RELATIONS FOR MOMENTS OF LOWER GENERALIZED ORDER STATISTICS FROM GENERALIZED EXPONENTIAL DISTRIBUTION AND CHARACTERIZATION

1. Introduction

In this chapter we have considered, discussed and established exact expressions and some recurrence relations satisfied by single and product moments of lower generalized order statistics from generalized exponential distribution. Further the results are deduced for moments of order statistics and lower record values and two theorems for characterizing generalized exponential distribution are stated and proved.

A random variable \( X \) is said to have generalized exponential distribution (Gupta and Kundu, 1999) if its pdf is of the form

\[
f(x) = \theta \lambda (1 - e^{-\lambda x})^{\theta-1} e^{-\lambda x}, \quad x > 0, \quad \theta, \lambda > 0
\]  

(4.1.1)

and the corresponding df is

\[
F(x) = (1 - e^{-\lambda x})^\theta, \quad x > 0, \quad \theta, \lambda > 0.
\]  

(4.1.2)

Here \( \theta \) is the shape parameter and \( \lambda \) is the scale parameter, when \( \theta = 1 \), this distribution corresponds to exponential distribution with the scale parameter \( \lambda \).

It is observed in Gupta and Kundu (1999) that the generalized exponential distribution can be used quite effectively in analyzing many lifetime data, particularly in place of gamma and Weibull distributions.

The concept of lower generalized order is as introduced in chapter I.

Part of the results of this chapter appeared in Khan and Kumar (2011a)
2. Relations for single moments

We note that \( f(x) \) and \( F(x) \) satisfy the relation

\[
\theta \lambda F(x) = (e^{\lambda x} - 1)f(x).
\]  
(4.2.1)

The relation in (4.2.1) will be used to derive some recurrence relations for the moments of lower generalized order statistics from the generalized exponential distribution.

The pdf of \( r \)-th gos is given as

\[
f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [F(x)]^{\gamma r - 1} f(x) g_m^{-1}(F(x)).
\]  
(4.2.2)

We shall first establish the exact expression for \( E[X^*(r,n,m,k)] \). Using (4.2.2), we have when \( m \neq -1 \)

\[
E[X^*(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma r - 1} f(x) g_m^{-1}(F(x))dx.
\]

\[
= \frac{C_{r-1}}{(r-1)!} J_j(\gamma r - 1, r - 1),
\]  
(4.2.3)

\[
J_j(a,b) = \int_0^\infty x^j [F(x)]^a f(x) g_m^b(F(x))dx.
\]  
(4.2.4)

On expanding \( g_m^b(F(x)) = \left[ \frac{1}{m+1} \right] \binom{m+1}{u} \) binomially in (4.2.4), we get when \( m \neq -1 \)

\[
J_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^b (-1)^u \binom{b}{u} \int_0^\infty x^j [F(x)]^{a+u(m+1)} f(x) dx.
\]  
(4.2.5)

By setting \( t = [F(x)]^{1/\theta} \) in (4.2.5), we obtain

\[
J_j(a,b) = \frac{\theta}{\lambda^j (m+1)^b} \sum_{u=0}^b (-1)^u \binom{b}{u} \times \int_0^1 [-\ln(1-t)]^j t^{\theta[a+u(m+1)+1]-1} dt.
\]  
(4.2.6)

On using the logarithmic expansion
\[-\ln(1-t)^j = \left( \sum_{p=1}^{\infty} \frac{t^p}{p} \right)^j = \sum_{p=0}^{\infty} \alpha_p(j) t^{j+p}, \quad |t| < 1, \quad (4.2.7)\]

where \( \alpha_p(j) \) is the coefficient of \( t^{j+p} \) in the expansion of

\[
\left( \sum_{p=1}^{\infty} \frac{t^p}{p} \right)^j
\]

[Balakrishnan and Cohen (1991), Shawky and Bakoban (2008b)] and integrating the resulting expression, we get

\[
J_j(a,b) = \frac{\theta}{\lambda^j (m+1)^b} \sum_{p=0}^{\infty} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \frac{\alpha_p(j)}{[\theta(a+u(m+1)+1] + j + p]].
\]

(4.2.8)

When \( m = -1 \), we have

\[
J_j(a,b) = \frac{0}{0}, \text{ as } \sum_{u=0}^{b} (-1)^u \binom{b}{u} = 0.
\]

On applying L’ Hospital rule and then using (2.2.10) (Ruiz, 1996) on the lines of (2.2.7) for \( m = -1 \) derived in Section 2 of Chapter II, we get

\[
\lim_{m \to -1} J_j(a,b) = \frac{b! \theta^{b+1}}{\lambda^j} \sum_{p=0}^{\infty} \frac{\alpha_p(j)}{[\theta(a+1) + j + p]^{b+1]].}
\]

(4.2.9)

Now substituting for \( J_j(\gamma_r -1,r-1) \) from (4.2.8) in (4.2.3) and simplifying, we obtain when \( m \neq -1 \)

\[
E[X^*_j(r,n,m,k)] = \frac{\theta C_{r-1}}{(r-1)! \lambda^j (m+1)^{r-1}} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}
\]

\[
\times \frac{\alpha_p(j)}{(\theta \gamma_{r-u} + j + p)}. \quad (4.2.10)
\]

Identity 2.1: For \( \gamma_r \geq 1, \ k \geq 1, \ 1 \leq r \leq n \) and \( m \neq -1 \)

\[
\sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}} = \frac{(r-1)!(m+1)^{r-1}}{\prod_{t=0}^{r} \gamma_t}. \quad (4.2.11)
\]
Relations for moments of lower generalized order statistics from generalized exponential distribution

Proof: At $j = 0$ in (4.2.10), we have

$$1 = \theta C_{r-1} \frac{\sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{\alpha_p(0)}{(\theta \gamma_{r-u} + p)}}{(r-1)!(m+1)^{r-1}}.$$ 

Note that, if $j = 0$, then

$$\alpha_p(0) = 1, \quad p = 0 \quad \text{and} \quad \alpha_p(0) = 0, \quad p > 0 \quad \text{(see Appendix)}$$

and hence the result given in (4.2.11).

Special cases

i) Putting $m = 0$, $k = 1$ in (4.2.10), the explicit formula for single moments of order statistics of the generalized exponential distribution can be obtained as

$$E[X_{n-r+1:n}^j] = \frac{\theta C_{r:n}}{\lambda^j} \sum_{p=0}^{\infty} \sum_{u=0}^{\infty} (-1)^u \binom{r-1}{u} \frac{\alpha_p(j)}{[\theta(n-r+1+u) + j + p]}.$$ 

That is

$$E[X_{r:n}^j] = \frac{\theta C_{r:n}}{\lambda^j} \sum_{p=0}^{\infty} \sum_{u=0}^{n-r} (-1)^u \binom{n-r}{u} \frac{\alpha_p(j)}{[\theta(r+u) + j + p]},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$ 

ii) Putting $m = -1$ in (4.2.10), we deduce the explicit expression for the moments of lower $k$ record values for the generalized exponential distribution in view of (4.2.3) and (4.2.9) as

$$E[(Z_r^{(k)})^j] = \frac{(\theta k)^r}{\lambda^j} \sum_{p=0}^{\infty} \frac{\alpha_p(j)}{(\theta k + j + p)^r}.$$ 

and for lower records, $k = 1$

$$E[X_{L(r)}^j] = \frac{\theta^r}{\lambda^j} \sum_{p=0}^{\infty} \frac{\alpha_p(j)}{(\theta + j + p)^r}.$$
A recurrence relation for single moments of $lgos$ from df (4.1.1) is obtained in the following theorem.

Theorem 2.1: For the distribution as given in (4.1.2) and for $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots$

$$E[X^{*j}(r, n, m, k)] - E[X^{*j}(r-1, n, m, k)] = \frac{j}{\theta \lambda \gamma_r} \{E[X^{*j-1}(r, n, m, k)] - E[\phi(X^{*}(r, n, m, k))]\}, \quad (4.2.12)$$

where

$$\phi(x) = x^{j-1} e^{\lambda x}.$$ 

Proof: From (4.2.2), we have

$$E[X^{*j}(r, n, m, k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{-r} g_m^{-1}(F(x)) f(x) \, dx.$$ 

(4.2.13)

Integrating by parts treating $[F(x)]^{-r} f(x)$ for integration and rest of the integrand for differentiation, we get

$$E[X^{*j}(r, n, m, k)] = E[X^{*j}(r-1, n, m, k)] - \frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^{j-1} [F(x)]^{-r} g_m^{-1}(F(x)) f(x) \, dx,$$

the constant of integration vanishes since the integral considered in (4.2.13) is a definite integral. On using (4.2.1), we obtain

$$E[X^{*j}(r, n, m, k)] = E[X^{*j}(r-1, n, m, k)] - \frac{j C_{r-1}}{\theta \lambda \gamma_r (r-1)!} \times \left\{ \int_0^\infty x^{j-1} e^{\lambda x} [F(x)]^{-r} g_m^{-1}(F(x)) f(x) \, dx \right.$$

$$- \left. \int_0^\infty x^{j-1} [F(x)]^{-r} f(x) g_m^{-1}(F(x)) \, dx \right\}$$

and hence the result.
Relations for moments of lower generalized order statistics from generalized exponential distribution

**Remark 2.1:** Putting \( m = 0, \ k = 1 \), in (4.2.12), we obtain a recurrence relation for single moments of order statistics of the generalized exponential distribution in the form

\[
E[X_{n-r+1:n}^j] - E[X_{n-r+2:n}^j] = \frac{j}{\theta \lambda (n-r+1)} \{E[X_{n-r+1:n}^{j-1}] - E[\phi(X_{n-r+1:n})]\}.
\]

Replacing \( (n-r+1) \) by \( (r-1) \), we have

\[
E[X_{r:n}^j] - E[X_{r-1:n}^j] = -\frac{j}{\theta \lambda (r-1)} \{E[X_{r-1:n}^{j-1}] - E[\phi(X_{r-1:n})]\}.
\]

**Remark 2.2** Setting \( m = -1 \) and \( k \geq 1 \) in (4.2.12), we get a recurrence relation for single moments of lower \( k \) record values from generalized exponential distribution in the form

\[
E[(Z_r^{(k)})^j] - E[(Z_{r-1}^{(k)})^j] = \frac{j}{\theta \lambda k} \{E[(Z_r^{(k)})^{j-1}] - E[\phi(Z_r^{(k)})]\}.
\]

### 3. Relations for product moments

The joint pdf of \( X^*(r,n,m,k) \) and \( X^*(s,n,m,k) \), \( 1 \leq r < s \leq n \), is given as

\[
f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [F(x)]^m f(x)g_{m}^{r-1}[F(x)]
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{y_s-1}f(y), \quad x > y. \tag{4.3.1}
\]

The explicit expressions for the product moments of \( l \) gos \( X^{*i}(r,n,m,k) \) and \( X^{*j}(s,n,m,k) \), \( 1 \leq r < s \leq n \), can be obtained when \( m \neq -1 \) as

\[
E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^x \int_0^y x^i y^j [F(x)]^m f(x)g_{m}^{r-1}[F(x)]
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{y_s-1} f(y) dy dx. \tag{4.3.2}
\]
On expanding $g_m^{-1}(F(x)) = \left[ \frac{1}{m+1} \{1 - (F(x))^{m+1}\} \right]^{-1}$ binomially in (4.3.2), we get

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \int_0^\infty \int_0^x x^i y^j [F(x)]^{m+u(m+1)} f(x)$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y) dy dx$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} J_{i,j}(m+u(m+1),s-r-1,\gamma_s-1),$$

where

$$J_{i,j}(a,b,c) = \int_0^\infty \int_0^x x^i y^j [F(x)]^a f(x)[h_m(F(y)) - h_m(F(x))]^b$$

$$\times [F(y)]^c f(y) dy dx.$$  \hspace{1cm} (4.3.3)

Expanding $[h_m(F(y)) - h_m(F(x))]^b$ binomially in (4.3.4) after noting that $h_m(F(y)) - h_m(F(x)) = g_m(F(y)) - g_m(F(x))$, we get when $m \neq -1$

$$J_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} \int_0^\infty x^i [F(x)]^{a+(b-v)(m+1)} f(x) I(x) dx,$$

where

$$I(x) = \int_0^x y^j [F(y)]^{c+v(m+1)} f(y) dy.$$  \hspace{1cm} (4.3.5)

By setting $z = [F(y)]^{1/\theta}$ in (4.3.6), we get

$$I(x) = \frac{\theta}{\lambda^j} \sum_{p=0}^{\infty} \frac{\alpha_p(j)[F(x)]^{c+v(m+1)+1+(j+p)/\theta}}{[\theta[c + v(m+1) + 1] + j + p]}.$$  \hspace{1cm} (4.3.6)

On substituting the above expression of $I(x)$ in (4.3.5), we find that
Relations for moments of lower generalized order statistics from generalized exponential...

\[ J_{i,j}(a,b,c) = \frac{\theta}{\lambda^j(m+1)^b} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \binom{b}{q} \frac{\alpha_p(j)}{[\theta(c + v(m+1) + 1) + j + p]} \times \int_0^\infty x^i [F(x)]^{a+c+b(m+1)+1+(j+p)/\theta} f(x) dx. \] (4.3.7)

Again by setting \( t = [F(x)]^{1/\theta} \) in (4.3.7) and simplifying the resulting expression, we obtain

\[ J_{i,j}(a,b,c) = \frac{\theta^2}{\lambda^{i+j}(m+1)^b} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \binom{b}{q} \frac{\alpha_p(j) \alpha_q(i)}{[\theta(c + v(m+1) + 1) + j + p][\theta(a + c + b(m+1) + 2) + i + j + p + q]} \] (4.3.8)

and when \( m = -1 \) that

\[ J_{i,j}(a,b,c) = 0, \quad \text{as} \quad \sum_{v=0}^{b} (-1)^v \binom{b}{v} = 0. \]

Therefore applying L’ Hospital rule and using (2.2.10), we find that

\[ \lim_{m \to -1} J_{i,j}(a,b,c) = \frac{b! \theta^{b+2}}{\lambda^{i+j}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\alpha_p(j)}{[\theta(c + 1) + j + p]^{b+1}} \times \frac{\alpha_q(i)}{[\theta(a + c + 2) + i + j + p + q]}. \] (4.3.9)

Now on substituting for \( J_{i,j}(m+u(m+1),s-r-1,\gamma_s-1) \) from (4.3.8) in (4.3.3) and simplifying, we obtain when \( m \neq -1 \)

\[ E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \frac{\theta^2 C_{s-1}}{(r-1)!(s-r-1)!{\lambda^{i+j}(m+1)}^{s-2}} \times \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \times \frac{\alpha_p(j) \alpha_q(i)}{(\theta \gamma_{s-v} + j + p)(\theta \gamma_{r-u} + i + j + p + q)}. \] (4.3.10)

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Identity 3.1: For $\gamma_r, \gamma_s \geq 1, k \geq 1, 1 \leq r < s \leq n$ and $m \neq -1$

$$\sum_{v=0}^{s-r-1} (-1)^v \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}} = \frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{t=r+1}^{s} \gamma_t}. \tag{4.3.11}$$

Proof: At $i = j = 0$ in (4.3.10), we have

$$1 = \frac{\theta^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} (-1)^u \binom{r-1}{u} \frac{\alpha_p(0)\alpha_q(0)}{(\theta \gamma_{s-v} + p)(\theta \gamma_{r-u} + p + q)}.$$

Note that, if $i = j = 0$, then

$$\alpha_p(0) = 1, \quad \alpha_q(0) = 1, \quad p, q = 0 \quad \text{and} \quad \alpha_p(0) = 0, \quad \alpha_q(0) = 0, \quad p, q > 0 \quad \text{(see Appendix)}.$$

Therefore,

$$\sum_{v=0}^{s-r-1} (-1)^v \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}} = \frac{(r-1)!(s-r-1)!(m+1)^{s-2}}{C_{s-1}} \times \frac{1}{\sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}}}.$$

Now on using (4.2.11), we get the result given in (4.3.11).

Special cases

i) Putting $m = 0, k = 1$ in (4.3.10), the explicit formula for the product moments of order statistics of the generalized exponential distribution is obtained as

$$E[X_{n-r+1:n}^i X_{n-s+1:n}^j] = \frac{\theta^2 C_{r,s:n}}{\lambda^{i+j}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \sum_{v=0}^{s-r-1} (-1)^u \binom{r-1}{u} \frac{\alpha_p(j)\alpha_q(i)}{[\theta(n-s+1+v) + j + p][\theta(n-r+1+u) + i + j + p + q]}.$$
That is
\[
E[X_{r:n}^i X_{s:n}^j] = \frac{\theta^2}{\lambda^{i+j}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (-1)^{u+v} \binom{n-s}{u} \alpha_p(i) \alpha_q(j) \\
\times \left( s - r - 1 \right)_v \frac{\alpha_p(i) \alpha_q(j)}{\theta^{(r+v) + i + p} [\theta^{(s+u) + i + j + p + q}]},
\]

where \( C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \).

ii) Putting \( m = -1 \) in (4.3.10), the explicit formula for the product moments of lower \( k \) record values for the generalized exponential distribution can be obtained in view of (4.3.3) and (4.3.9) as
\[
E[(Z_r^{(k)})^i (Z_s^{(k)})^j] = \frac{(\theta k)^s}{\lambda^{i+j}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\alpha_p(j) \alpha_q(i)}{(\theta k + j + p)^{s-r} (\theta k + i + j + p + q)^r}
\]

and hence
\[
E[X_{L(r)}^i X_{L(s)}^j] = \frac{\theta^s}{\lambda^{i+j}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\alpha_p(j) \alpha_q(i)}{(\alpha + j + p)^{s-r} (\alpha + i + j + p + q)^r}.
\]

Making use of (4.2.1), we can derive recurrence relations for product moments of lower \( k \) order statistics from (4.1.2).

**Theorem 3.1:** For the distribution as given in (4.1.2) and for \( 1 \leq r < s \leq n, n \geq 2 \) and \( k = 1,2,\ldots \)
\[
E[X^{*i}(r,n,m,k) X^{*j}(s,n,m,k)] - E[X^{*i}(r,n,m,k) X^{*j}(s-1,n,m,k)] \\
= \frac{j}{\theta \lambda} \left\{ E[X^{*i}(r,n,m,k) X^{*j-1}(s,n,m,k)] \\
- \phi(X^*(r,n,m,k) X^*(s,n,m,k)) \right\},
\]

where
\[
\phi(x, y) = x^i y^{j-1} e^{\lambda y}.
\]
From (4.3.1), we have

\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \]
\[ \times \int_0^\infty x^i [F(x)]^m f(x) g_m^{-1}(F(x)) I(x) dx, \quad (4.3.13) \]

where

\[ I(x) = \int_0^x y^j [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} - 1 f(y) dy. \]

Solving the integral in \( I(x) \) by parts and substituting the resulting expression in (4.3.13), we get

\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] = E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \]
\[ - \frac{j C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_0^\infty \int_0^x x^i y^{j-1} [F(x)]^m f(x) g_m^{-1}(F(x)) \]
\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} - 1 f(y) dy dx, \]

the constant of integration vanishes since the integral in \( I(x) \) is a definite integral. On using the relation (4.2.1), we obtain

\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] = E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \]
\[ - \frac{j C_{s-1}}{\alpha \lambda \gamma_s (r-1)!(s-r-1)!} \left\{ \int_0^\infty \int_0^x x^i y^{j-1} e^{\lambda y} [F(x)]^m f(x) g_m^{-1}(F(x)) \right. \]
\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} - 1 f(y) dy dx \]
\[ - \int_0^\infty \int_0^x x^i y^{j-1} [F(x)]^m f(x) g_m^{-1}(F(x)) \]
\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s} - 1 f(y) dy dx \right\} \]

and hence the result.

**Remark 3.1:** Putting \( m = 0, \ k = 1 \) in (4.3.12), we obtain recurrence relations for product moments of order statistics of the generalized exponential distribution in the form
\[ E[X_{n-r+1:n}^i X_{n-s+1:n}^j] - E[X_{n-r+1:n}^i X_{n-s+1:n}^{j+1}] = \frac{j}{\theta \lambda (n-s+1)} \times \{ E[X_{n-r+1:n}^i X_{n-s+1:n}^{j-1}] - E[\phi(X_{n-r+1:n} X_{n-s+1:n})] \}. \]

That is

\[ E[X_{r:n}^i X_{s:n}^j] - E[X_{r-1:n}^i X_{s:n}^j] = \frac{i}{\theta \lambda (r-1)} \times \{ E[X_{r-1:n}^{i-1} X_{s:n}^j] - E[\phi(X_{r-1:n} X_{s:n})] \}. \]

**Remark 3.2:** Setting \( m = -1 \) and \( k \geq 1 \), in (4.3.12), we obtain the recurrence relations for product moments of lower \( k \) record values from generalized exponential distribution in the form

\[ E[(Z_r^{(k)})^i (Z_s^{(k)})^j] - E[(Z_r^{(k)})^i (Z_{s-1}^{(k)})^j] = \frac{j}{\theta \lambda k} \{ E[(Z_r^{(k)})^i (Z_s^{(k)})^{j-1}] - E[\phi(Z_r^{(k)})(Z_s^{(k)})] \}. \]

**Remark 3.3:** At \( j = 0 \) in (4.3.10), we have

\[ E[X^{*i}(r,n,m,k)] = \frac{\theta^2 C_{s-1}}{(r-1)! (s-r-1)! \lambda^i (m+1)^{s-2}} \times \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \frac{\alpha_p(0) \alpha_q(i)}{\theta \gamma_{s-v} + p}(\theta \gamma_{r-u} + i + p + q). \]

On using \( \alpha_p(0) \) from Appendix in above equation, we have

\[ E[X^{*i}(r,n,m,k)] = \frac{\theta C_{s-1}}{(r-1)! (s-r-1)! \lambda^i (m+1)^{s-2}} \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \frac{\alpha_q(i)}{\gamma_{s-v}(\theta \gamma_{r-u} + i + q)}. \]  

(4.3.14)

Making use of (4.3.11) in (4.3.14) and simplifying the resulting expression, we get
\[ E[X^{\gamma_i}(r,n,m,k)] = \frac{\theta C_{r-1}}{(r-1)!} \sum_{q=0}^{\infty} \sum_{u=0}^{r-1} (-1)^u \left( \frac{r-1}{u} \right) \frac{\alpha_q(i)}{\theta \gamma_{r-u} + i + q} \]

as obtained in (4.2.10).

**Remark 3.4:** At \( i = 0 \), Theorem 3.1 reduces to Theorem 2.1.

4. Characterization

Let \( X^*(r,n,m,k), r=1,2,\ldots,n \) be lgos from a continuous population with \( df \) \( F(x) \) and \( pdf \) \( f(x) \), then the conditional \( pdf \) of \( X^*(s,n,m,k) \) given \( X^*(r,n,m,k) = x \), \( 1 \leq r < s \leq n \), in view of (4.3.1) and (4.2.2), is

\[
\begin{align*}
  f_{X^*(s,n,m,k)\mid X^*(r,n,m,k)}(y \mid x) &= \frac{C_{s-1}}{(s-r-1)!C_{r-1}} [F(x)]^{m-r,1} \\
  &\times [(h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{y_s-1} f(y). 
\end{align*}
\]

**Theorem 4.1:** Let \( X \) be a non negative random variable having an absolutely continuous distribution function \( F(x) \) with \( F(0) = 0 \) and \( 0 < F(x) < 1 \) for all \( x > 0 \), then

\[
E[X^*(s,n,m,k) \mid X^*(r,n,m,k) = x] = \sum_{p=1}^{\infty} \frac{(1-e^{-\lambda x})^p}{\lambda^p} \frac{Y_{r+j}}{Y_{r+j} + p / \theta} 
\]

if and only if

\[
F(x) = (1 - e^{-\lambda x})^\theta, \quad x > 0, \quad \theta, \lambda > 0. 
\]

**Proof:** From (4.4.1), we have

\[
\begin{align*}
  E[X^*(s,n,m,k) \mid X^*(r,n,m,k) = x] &= \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \\
  &\times \int_{x}^{0} y \left[ 1 - \left( \frac{F(y)}{F(x)} \right)^{m+1} \right]^{s-r-1} \left( \frac{F(y)}{F(x)} \right)^{y_s-1} f(y) \frac{dy}{F(x)}. 
\end{align*}
\]
By setting \( u = \frac{F(y)}{F(x)} = \left( \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda y}} \right)^\theta \) from (4.1.2) in (4.4.3), we obtain

\[
E[X^*(s,n,m,k) | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r-1}} \times \int_0^1 \left[ -\ln \{1 - (1 - e^{-\lambda x})u^{1/\theta}\} \right] u^{\gamma_{s-1} - 1}(1 - u^{m+1})^{s-r-1} du
\]

\[
= \frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r-1}} \sum_{p=1}^{\infty} \frac{(1 - e^{-\lambda x})^p}{\lambda^p} \int_0^1 u^{(p/\theta) + \gamma_{s-1} - 1}(1 - u^{m+1})^{s-r-1} du.
\]  

(4.4.4)

Again by setting \( t = u^{m+1} \) in (4.4.4), we get

\[
E[X^*(s,n,m,k) | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r}} \times \sum_{p=1}^{\infty} \frac{(1 - e^{-\lambda x})^p}{\lambda^p} \int_0^1 t^{\frac{p+\theta k}{\theta(m+1)} + n-s-1} (1 - t)^{s-r-1} dt
\]

\[
= \frac{C_{s-1}}{C_{r-1}(m + 1)^{s-r}} \sum_{p=1}^{\infty} \frac{(1 - e^{-\lambda x})^p}{\lambda^p} \frac{\Gamma \left( \frac{p + \theta k}{\theta(m+1)} + n-s \right)}{\Gamma \left( \frac{p + \theta k}{\theta(m+1)} + n-r \right)}
\]

\[
= \frac{C_{s-1}}{C_{r-1}} \sum_{p=1}^{\infty} \frac{(1 - e^{-\lambda x})^p}{\lambda^p} \prod_{j=1}^{s-r} (\gamma_{r+j} + p/\theta)
\]

and hence the result given in (4.4.2).

To prove sufficient part, we have from (4.4.1) and (4.4.2)

\[
\frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r-1}} \int_0^x y[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1} dy
\]

\[
	imes [F(y)]^{\gamma_{s-1}} f(y) dy = [F(x)]^{\gamma_{r+1}} H_r(x),
\]  

(4.4.5)

where
\[ H_r(x) = \sum_{p=1}^{\infty} \frac{(1-e^{-\lambda x})^p s-r}{\lambda p} \prod_{j=1}^{\gamma_r+j} \left( \frac{p}{\gamma_r+j+p/\theta} \right). \]

Differentiating (4.4.5) both sides with respect to \( x \), we get

\[
\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_0^x y[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-2} \times (F(y))^{\gamma_s-1} f(y) dy \\
= H'_r(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x)
\]

or

\[
\gamma_{r+1} H_{r+1}(x)[F(x)]^{\gamma_{r+2}+m} f(x) \\
= H'_r(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x).
\]

Therefore,

\[
\frac{f(x)}{F(x)} = \frac{H'_r(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]} \\
= \frac{\theta \lambda e^{-\lambda x}}{1-e^{-\lambda x}}
\]

which proves that

\[ F(x) = (1-e^{-\lambda x})^\theta, \quad x > 0, \quad \theta, \lambda > 0. \]

**Theorem 4.2:** Let \( X \) be a non-negative random variable having an absolutely continuous distribution function \( F(x) \) with \( F(0) = 0 \) and \( 0 < F(x) < 1 \) for all \( x > 0 \), then

\[
E[X^j (r,n,m,k)] = E[X^j (r-1,n,m,k)] - \frac{j}{\theta \lambda \gamma_r} E[\phi(X^*(r,n,m,k))]
\]

\[
+ \frac{j}{\theta \lambda \gamma_r} E[X^{j-1}(r,n,m,k)]
\]

(4.4.6)

if and only if

\[ F(x) = (1-e^{-\lambda x})^\theta, \quad x > 0, \quad \theta, \lambda > 0. \]
**Proof**: The necessary part follows immediately from equation (4.2.12). On the other hand if the recurrence relation in equation (4.4.6) is satisfied, then on using equation (4.2.2), we have

\[
\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r^{-1}} f(x) g_m^{r-1}(F(x)) \, dx
\]

\[
= \frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j+1}[F(x)]^{\gamma_r+m} f(x) g_m^{r-2}(F(x)) \, dx
\]

\[
- \frac{jC_{r-1}}{\theta \lambda \gamma_r(r-1)!}\int_0^\infty x^{j-1} e^{\lambda x}[F(x)]^{\gamma_r^{-1}} f(x) g_m^{r-1}(F(x)) \, dx
\]

\[
+ \frac{jC_{r-1}}{\theta \lambda \gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r^{-1}} f(x) g_m^{r-1}(F(x)) \, dx. \quad (4.4.7)
\]

Integrating the first integral on the right hand side of equation (4.4.7) by parts and simplifying the resulting expression, we get

\[
\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r^{-1}} g_m^{r-1}(F(x)) \, dx
\]

\[
\times \left\{ F(x) - \frac{e^{\lambda x}}{\theta \lambda} f(x) + \frac{1}{\theta \lambda} f(x) \right\} \, dx = 0. \quad (4.4.8)
\]

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, 1984) to equation (4.4.8), we get

\[
\frac{f(x)}{F(x)} = \frac{\theta \lambda}{(e^{\lambda x} - 1)}
\]

which proves that

\[
F(x) = (1 - e^{-\lambda x})^\theta, \quad x > 0, \quad \theta, \lambda > 0.
\]