot{\tau}_{ij} = -\frac{k}{\eta_i} \rho_f, \quad (2.5.3) \]

Substituting the value of \( \tau_{ij} \) as given in the relations (2.5.1) and considering a homogeneous isotropic material for which the coefficients \( \lambda, \mu, \alpha, M \) and the porosity \( f \) are constants, we get from the equilibrium equations (2.5.2)

\[ \rho e \sigma_{MN} = \frac{16}{7} \cdot 7.5 \]
\[ \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \varepsilon_{ij, \ell} - \alpha \phi = 0 \quad (2.5.4) \]

where we have used the relation
\[ \varepsilon_{ij, \ell} = \frac{1}{2} (\chi_{ij, \ell} + \phi_{ij, \ell}). \]

Applying the divergence operator to the relation (2.5.3) and then using the last equation of (2.5.1) and the relation
\[ \mathbf{u}_L = f (\mathbf{u}_L - \mathbf{u}_L) \]
one gets
\[ \nabla^2 \phi = \frac{d}{M_f} \left[ \alpha \mathbf{M} \dot{\varepsilon} + \dot{\mathbf{F}}_f \right] \quad (2.5.5) \]
where \( d = \frac{\eta_f f^2}{k} \) is called the Darcy's constant.

If instead we use the stress-strain relations (2.3.16), then we find that the equations of equilibrium (2.5.2) and Darcy's law (2.5.3) are given respectively by
\[ N \nabla^2 \mathbf{u} + (N + 5) \varepsilon_{ij, \ell} + \frac{\alpha + 1}{\mathbf{R}} \sigma_{ij, \ell} = 0 \quad (2.5.6) \]

and
\[ \nabla^2 \sigma = \frac{d}{\mathbf{R}} \left[ \dot{\sigma} - (\alpha + \mathbf{R}) \dot{\varepsilon} \right] \quad (2.5.7) \]
where
\[ S = \mathbf{A} - \frac{\alpha^2}{\mathbf{R}}, \quad d = \frac{\eta_f f^2}{k}. \]

The two above forms (2.5.4) and (2.5.6) of equations of equilibrium in terms of displacement and fluid pressure and the forms (2.5.5) and (2.5.7) of Darcy's law have been used in the present thesis.

If in particular, both the pore fluid and the matrix material are incompressible then \( \alpha = 1 \) and \( M = \infty \). The equation of equi-
librium (2.5.4) in this case are simplified to
\[ \mu \nabla^2 \mathbf{u}_L + (\lambda + 2\mu) \mathbf{e}_L - \mathbf{\beta}_L = 0 \]  \hspace{1cm} (2.5.8)
and the Darcy's law (2.5.5) to
\[ \frac{k}{\eta_1} (\lambda + 2\mu) \nabla^2 \mathbf{e} = \dot{\mathbf{e}} \]  \hspace{1cm} (2.5.9)
where we have used the equation (2.5.8). Introducing the Poisson's ratio \( \nu' \left( = \frac{\nu_1}{\lambda / 2(\lambda + \mu)} \right) \) and the shear modulus \( \alpha' \left( = \frac{\mu}{\lambda} \right) \) of the matrix material, the equations (2.5.8) and (2.5.9) can be written as
\[ \nabla^2 \mathbf{u}_L + (2\mu' - 1) \mathbf{e}_L - \frac{1}{\mu'} \mathbf{\beta}_L = 0 \]  \hspace{1cm} (2.5.10)
and
\[ c' \nabla^2 \mathbf{e} = \dot{\mathbf{e}} \]  \hspace{1cm} (2.5.11)
where \( \mu' = \frac{1 - \nu'}{1 - 2\nu'} \) is a non-dimensional elastic ratio and \( c' = 2\alpha' \left( k / \eta_1 \right) \) is the well-known 'co-efficient of consolidation'. Equations (2.5.10) and (2.5.11) with \( \gamma_1 = 1 \) have been used to solve some specific boundary value problems of poroelasticity by MoNamee and Gibson [25, 26] and Gibson, Schiffman and Pu [28], where they have taken the compressive strains as positive.

The field equations for consolidating elastic material given by (2.5.4) to (2.5.7) are similar to the displacement equations of classical linear elasticity apart from the fact that the equations involve a time-dependent pore fluid pressure which satisfies a diffusion type equation. Also the displacement field of the solid and pore fluid pressure field are coupled in the sense that
one cannot be determined independently of the other. As a result, the displacements $u_i$ and the pore pressure $p_f$ are to be determined simultaneously as functions of space and time coordinates. In particular, if the loading conditions are such that a steady state condition is reached (after an infinite time has elapsed from the instant of loading) for the pore fluid pressure $p_f$ and the dilatation $\varepsilon$ then the equations become time-independent and they are similar to those of un\textit{coupled steady state} thermoelasticity where $p_f$ plays the role of temperature. If the steady state value for the pore pressure is zero everywhere in the medium, then the equations reduce to Navier's equations of elasticity. In this sense, the theory of quasi-static poroelasticity may be considered as the generalisations of the classical theory of elasticity and the un\textit{coupled steady state} theory of thermoelasticity.

2.6. **Boundary conditions**

In the solutions of the particular boundary value problems in the present theory, one has to consider boundary conditions on the pore pressure or its gradient apart from the usual boundary conditions on stresses and displacements. The two physically realistic boundary conditions on pore fluid pressure occur when the boundary is either fully permeable or impermeable to the flow of pore fluid. The following types of boundary conditions have been used in the sequel:

(i) The solid stresses $\sigma_{ij}$ or the total stresses $\tau_{ij}$ are prescribed on the boundary or on a part of the boundary as
functions of the space and time coordinates.

(ii) The solid displacements $u_i$ are prescribed on the whole surface or a part of the surface.

(iii) The fluid pressure $p_f$ or its derivative is given on the boundary. In particular, if the surface is fully permeable, then $p_f = 0$ on the boundary and if it is impermeable then $\partial p_f / \partial n$ is taken to be zero, $\delta n$ being an element of normal to the boundary.

For composite materials, the displacements $u_i$, the pore pressure $p_f$, total stresses $\tau_{ij}$ and the volume flow vector $\omega_i$ (as defined in equation (2.2.5)) are continuous at the interface of the different materials.

2.7. Thermoelastic analogy

It was pointed out by Biot [14] that in the absence of body forces, the equations of quasi-static poroelasticity are analogous with those of coupled quasi-static thermoelasticity (in which the body forces and heat sources are supposed to be absent). It is possible to extend this analogy to the physical variables and the coefficients involved in the two theories.

In the absence of body forces and heat sources, the stress-strain relations, the entropy acquired by an element, the equations of equilibrium and the equation of heat conduction for a thermoelastic medium (taking into consideration the coupling of temperature and strain fields) are [14, 46]:

...
respectively, where

$$\rho_s = (3\lambda + 2\mu) \alpha_o,$$  \hspace{1cm} (2.7.5)

$\alpha$, being the co-efficient of linear thermal expansion, $\theta$ is the increment of temperature over the initial (stress free) temperature $T_0$, $\lambda$, the specific entropy acquired by an element, $h_o$ the amount of heat absorbed, $k_o$ the thermal conductivity, $c_v$ the specific heat per unit mass, $\rho_o$ is the density of the material and $\lambda, \mu$ are Lamé's co-efficients of the thermoelastic medium.

A comparison of the relations (2.7.1) with the first of the equations (2.3.5) gives the correspondence relation

$$p_4 \leftrightarrow \theta,$$
$$\lambda \leftrightarrow \rho_o,$$  \hspace{1cm} (2.7.6)

while the last of the equations (2.3.5) gives with (2.7.2) the relation

$$M \leftrightarrow \frac{T_0}{\rho_o c_v},$$  \hspace{1cm} (2.7.7)

$$\gamma_o \leftrightarrow \beta.$$

The equation (2.7.4) corresponds to the equation (2.5.5) with the
where \( \varepsilon \) is the non-dimensional thermoelastic coupling parameter defined by

\[
\varepsilon = \frac{(3 \lambda + 2 \mu)^2 \alpha_c^2}{\rho \omega C_v (\lambda + 2 \mu)} .
\]  

(2.7.10)

If the term proportional to \( \varepsilon \) (which is generally called the coupling term) in equation (2.7.9) is negligible compared to unity then by making \( \varepsilon \) tend to zero we get the uncoupled heat equation in the form

\[
k_c \nabla^2 \theta = \rho_c C_v \dot{\theta} .
\]  

(2.7.11)

It follows from the relations (2.7.5) to (2.7.7) with (2.7.10) that the thermoelastic coupling parameter has the following correspondence

\[
\varepsilon \longleftrightarrow \frac{\alpha_c^2 \mu}{\lambda + 2 \mu} .
\]  

(2.7.12)

The analogy between the theories of thermoelasticity and poroelasticity can be carried further to the solutions of particular problems of one theory and those of the corresponding analogous prob-
lems in the other. But the solutions of the coupled quasi-static thermoelasticity are not found in the existing literature [46]. This is due to the fact that the coupling effect in quasi-static problems in thermoelasticity being very small is generally disregarded as a first approximation, thus decoupling the temperature and the deformation fields. On the contrary, the pore pressure and deformation fields in the quasi-static problems of poroelasticity cannot be decoupled because of the inherent coupling between the two fields from the standpoint of general theory of three-dimensional consolidation. Hence the solutions of the problems of poroelasticity presented in the present thesis may be considered as solutions of the corresponding analogous problems in thermoelasticity.

2.8. Generalisations to dynamical poroelasticity, viscoelasticity etc.

To consider the problems of elastic wave propagation in a porous solid the theory of quasi-static poroelasticity was generalised by Biot [15, 16] by adding suitable inertia terms in the equations (2.5.2) which were so far neglected in the quasi-static theory and the propagation of three types of body waves in an infinite medium was predicted. The dynamical equations of poroelasticity are given by

\[ N \nabla^2 u_l + (A+N) \varepsilon_{l,l} + \varepsilon_{l} = \frac{\partial^2}{\partial t^2} \left( \rho_{11} u_l + \rho_{12} u_l \right) + \frac{1}{\gamma} \left( u_l - U_l \right) \]

\[ \frac{\partial \sigma_{l,l}}{\partial x_l} = \frac{2}{\gamma t^2} \left( \rho_{11} u_l + \rho_{12} u_l \right) - \frac{1}{\gamma} \left( u_l - U_l \right) \]

where \( \rho_{11} \), \( \rho_{12} \), \( \rho_{32} \) are the mass co-efficients such that

\[ \rho_{11} > 0, \quad \rho_{22} > 0, \quad \rho_{31} < 0, \quad \rho_{11} \rho_{22} - \rho_{12}^2 > 0, \quad \rho_{11} + \rho_{32} + 2 \rho_{12} > 0 \]
Viscoelastic theory of porous solids \([13, 16]\) results from the elastic theory by replacing the elastic co-efficients by certain operators. As for example, \(\mu\) and \(\nu\) in the stress-strain relations \((2.3.6)\) should be replaced by \(\mu^*\) and \(\nu^*\) where

\[
\begin{align*}
\mu^* &= \rho \int_0^\infty \frac{\mu(\tau)}{\rho + \tau} \, d\tau + \mu + \rho \mu', \\
\nu^* &= \rho \int_0^\infty \frac{\nu(\tau)}{\rho + \tau} \, d\tau + \nu + \rho \nu',
\end{align*}
\]

\(\rho \equiv \partial^2 / \partial t^2\); \(\mu, \mu', \nu, \nu'\) are constants and \(\mu(\tau), \nu(\tau)\) represent the continuous spectrums of the relaxation parameter \(\tau\).

Further generalisations of the quasi-static theory of poroelasticity include the theory of stability and consolidation of a porous medium under initial stress \([17]\) and the theory of finite deformation of porous solids \([18]\). A continuum theory for a heat conducting porous solid saturated by a mixture of fluids which are viscous, compressible and heat conducting has every recently been developed by Bedford and Ingram \([19]\) with the help of continuum theory of mixtures.