CHAPTER 0
GENERAL PREREQUISITES

This chapter, for ready reference, consists of definitions, lemmas, theorems and corollaries which exists in literature of General topology. These have been frequently used in this thesis. Other current definitions, lemmas and theorems in the thesis but not stated in this chapter have been reproduced at the places where they are required.

NOTATION USED

Throughout the thesis (X, τ), (Y, σ), etc. (or simply X, Y etc.) always denote topological spaces. The family of τ-closed sets in (X, τ) is denoted by \( \mathcal{F}(\tau) \) while the closure and interior of \( A \subset X \) are respectively denoted by \( \text{Cl}_X(A) \) and \( \text{Int}_X(A) \) or simply by \( \text{Cl}(A) \) and \( \text{Int}(A) \) when there is no chance of confusion. By \( A^C \), we denote the complement of \( A \). In \( (X, \tau) \), for a set \( A \subset X \), the family \( \{ U \in \tau : A \subset U \} \) is designated by \( \Sigma(A) \) and for a point \( x \in X, \Sigma(x) = \{ U \in \tau : x \in U \} \). When \( A \) is a subspace of a topological space \( (X, \tau) \) we write it as either \( (A, \tau_A) \) or \( (A, \tau/A) \). The restriction of a function \( f : X \to Y \) to \( A \subset X \) has been denoted by \( f_A \) or \( f/A \) throughout the thesis. Notations not explained here but used in the thesis taken from Pervin [68], Dugundji [21] and Kelley [38].

KNOWN DEFINITIONS

**Definition 0.1.** In \((X, \tau), A \subset X\) is called

(i) a preopen set (briefly p.o. set) \([47]\) iff \( A \subset \text{Int}(\text{Cl}(A)) \);

(ii) a semi-open set (briefly s.o. set) \([40]\) iff there exists \( O \in \tau \) such that \( O \subset A \subset \text{Cl}(O) \); equivalently, \( A \) is s.o. \([40]\) iff \( A \subset \text{Cl}(\text{Int}(A)) \);

(iii) an \( \alpha \)-set \([55]\) iff \( A \subset \text{Int}(\text{Cl}(\text{Int}(A))) \);

(iv) a regular open set (briefly r.o. set) \([85]\) iff \( A = \text{Int}(\text{Cl}(A)) \);
(v) a generalised closed set \([41]\) (briefly g-closed set) iff

\[ \text{Cl}(A) \subseteq O \]

whenever \(A \subseteq O \in \tau;\)

(vi) a semi-preopen set (briefly s.p.o. set) \([1]\) iff there exists a preopen set \(U \in X\) such that \(U \subseteq A \subseteq \text{Cl}(U)\). Equivalently, \(A\) is s.p.o. iff

\[ A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A))). \]

The family of all p.o. (resp. s.o., r.o., s.p.o.) sets is denoted by \(\text{PO}(X)\) (resp. \(\text{SO}(X), \text{RO}(X), \text{SPO}(X)\)). For each \(x \in X\), the family of p.o. (resp. s.o., r.o., s.p.o.) sets containing \(x\) is denoted by \(\text{PO}(X, x)\) (resp. \(\text{SO}(X, x), \text{RO}(X, x), \text{SPO}(X, x)\)). The family of \(\alpha\)-sets is denoted by \(\alpha(X)\) (or \(\tau^\alpha\)).

The family of all \(\alpha\)-sets is denoted by \(\alpha(X, x)\).

**Definition 0.2.** The complement of a p.o. (resp. s.o., r.o., s.p.o.) set is called preclosed \([47]\) (resp. semi-closed \([11]\), regular closed \([85]\), semi-preclosed \([1]\)). Equivalently a set \(F\) is

(i) preclosed \([47]\) iff \(\text{Cl}(\text{Int}(F)) \subseteq F;\)

and (ii) semi-preclosed \([1]\) iff \(\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq F.\)

The family of all preclosed (resp. semi-closed, semi-preclosed) set is denoted by \(\text{PC}(X)\) (resp. \(\text{SC}(X), \text{SPF}(X)\)).

**Definition 0.3.** The preclosure \([22]\) (resp. semi-closure \([11, 16]\), semi-preclosure \([1]\)) of \(A \subseteq X\) is denoted by \(\text{pcl}(A)\) (resp. \(\text{scl}(A), \text{spcl}(A)\)) and is defined by

\[ \text{pcl}(A) = \bigcap \{B : B \text{ is preclosed and } B \supseteq A\} \]

(resp. \(\text{scl}(A) = \bigcap \{B : B \text{ is semi-closed and } B \supseteq A\},\)

\[ \text{spcl}(A) = \bigcap \{B : B \text{ is semi-preclosed and } B \supseteq A\}).\]

The preinterior \([9]\) (resp. semi-preinterior \([1]\)) of \(A \subseteq X\) is denoted by \(\text{pint}(A)\) (resp. \(\text{spint}(A)\)) and is defined by

\[ \text{pint}(A) = \bigcup \{B : B \in \text{PO}(X) \text{ and } B \subseteq A\} \]

(resp. \(\text{spint}(A) = \bigcup \{B : B \in \text{SPO}(X) \text{ and } B \subseteq A\}).\)
**Definition 0.4.** Let \((X, \tau)\) be a topological space, \(A \subseteq X, x \in X\). Then \(A\) is a

(i) **pre-neighbourhood** (briefly **pre-nbd**) \([36]\) of \(x\) if there exists a 
\[ U \in \PO(X, x) \text{ such that } x \in U \subseteq A; \]

(ii) **semi-pre neighbourhood** (briefly **sp-nbd**) \([29]\) of \(x\) if there exists a 
\[ U \in \SPO(X, x) \text{ such that } x \in U \subseteq A. \]

The family of all pre-nbds of a point \(x \in X\) is denoted by \(\mathcal{N}_p(x)\).

**Definition 0.5.** A subset \(A\) of \(X\) is said to be **preclopen** \([36]\) if 
\[ A \in \PO(X) \cap \PC(X). \]

**Definition 0.6.** Let \(X\) be a set, \(\tau\) and \(\sigma\) be topologies for \(X\). Then \(\tau\) is said to be **weakly equivalent** \([27]\) to \(\sigma\) provided if \(U \in \tau\) with \(U \neq \emptyset\), then there exists a \(V \in \sigma\) such that \(V \neq \emptyset\) and \(V \subseteq U\) and if \(U \in \sigma\) with \(U \neq \emptyset\) then there exists a \(V \in \tau\) such that \(V \neq \emptyset\) and \(V \subseteq U\).

**Definition 0.7.** Let \(X\) and \(Y\) be two spaces. A set \(S \subseteq X \times Y\) is termed strongly \(p\)-\(\theta\)-closed \([66]\) (resp. strongly pre-\(\theta\)-closed \([63]\)) with respect to \(X \times Y\) if for each \((x, y) \in S\), there exist \(U \in \PO(X, x)\) and \(V \in \PO(Y, y)\) such that \([\text{Cl}_X(U) \times \text{Cl}_Y(V)] \cap S\) (resp. \([\text{pcl}_X(U) \times \text{pcl}_Y(V)] \cap S\)) = \(\emptyset\).

**Definition 0.8.** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is called

(i) **weakly continuous** \([39]\) (resp. almost continuous in the sense of Singal \([77]\)) briefly **w.c.** (resp. **a.c.S.**) iff for each \(x \in X\) and each \(V \in \Sigma(f(x))\), there exists a \(U \in \Sigma(x)\) (resp. \(U \in \Sigma(x)\)) such that 
\[ f[U] \subseteq \text{Cl}_Y(V) \text{ (resp. } f[U] \subseteq \text{Int}_Y(\text{Cl}_Y(V)) ; \]
equivalently, if for each \(V \in \RO(Y), f^{-1}[V] \in \tau\) ;

(ii) **quasi continuous** \([51]\) at \(x \in X\) iff for each \(U \in \Sigma(x)\) in \(X\) and each \(V \in \Sigma(f(x))\) in \(Y\) there exists a non-empty \(G \in \tau\) such that \(G \subseteq U\) and \(f[G] \subseteq V\); if \(f\) is quasi continuous at every point of \(X\), then it is called quasi continuous;
(iii) somewhat continuous [27] provided for each $V \in \sigma$ with $f^{-1}[V] \neq \emptyset$
there exists a $U \in \tau$ with $U \neq \emptyset$ such that $U \subseteq f^{-1}[V]$;
(iv) precontinuous [47] (resp. almost continuous in the sense of Husain [31]) briefly pc (resp. a.c.H.) iff for each $V \in \sigma$ (resp. for each $x \in X$
and $V \in \Sigma(f(x))$) $f^{-1}[V] \in PO(X)$ (resp. $Cl_x(f^{-1}[V])$ is a
neighbourhood of $x$ in $X$);
(v) weakly $\alpha$-continuous [57] briefly w.$\alpha$.c. iff for each $x \in X$ and each
$V \in \Sigma(f(x))$ there exists a $U \in \alpha(X,x)$ such that $f[U] \subseteq Cl_Y(V)$;
(vi) quasi-precontinuous [64] briefly qpc iff for each $x \in X$ and each
$V \in \Sigma(f(x))$ there exists a $U \in PO(X,x)$ such that $f[U] \subseteq Cl_Y(V)$.
(vii) semi-precontinuous [73] if for each $V \in \sigma$, $f^{-1}[V] \subseteq SPO(X)$.
(viii) almost precontinuous [53] briefly apc iff for each $x \in X$ and for each
$V \in RO(Y,f(x))$ there exists a $U \in PO(X,x)$ such that $f[U] \subseteq V.$

**Definition 0.9.** A mapping $f : (X, \tau) \to (Y, \sigma)$ is called

(i) somewhat open [27] if for each $U \in \tau$ with $U \neq \emptyset$ there exists a $V \in \sigma$
with $V \neq \emptyset$ such that $V \subseteq f[U]$;
(ii) preopen [47] iff $f[A] \subseteq PO(Y)$ for all $A \in \tau$;
(iii) p-open [34] iff $f[A] \subseteq PO(Y)$ for all $A \in PO(X)$;
(iv) preirresolute [74] briefly pi iff $f^{-1}[V] \subseteq PO(X)$ for each $V \in PO(Y)$;
(Mashhour et al. [42] termed preirresoluteness as M-precontinuity);
(v) quasi preirresolute [62] briefly qpi iff for each $x \in X$ and for each
$V \in PO(Y,f(x))$ there exists $U \in PO(X,x)$ such that
$f[U] \subseteq pcl_Y(V)$;
(vi) semi-preclosed [67] if $f[F]$ is semi-preclosed in $Y$, for each closed
$F$ in $X$. 

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Definition 0.10 [32]. Let \( f : (X, \tau) \to (Y, \sigma) \) be any function. Then the subset \( G(f) = \{(x, f(x)) : x \in X\} \) of the product space \((X \times Y, \tau \times \sigma)\) is called the graph of \( f \).

Definition 0.11 [32]. Let \( X \) and \( Y \) be topological spaces. A function \( f : (X, \tau) \to (Y, \sigma) \) is said to have the closed graph if its graph \( G(f) \) is closed in the product space \((X \times Y, \tau \times \sigma)\).

Definition 0.12. Let \( X \) and \( Y \) be topological spaces. A function \( f : (X, \tau) \to (Y, \sigma) \) is said to have the semi-closed [19] graph if for each \((x, y) \in X \times Y - G(f)\), there exist \( U \in \text{SO}(X, x)\), \( V \in \text{SO}(Y, y)\) such that
\[
[U \times V] \cap G(f) = \phi.
\]

Definition 0.13. Let \( X \) and \( Y \) be topological spaces. A function \( f : (X, \tau) \to (Y, \sigma) \) is said to have the strongly closed [43] (resp. strongly semi-closed [20]) graph if for each \((x, y) \in X \times Y - G(f)\), there exist \( U \in \Sigma(x)\), \( V \in \Sigma(y)\) (resp. \( U \in \text{SO}(X, x)\), \( V \in \text{SO}(Y, y)\)) such that
\[
[U \times \text{Cl}(V)] \cap G(f) = \phi \quad \text{(resp. } [U \times \text{scl}(V)] \cap G(f) = \phi).\]

Definition 0.14. A function \( f : X \to Y \) is said to have a strongly p-\( \theta \)-closed [66] (resp. pre-\( \theta \)-closed [63] ) graph if the graph \( G(f) \) is strongly p-\( \theta \)-closed (resp. pre-\( \theta \)-closed) in \( X \times Y \).

Definition 0.15 [30]. Let \( f : X \to Y, x \in X \). The cluster set of \( f \) at \( x \), denoted by \( C(f; x) \), is defined as the set of all points \( y \) in \( Y \) such that there exists a net \( \langle x_\alpha \rangle_{\alpha \in D} \) in \( X \) with \( x_\alpha \to x \) and \( f(x_\alpha) \to y \).

Definition 0.16 [50]. Let \( X \) be a topological space and \( \langle x_\alpha \rangle_{\alpha \in D} \) be a net in \( X \). Then \( \langle x_\alpha \rangle_{\alpha \in D} \) is said to be preconvergent to a point \( x \in X \), denoted by \( x_\alpha \rightharpoonup x \) iff \( \langle x_\alpha \rangle_{\alpha \in D} \) is eventually in every \( V \in \text{PO}(X, x) \).
**Definition 0.17** [66]. A space \((X, \tau)\) will be said to have the property P if the closure is preserved under finite intersection or equivalently, the closure of intersection of any two subsets equals the intersection of their closures.

**Definition 0.18.** Let \((X, \tau)\) be a topological space.

(i) \(A \subseteq X\) is called quasi-H-closed relative to \(X\) [72] iff for every cover \(\{V_\alpha : \alpha \in \Lambda, V_\alpha \in \tau\}\) of \(A\) there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that

\[A \subseteq \bigcup \{\text{Cl}_X (V_\alpha) : \alpha \in \Lambda_0\}\].

(ii) Two subsets \(A\) and \(B\) of \(X\) are called preseparated [82] iff

\[A \cap \text{pcl} (B) = \text{pcl} (A) \cap B = \emptyset\].

**Definition 0.19.** \(X\) is said to be

(i) preconnected [61] if it is not the union of two preseparated sets;

(ii) locally P-connected [82] if for every point \(x \in X\) and every \(O \in \Sigma (x)\)

there exists an open preconnected set \(G\) such that \(x \in G \subseteq O\);

(iii) precompact [48] if every p.o. cover of \(X\) admits a finite subcover (Mashhour et al. [48] defined this space as strongly compact);

(iv) strongly P-closed [62] if every p.o. cover of \(X\) has a finite subfamily such that the union of their preclosures covers \(X\).

**Definition 0.20.** \(X\) is called

(i) Urysohn [85] (resp. pre-Urysohn [66]) iff for every pair of points \(x, y \in X\) such that \(x \neq y\) there exist \(U \in \Sigma (x), V \in \Sigma (y)\) (resp. \(U \in \text{PO} (X, x), V \in \text{PO} (X, y)\) such that \(\text{Cl} (U) \cap \text{Cl} (V) = \emptyset\)

(resp. \(\text{pcl} (U) \cap \text{pcl} (V) = \emptyset\));

(ii) pre-\(T_1\) [36] (resp. sp-\(T_1\) [29]) iff for every pair of points \(x, y \in X\) such that \(x \neq y\), there exists a \(U \in \text{PO} (X, x)\) (resp. \(U \in \text{SPO} (X, x)\)) not containing \(y\) and a \(V \in \text{PO}(X, y)\) (resp. \(V \in \text{SPO}(X, y)\)) not containing \(x\);
(iii) pre-T$_2$ [36] (resp. sp-T$_2$ [29]) iff for every pair of points $x, y \in X$ such that $x \neq y$, there exist $U \in \text{PO} (X, x)$, $V \in \text{PO} (X, y)$ (resp. $U \in \text{SPO} (X, x)$, $V \in \text{SPO} (X, y)$) such that $U \cap V = \emptyset$;

(iv) pre-regular [63] if for each $F \in \Sigma (X)$ and each $x \not\in F$ there exist disjoint p.o. sets $U, V$ such that $x \in U$ and $F \subseteq V$;

(v) sp-regular [67] if for each closed set $F$ of $X$ and each $x \not\in F$ there exist $U, V \in \text{SPO} (X)$ such that $F \subseteq U$, $x \in V$;

(vi) a R$_0$-space [76] iff for each $G \in \Sigma (x)$ implies $\text{Cl} (\{x\}) \subseteq G$;

(vii) a R$_1$-space [17] iff for $x, y \in X$ such that $\text{Cl} (\{x\}) \neq \text{Cl} (\{y\})$ there exist disjoint open sets $U$ and $V$ such that $\text{Cl} (\{x\}) \subseteq U$ and $\text{Cl} (\{y\}) \subseteq V$;

(viii) a door space [38] iff for every $A \subseteq X$ either $A \in \tau$ or $A \in \mathcal{F}(\sigma)$;

(ix) a semi-door space [83] iff for every $A \subseteq X$ either $A \in \text{SO}(X)$ or $A \in \text{SC} (X)$;

(x) weakly-R$_0$ [44] iff $\bigcap_{x \in X} \text{Cl} (\{x\}) = \emptyset$.

(xi) submaximal [13] if every dense subset of $X$ is open.

(xii) irreducible [12] if every open subset of $X$ is dense.

**Lemma 0.1** [9]. In a topological space $X$ for any $A \subseteq X$, $\text{pcl} (A) \subseteq \text{Cl} (A)$.

**Lemma 0.2** [66]. In a topological space $X$ if $A \in \text{PO} (X)$, $B \in \alpha (X)$, then $A \cap B \in \text{PO} (X)$.

**Lemma 0.3** [29]. In a topological space $X$ if

$A \in \text{SPO} (X)$, $B \in \alpha (X)$, then $A \cap B \in \text{SPO} (X)$.

**Lemma 0.4** [63]. If $A \subseteq Y \subseteq X$ and $Y \in \alpha (X)$, then $\text{pcl}_Y (A) = \text{pcl}_X (A) \cap Y$.

**Lemma 0.5** [1]. Let $A$ be a subset of $X$, then
(i) $\text{spint} (A) = A \cap \text{Cl} (\text{Int} (\text{Cl} (A)))$;

(ii) $\text{spcl} (A) = A \cup \text{Int} (\text{Cl} (\text{Int} (A)))$.

**Lemma 0.6** [36]. Let $A \subseteq Y \subseteq X$ and $Y \in \text{PO} (X)$, then

\[ A \in \text{PO} (X) \iff A \in \text{PO} (Y). \]

**Lemma 0.7** [29]. Let $A \subseteq Y \subseteq X$ and $Y \in \text{PO} (X)$, then

\[ A \in \text{SPO} (X) \iff A \in \text{SPO} (Y). \]

**Lemma 0.8** [47]. If $A \in \text{PO} (X)$ and $B \in \text{PO} (Y)$, then $A \times B \in \text{PO} (X \times Y)$.

**Lemma 0.9** [29]. If $A \in \text{SPO} (X)$ and $B \in \text{SPO} (Y)$ then, $A \times B \in \text{SPO} (X \times Y)$.

**Lemma 0.10** [29]. Every one pointic set in a topological space is either semi-preclosed or open.

**Lemma 0.11.** A function $f : (X, \tau) \to (Y, \sigma)$ has a closed [42] (resp. semi-closed [19]) graph if for each $(x, y) \in X \times Y - G(f)$ there exist $U \in \Sigma (x)$, $V \in \Sigma (y)$ (resp. $U \in \text{SO} (X, x)$, $V \in \text{SO} (Y, y)$) such that $f [U] \cap V$ (resp. $f [U] \cap \text{Cl} (V)$) $= \phi$.

**Lemma 0.12.** A function $f : (X, \tau) \to (Y, \sigma)$ has a strongly closed [43] (resp. strongly semi-closed [20]) graph if for each $(x, y) \in X \times Y - G(f)$, there exist $U \in \Sigma (x)$, $V \in \Sigma (y)$ (resp. $U \in \text{SO} (X, x)$, $V \in \text{SO} (Y, y)$) such that $f [U] \cap \text{Cl} (V)$ (resp. $f [U] \cap \text{scl} (V)$) $= \phi$.

**Lemma 0.13** [1]. Let $A$ be a subset of $X$, then $\text{pint} (A) = A \cap \text{Int} (\text{Cl} (A))$.

**Theorem 0.1** [48]. Every pc subset of a strongly compact space is strongly compact.

**Theorem 0.2** [67]. A topological space $X$ is sp-regular iff for each $x \in X$ and $U \in \Sigma (x)$ there exists $V \in \text{SPO} (X)$ such that

\[ x \in V \subseteq \text{spcl} (V) \subseteq U. \]

**Theorem 0.3** [29]. A topological space $X$ is sp-$T_1$ iff every one pointic set is semi-preclosed.