CHAPTER V
MORE ON SEPARATION AXIOMS

Chapter V is the continuation of our study in separation axioms. Chapter IV has been devoted to the study of only one type of separation axiom viz, sp-Urysohn spaces. But in this chapter we turn our attention to four other types of separation axioms. These separation axioms are rich with properties and deeply related with other spaces in the literature. Prompted by these convictions we turn our attention to the study of the following separation axioms which are termed as

(i) Semi-pre R₀ (briefly sp-R₀) space,
(ii) Semi-pre R₁ (briefly sp-R₁) space,
(iii) Weakly semi-pre R₀ (briefly wsp-R₀) space,
(iv) Semi-pre door (briefly sp-door) space.

This chapter consists of two parts. Part I is concerned with the study of (i) – (iii) separation axioms mentioned above. Part-II deals with a brief investigation into sp-door spaces.

PART-I

sp-R₀ and sp-R₁ spaces

We begin this part with the following definition:

**Definition 5.1.** Let \((X, \tau)\) be a topological space and \(A \subset X\). Then the semi-pre Kernel of \(A\) (briefly sp-Ker \((A)\)) is defined to be the set

\[
\text{sp-Ker} \((A) = \bigcap \{ U : U \in \text{SPO} \((X), A \subset U\}.\]

**Remark 5.1.** Ghosh [29] defined semi-pre Kernel of a point \(x \in X\) as

\[
\text{sp-Ker} \((\{x\}) = \bigcap \{ y \in X : x \in \text{spcl} \((\{y\})\}.\]
The following lemma is useful for the study of sp-R₀ spaces. We state and prove it first.

**Lemma 5.1.** Let X be a topological space and x, y ∈ X. Then

\[ y \in \text{sp-Ker} \left( \{x\} \right) \iff \text{spcl} \left( \{y\} \right). \]

**Proof. Necessity.** Let \( y \in \text{sp-Ker} \left( \{x\} \right) \)

\[ \Rightarrow y \in \cap \{U : U \in \text{SPO} \left( X \right), \{x\} \subset U\} \]

\[ \Rightarrow \{y\} \cap U \neq \emptyset \text{ for every } U \in \text{SPO} \left( X, x \right). \]

Therefore, by lemma 4.1,

\[ x \in \text{spcl} \left( \{y\} \right). \]

**Sufficiency.** Suppose

\[ x \in \text{spcl} \left( \{y\} \right) \]

\[ \Rightarrow \{y\} \cap U \neq \emptyset \text{ for every } U \in \text{SPO} \left( X, x \right) \]

\[ \Rightarrow y \in \cap \{U : U \in \text{SPO} \left( X \right), \{x\} \subset U\} \]

\[ \Rightarrow y \in \text{sp-Ker} \left( \{x\} \right). \]

**Remark 5.1.** From the above lemma, one now observes that for a one pointic set, Ghosh’s definition coincide with Definition 5.1.

Next we prove a fundamental result on sp-Kernel of a set in the form of following lemma which is interesting in its own right.

**Lemma 5.2.** Let X be a topological space and \( A \subset X \). Then

\[ \text{sp-Ker} \left( A \right) = \{x \in X : \text{spcl} \left( \{x\} \right) \cap A \neq \emptyset \}. \]

**Proof.** Let \( x \in \text{sp-Ker} \left( A \right) \) and suppose

\[ \text{spcl} \left( \{x\} \right) \cap A = \emptyset. \]

Clearly

\[ \{x\} \in \text{spcl} \left( \{x\} \right) \]

\[ \Rightarrow x \notin X - \text{spcl} \left( \{x\} \right). \]

We observe that
A ⊂ X − spcl ({x}) ∈ SPO (X),

but X − spcl ({x}) does not contain x. This contradicts our assumption that x ∈ sp-Ker (A).

Thus

\[ \text{spcl} ({x}) \cap A \neq \emptyset \]
\[ \Rightarrow \text{sp-Ker} (A) \subset \{ x \in X : \text{spcl} ({x}) \cap A \neq \emptyset \}. \]

To prove the reverse inclusion let x ∈ X be such that

\[ \text{spcl} ({x}) \cap A \neq \emptyset. \]

If possible let

\[ x \notin \text{sp-Ker} (A). \]

Then there exists a \( U_0 \in SPO (X) \) with \( A \subset U \) such that

\[ x \notin U_0. \]

Again since \( \text{spcl} ({x}) \cap A \neq \emptyset \) there exists \( y \in X \) such that

\[ y \in \text{sp-Ker} ({x}) \cap A. \]

Now \( y \in \text{spcl} ({x}) \) implies by, Lemma 5.1, that

\[ x \in \text{sp-Ker} ({y}) \]
\[ \Rightarrow x \in \cap \{ U : U \in SPO (X), \{ y \} \subset U \} \]
\[ \subset \cap \{ U : U \in SPO (X), A \subset U \}, \text{as } y \in A \]
\[ \Rightarrow \text{a contradiction to the above assertion that } x \notin U_0 \]
\[ \Rightarrow x \in \text{sp-Ker} (A), \]

whence

\[ \{ x \in X : \text{spcl} ({x}) \cap A \neq \emptyset \} \subset \text{sp-Ker} (A). \]

Hence \( \text{sp-Ker} (A) = \{ x \in X : \text{spcl} ({x}) \cap A \neq \emptyset \}. \)

**SEMI-PRE R₀ SPACES**

Equipped with all these results we are now in a position to introduce a

\( \sigma \)

new separation axiom termed \( \text{semi-pre R₀} \) given below.
Definition 5.2. A topological space \((X, \tau)\) is said to be a semi-pre \(R_0\) (briefly sp-\(R_0\)) space if
\[
\text{spcl}\{x\} \subseteq U \text{ for every } U \in \text{SPO}(X, x).
\]
A useful characterisation of sp-\(R_0\) space is the following theorem.

Theorem 5.1. A topological space \((X, \tau)\) is sp-\(R_0\) iff for each \(U \in \text{SPO}(X, x)\)
\[
\text{Int}(\text{Cl}(\text{Int}\{x\})) \subseteq U.
\]

Proof. Let \(U \in \text{SPO}(X, x)\). The sp-\(R_0\)-ness of \(X\) produces
\[
\text{spcl}\{x\} \subseteq U.
\]
Then Lemma 0.5 yields
\[
\{x\} \cup \text{Int}(\text{Cl}(\text{Int}\{x\})) \subseteq U
\]
\[
\Rightarrow \text{Int}(\text{Cl}(\text{Int}\{x\})) \subseteq U.
\]
Conversely, let the given condition hold. Also let \(x \in X\) and \(U \in \text{SPO}(X, x)\).
Then
\[
\text{Int}(\text{Cl}(\text{Int}\{x\})) \subseteq U
\]
\[
\Rightarrow \{x\} \cup \text{Int}(\text{Cl}(\text{Int}\{x\})) \subseteq U.
\]
Again, by Lemma 0.5, one obtains
\[
\text{spcl}\{x\} \subseteq U.
\]

Some Basic Properties of sp-\(R_0\) Spaces

Our next theorem establishes the relation between a sp-\(R_0\) space and a sp-\(T_1\) space.

Theorem 5.2. A topological space is sp-\(R_0\) iff it is sp-\(T_1\).

Proof. Let \(X\) be a sp-\(R_0\) space and \(x \in X\). Then, by Lemma 0.10,
\[
\{x\} \in \text{SPO}(X) \text{ or } \{x\} \in \text{SPF}(X).
\]
If \(\{x\} \in \text{SPO}(X)\) then \(\text{spcl}\{x\} \subseteq \{x\}\)
\[
\Rightarrow \text{spcl}\{x\} = \{x\}
\]
\[
\Rightarrow \{x\} \in \text{SPF}(X).
\]
Thus every one pointic set in $X$ is semi preclosed. Hence by Theorem 0.3, $X$ is sp-$T_1$.

Conversely, let $X$ be sp-$T_1$ and $U \in \text{SPO}(X, x)$. The sp-$T_1$-ness of $X$ gives

$$\{x\} \in \text{SPF}(X)$$

whence one deduces

$$\text{spcl}(\{x\}) \subset \{x\} \subset U$$

$$\Rightarrow X \text{ is sp-}R_0.$$

**Theorem 5.3.** For a topological space $(X, \tau)$ the following statements are equivalent:

(i) $(X, \tau)$ is sp-$R_0$;

(ii) For any $F \in \text{SPF}(X)$, $x \notin F$ there exists a $U \in \text{SPO}(X)$ such that $x \notin U$ and $F \subset U$;

(iii) If $F \in \text{SPF}(X)$ and $x \notin F$ then $F \cap \text{spcl}(\{x\}) = \emptyset$;

(iv) For any two points $x$ and $y$ of $X$

$$\text{spcl}(\{x\}) \neq \text{spcl}(\{y\}) \Rightarrow \text{spcl}(\{x\}) \cap \text{spcl}(\{y\}) = \emptyset.$$

**Proof.** (i) $\Rightarrow$ (ii). Let $(X, \tau)$ be sp-$R_0$, $F \in \text{SPF}(X)$ and $x \notin F$. Then

$$X - F \in \text{SPO}(X, x)$$

$$\Rightarrow \text{spcl}(\{x\}) \subset X - F$$

$$\Rightarrow X - \text{spcl}(\{x\}) \supset F.$$ 

Set

$$U = X - \text{spcl}(\{x\}).$$

Clearly

$$U \in \text{SPO}(X), x \notin U \text{ and } F \subset U.$$

(ii) $\Rightarrow$ (iii). Let $F \in \text{SPF}(X), x \notin F$. By (ii) there exists $U \in \text{SPO}(X)$ such that $F \subset U$ and $x \notin U$. 

Now
\[ U \cap \{x\} = \emptyset \]
\[ \Rightarrow U \cap \text{spcl} (\{x\}) = \emptyset \]
\[ \Rightarrow \mathcal{F} \cap \text{spcl} (\{x\}) = \emptyset. \]

(iii) \(\Rightarrow\) (iv). Suppose
\[ \text{spcl} (\{x\}) \neq \text{spcl} (\{y\}) \text{ for } x, y \in X. \]
Obviously, \(x \neq y\). Now distinct semi-pre-closures indicates that there exists a
\[ z \in \text{spcl} (\{x\}) \text{ but } z \notin \text{spcl} (\{y\}) \]
or
\[ z \notin \text{spcl} (\{x\}) \text{ but } z \in \text{spcl} (\{y\}). \]
To fix our ideas let
\[ z \in \text{spcl} (\{x\}) \text{ and } z \notin \text{spcl} (\{y\}). \]
Since \(z \notin \text{spcl} (\{y\})\) there exists a \(V \in \text{SPO} (X, z)\) such that
\[ \{y\} \cap V = \emptyset \Rightarrow y \notin V. \]
Again \(z \in \text{spcl} (\{x\})\) yields that \(x \in V\). Therefore,
\[ V \in \text{SPO} (X, x) \text{ and hence } x \notin \text{spcl} (\{y\}). \]
If we write \(F = \text{spcl} (\{y\})\), then \(x \notin F\).
Now using (iii) we deduce
\[ \text{spcl} (\{x\}) \cap \text{spcl} (\{y\}) = \emptyset. \]
(iv) \(\Rightarrow\) (i). Suppose (iv) holds.
Let \(x \in V \in \text{SPO} (X)\). Also let \(y \notin V\),
This, then, assures that \(x \neq y\) and \(x \notin \text{spcl} (\{y\})\).
Thus
\[ \text{spcl} (\{x\}) \neq \text{spcl} (\{y\}). \]
By (iv),
\[ \text{spcl} (\{x\}) \cap \text{spcl} (\{y\}) = \emptyset \text{ for each } y \in X - V. \]
Let y run over $X - V$. Then from above
\[ \text{spcl}\{\{x\}\} \cap \bigcup_{y \in X - V} \text{spcl}\{\{y\}\} = \emptyset. \]

Since $X - V$ is a semi-preclosed set containing \{y\} by Lemma 4.1
\[ \text{spcl}\{\{y\}\} \subseteq X - V \]
\[ \Rightarrow \bigcup_{y \in X - V} \text{spcl}\{\{y\}\} \subseteq X - V \quad \forall y \in X - V \]
\[ \Rightarrow X - V \subseteq \bigcup_{y \in X - V} \text{spcl}\{\{y\}\} \subseteq X - V \]
\[ \Rightarrow X - V = \bigcup_{y \in X - V} \text{spcl}\{\{y\}\}. \]

So from above
\[ \text{spcl}\{\{x\}\} \cap (X - V) = \emptyset \]
\[ \Rightarrow \text{spcl}\{\{x\}\} \subseteq V. \]

Hence $(X, \tau)$ is sp-$R_0$.

**RELATION BETWEEN sp-KERNEL AND SEMI-PRECLOSURE**

Before going to reveal the next property of sp-$R_0$ space we establish an interesting relation between sp-Kernel and semi-preclosure of an one pointic set. This is embodied in the next Lemma.

**Lemma 5.3.** Let $X$ be a topological space. Then for any two points $x, y \in X$
\[ \text{sp-Ker}\{\{x\}\} \neq \text{sp-Ker}\{\{y\}\} \iff \text{spcl}\{\{x\}\} \neq \text{spcl}\{\{y\}\}. \]

**Proof.** Suppose
\[ \text{sp-Ker}\{\{x\}\} \neq \text{sp-Ker}\{\{y\}\}. \]

This then guarantees the existence of a point $z$ such that either
\[ z \in \text{sp-Ker}\{\{x\}\} \text{ but } z \notin \text{sp-Ker}\{\{y\}\} \]

or
\[ z \notin \text{sp-Ker}\{\{x\}\} \text{ but } z \in \text{sp-Ker}\{\{y\}\}. \]

To fix our ideas let $z \in \text{sp-Ker}\{\{x\}\}$.

Since $z \in \text{sp-Ker}\{\{x\}\}$, by Lemma 5.2, one obtains
\{x\} \cap \text{spcl}(\{z\}) \neq \emptyset
\Rightarrow x \in \text{spcl}(\{z\}).

Again the fact that \( z \notin \text{sp-Ker}(\{y\}) \) ensures the emptiness of the set \( \{y\} \cap \text{spcl}(\{z\}) \). Now
\[ x \in \text{spcl}(\{z\}) \Rightarrow \text{spcl}(\{x\}) \subset \text{spcl}(\{z\}). \]
Thus from the foregoing
\[ \{y\} \cap \text{spcl}(\{x\}) = \emptyset \]
\[ \Rightarrow \text{spcl}(\{x\}) \neq \text{spcl}(\{y\}). \]

Next assume that
\[ \text{spcl}(\{x\}) \neq \text{spcl}(\{y\}). \]
Thus there exists a point \( z \in X \) such that
\[ z \in \text{spcl}(\{x\}) \text{ but } z \notin \text{spcl}(\{y\}). \]

By Lemma 4.1, of the thesis there exists a \( U \in \text{SPO}(X, z) \) such that
\[ U \cap \{y\} = \emptyset. \]

Now \( z \in \text{spcl}(\{x\}) \)
\[ \Rightarrow \{x\} \cap U \neq \emptyset \]
\[ \Rightarrow x \in U. \]
Therefore from above
\[ x \notin \text{spcl}(\{y\}) \]
\[ \Rightarrow y \notin \text{sp-Ker}(\{x\}). \]
Hence
\[ \Rightarrow \text{sp-Ker}(\{x\}) \neq \text{sp-Ker}(\{y\}). \]

Before we take up the next property of \text{sp-Ker} we consider the following example.

\textbf{Example 5.1.} Let \( X = \{a, b, c\} \) be the set with the topology
\[ \tau = \{\emptyset, X, \{a\}\}. \]
Then
SPO (X) = {ϕ, X, {a}, {a, b}, {a, c}}.

Clearly X is not sp-R₀. For a, b ∈ X we note that

sp-Ker ({a}) = {a} and sp-Ker ({b}) = {a, b},
i.e. sp-Ker ({a}) ∩ sp-Ker ({b}) ≠ ϕ.

Thus in an arbitrary topological space sp-Ker of any two points may not be disjoint. But if the space is sp-R₀ then the disjointness of sp-Ker of any two points characterises sp-R₀ space. In fact we have the following:

**CHARACTERISATION OF sp-R₀ SPACE**

**Theorem 5.4.** A topological space (X, τ) is sp-R₀ iff x, y ∈ X and

sp-Ker ({x}) ≠ sp-Ker ({y}) ⇒ sp-Ker ({x}) ∩ sp-Ker ({y}) = ϕ.

**Proof. Necessity.** Let X be sp-R₀ and x, y ∈ X be such that

sp-Ker ({x}) ≠ sp-Ker ({y}).

Lemma 5.3 yields

spcl ({x}) ≠ spcl ({y}). … (1)

If possible suppose

sp-Ker ({x}) ∩ sp-Ker ({y}) ≠ ϕ.

Let

z ∈ sp-Ker ({x}) ∩ sp-Ker ({y}).

Since z ∈ sp-Ker ({x}) by Lemma 5.1, x ∈ spcl ({z}).

This gives

⇒ spcl ({x}) ∩ spcl ({z}) ≠ ϕ.

The sp-R₀-ness of X and Theorem 5.3 together indicate that

spcl ({x}) = spcl ({z}).

Pursuing the same reasoning

spcl ({x}) = spcl ({z}) = spcl ({y})

which contradicts (1).

Hence sp-Ker ({x}) ∩ sp-Ker ({y}) = ϕ.
Sufficiency. Let \((X, \tau)\) be a topological space such that the given condition holds. Let \(x, y \in X\) be such that
\[
\text{spcl} \{\{x\}\} \neq \text{spcl} \{\{y\}\}.
\]
Then by Lemma 5.3
\[
\text{sp-Ker} \{\{x\}\} \neq \text{sp-Ker} \{\{y\}\}.
\]
Therefore, by hypothesis
\[
\text{sp-Ker} \{\{x\}\} \cap \text{sp-Ker} \{\{y\}\} = \emptyset.
\]
We assert that
\[
\text{spcl} \{\{x\}\} \cap \text{spcl} \{\{y\}\} = \emptyset.
\]
If possible suppose
\[
\text{spcl} \{\{x\}\} \cap \text{spcl} \{\{y\}\} \neq \emptyset.
\]
This indicates that there exists a
\[
z \in \text{spcl} \{\{x\}\} \cap \text{spcl} \{\{y\}\} \quad \Rightarrow z \in \text{spcl} \{\{x\}\} \text{ and } z \in \text{spcl} \{\{y\}\}.
\]
This then yields that
\[
x \in \text{sp-Ker} \{\{z\}\} \text{ and also } y \in \text{sp-Ker} \{\{z\}\}.
\]
Therefore
\[
\text{sp-Ker} \{\{x\}\} \cap \text{sp-Ker} \{\{z\}\} \neq \emptyset
\]
and
\[
\text{sp-Ker} \{\{y\}\} \cap \text{sp-Ker} \{\{z\}\} \neq \emptyset.
\]
Then by hypothesis
\[
\text{sp-Ker} \{\{x\}\} = \text{sp-Ker} \{\{z\}\} = \text{sp-Ker} \{\{y\}\} \quad \Rightarrow \text{a contradiction to the foregoing.}
\]
Therefore the equality
\[
\text{spcl} \{\{x\}\} \cap \text{spcl} \{\{y\}\} = \emptyset
\]
is established. Hence by Theorem 5.3, \((X, \tau)\) is \(\text{sp-R}_0\).

The next theorem offers another set of characterisation of \(\text{sp-R}_0\) space.
Theorem 5.5. For a topological space \((X, \tau)\) the following statements are equivalent:

(i) \((X, \tau)\) is sp-R_0 space;

(ii) For any non-empty set \(A\) and any \(G \in SPO(X)\) with \(A \cap G \neq \emptyset\), there exists a \(F \in SPF(X)\) such that

\[
A \cap F \neq \emptyset \text{ and } F \subseteq G;
\]

(iii) \(G \in SPO(X) \implies G = \bigcup \{F : F \in SPF(X), F \subseteq G\}\);

(iv) \(F \in SPF(X) \implies F = \bigcap \{G : G \in SPO(X), F \subseteq G\}\);

(v) \(x \in X \implies \text{spcl} (\{x\}) \subseteq \text{sp-Ker} (\{x\})\).

Proof. (i) \implies (ii). Let (i) hold. Suppose

\[
A \neq \emptyset, \ G \in SPO(X) \text{ and } A \cap G \neq \emptyset.
\]

Let \(x \in A \cap G\). Put \(F = \text{spcl} (\{x\})\). Then

\[
F \in SPF(X, x).
\]

Since \(X\) is sp-R_0

\[
x \in G \in SPO(X)
\]

\[
\implies \text{spcl} (\{x\}) \subseteq G
\]

\[
\implies F \subseteq G.
\]

Finally \(x \in F\) and \(x \in A\) ensures that \(A \cap F \neq \emptyset\).

(ii) \implies (iii). Assume \(G \in SPO(X)\). Clearly

\[
\bigcup \{F : F \in SPF(X), F \subseteq G\} \subseteq G.
\]

To prove the reverse inclusion suppose \(x \in G\).

By hypothesis there exists a \(F \in SPF(X)\) such that

\[
x \in F \text{ and } F \subseteq G,
\]

which, in its turn, imply that

\[
x \in F \subseteq \bigcup \{F : F \in SPF(X), F \subseteq G\}.
\]
Hence
\[ G = \bigcup \{ F : F \in SPF(X), F \subseteq G \}. \] ...(1)

(iii) \Rightarrow (iv). Assume that (iii) holds. An application of De Morgen's Law to (1) yields the desired result.

(iv) \Rightarrow (v). Let \( x, y \in X \) with the property
\[ y \notin sp-Ker(\{x\}). \]
Then there exists a \( U \in SPO(X, x) \) such that \( y \notin U \).
Hence
\[ \{y\} \cap U = \emptyset \]
whence
\[ spcl(\{y\}) \cap U = \emptyset. \]
Now, by (iv), we have
\[ spcl(\{y\}) = \bigcap \{ G : G \in SPO(X), spcl(\{y\}) \subseteq G \}. \]
So, from above
\[ \bigcap \{ G : G \in SPO(X), spcl(\{y\}) \subseteq G \} \cap U = \emptyset. \]
This, then, ensures the existence of a \( G \in SPO(X) \) such that
\[ spcl(\{y\}) \subseteq G \text{ and } x \notin G. \]
So
\[ \{x\} \cap G = \emptyset \]
whence
\[ spcl(\{x\}) \cap G = \emptyset. \]
This gives
\[ y \notin spcl(\{x\}). \]
Consequently
\[ spcl(\{x\}) \subseteq sp-Ker(\{x\}). \]
(v) \Rightarrow (ii). Let \( G \in SPO(X) \). By definition
\[ sp-Ker(\{x\}) \subseteq G. \]
Also by (v)
\[ \text{spcl} \left( \{x\} \right) \subseteq \text{sp-Ker} \left( \{x\} \right). \]

So,
\[ x \in \text{spcl} \left( \{x\} \right) \subseteq \text{sp-Ker} \left( \{x\} \right) \subseteq G \]
\[ \Rightarrow \text{spcl} \left( \{x\} \right) \subseteq G. \]

Thus \( x \in G \Rightarrow \text{spcl} \left( \{x\} \right) \subseteq G \)
\[ \Rightarrow X \text{ is sp-R}_0. \]

The following corollary is interesting in its own right and a very useful tool to characterise sp-R_0 space.

**Corollary 5.1.** \((X, \tau)\) is a sp-R_0 space iff
\[ \text{spcl} \left( \{x\} \right) = \text{sp-Ker} \left( \{x\} \right) \text{ for all } x \in X. \]

**Proof. Necessity.** Let \((X, \tau)\) be sp-R_0 and \( y \in \text{sp-Ker} \left( \{x\} \right) \).

Then Lemma 5.1 gives
\[ x \in \text{spcl} \left( \{y\} \right). \]

Hence
\[ \text{spcl} \left( \{x\} \right) \cap \text{spcl} \left( \{y\} \right) \neq \phi. \]

Therefore, by Theorem 5.3,
\[ \text{spcl} \left( \{x\} \right) = \text{spcl} \left( \{y\} \right) \]
from which one concludes
\[ y \in \text{spcl} \left( \{x\} \right) \]
\[ \Rightarrow \text{sp-Ker} \left( \{y\} \right) \subseteq \text{spcl} \left( \{x\} \right). \]

The reverse inclusion because of sp-R_0-ness, directly follows from Theorem 5.5 (v).

Hence
\[ \text{spcl} \left( \{x\} \right) = \text{sp-Ker} \left( \{y\} \right). \]
Sufficiency. Suppose \( \text{spcl} \{\{x\}\} = \text{sp-Ker} \{\{y\}\} \). Then that \( X \) is sp-\( R_0 \) follows readily from (v) of Theorem 5.5.

We shall now draw an end to our present discussion on sp-\( R_0 \) space after proving Theorem 5.6.

**Theorem 5.6.** For a topological space \( X \), the following statements are equivalent:

(i) \( X \) is sp-\( R_0 \);

(ii) \( F \in \text{SPF} (X) \) then \( F = \text{sp-Ker} (F) \);

(iii) \( F \in \text{SPF} (X) \) and \( x \in F \), then

\[
\text{sp-Ker} \{\{x\}\} \subseteq F;
\]

(iv) If \( x \in X \), then \( \text{sp-Ker} \{\{x\}\} \subseteq \text{spcl} \{\{x\}\} \).

**Proof.** (i) \( \Rightarrow \) (ii). This follows from Theorem 5.5 (iv) and the definition of sp-Ker

(ii) \( \Rightarrow \) (iii). Let \( F \in \text{SPF} (X) \) and \( x \in F \). Clearly

\[
\{x\} \subseteq F
\]

\[
\Rightarrow \text{sp-Ker} \{\{x\}\} \subseteq \text{sp-Ker} (F) = F.
\]

(iii) \( \Rightarrow \) (iv). Suppose \( x \in X \). Obviously

\[
\text{spcl} \{\{x\}\} \in \text{SPF} (X) \text{ and } x \in \text{spcl} \{\{x\}\}.
\]

Therefore by (iii)

\[
\text{sp-Ker} \{\{x\}\} \subseteq \text{spcl} \{\{x\}\}.
\]

(iv) \( \Rightarrow \) (i). Let \( x \in \text{spcl} \{\{y\}\} \). Then, by Lemma 5.1,

\[
y \in \text{sp-Ker} \{\{x\}\}.
\]

By hypothesis

\[
\text{sp-Ker} \{\{x\}\} \subseteq \text{spcl} \{\{x\}\}.
\]

which, in its turn, indicates that
This then implies, by Lemma 5.1, that
\[ x \in \text{sp-Ker } \{y\}. \]
From this one infers that
\[ \text{spcl } \{y\} \subset \text{sp-Ker } \{y\}. \]
Hence by Theorem 5.5, \( X \) is sp-R_0.

**SEMI - PRE R_1 SPACES**

We now dwell on some salient features of sp-R_1 spaces. We start with the definition of a sp-R_1 space, which runs as follows:

**Definition 5.3.** A topological space \( X \) is said to be semi-pre-R_1 (briefly sp-R_1) if for every pair of points \( x, y \in X \) with \( \text{spcl } \{x\} \neq \text{spcl } \{y\} \) there exist two disjoint sets \( U \in \text{SPO} (X, x), V \in \text{SPO} (X, y) \) such that
\[ \text{spcl } \{x\} \subset U \text{ and } \text{spcl } \{y\} \subset V. \]

Now the natural query that arises in our mind is: Is there any relation between sp-R_1 space and sp-R_0 space? The next theorem is the outcome of our endeavour to address the above query.

**Theorem 5.7.** Every sp-R_1 space is sp-R_0.

**Proof.** Let \( U \in \text{SPO} (X, x) \) and \( y \notin U \). This gives
\[ x \notin \text{spcl } \{y\} \]
\[ \Rightarrow \text{spcl } \{x\} \neq \text{spcl } \{y\}. \]
Since \( X \) is sp-R_1 there exists \( V \in \text{SPO} (X, y) \) such that
\[ \text{spcl } \{y\} \subset V \text{ and } x \notin V. \]
Thus
\[ y \notin \text{spcl } \{x\}. \]
The non-containment condition regarding \( y \) induces
\[ \text{spcl } \{x\} \subset U. \]
Hence $X$ is sp-$R_0$.

**Theorem 5.8.** A topological space is sp-$R_1$ iff it is sp-$T_2$.

**Proof.** Let $X$ be sp-$R_1$. Theorem 5.7 ensures that $X$ is sp-$R_0$ and hence by Theorem 5.2, $X$ is sp-$T_1$. We assert that $X$ is sp-$T_2$. To this end let $x, y \in X$ with $x \neq y$.

sp-$T_1$-ness of $X$ guarantees by Theorem 0.3 that $\text{spcl}(\{x\}) = \{x\}$ and $\text{spcl}(\{y\}) = \{y\}$.

Thus

$\text{spcl}(\{x\}) \neq \text{spcl}(\{y\})$.

Therefore sp-$R_1$-ness of $X$ provides two disjoint s.p.o. sets $U$ and $V$ such that $x \in U$ and $y \in V$.

Hence $X$ is sp-$T_2$.

In the Chapter IV of the thesis we define semi-pre-$\theta$-closure of a set. Using that notion, we now obtain a necessary and sufficient condition for sp-$R_1$-ness of an arbitrary topological space. The following two lemmas will be used in the proof of the above theorem.

**Lemma 5.4.** For any subset $A$ of a topological space $\text{spcl}(A) \subseteq \text{spcl}_0(A)$.

Proof is straightforward and is omitted.

**Lemma 5.5.** Let $(X, \tau)$ be a topological space and $x, y \in X$. Then $y \in \text{spcl}_0(\{x\})$ iff $x \in \text{spcl}_0(\{y\})$.

**Proof.** Let $y \in \text{spcl}_0(\{x\})$.

If possible suppose $x \notin \text{spcl}_0(\{y\})$.

This guarantees the existence of a $U \in \text{SPO}(X, x)$
such that
\[ \text{spcl}(U) \cap \{y\} = \emptyset \]

\[ \Rightarrow y \not\in \text{spcl}(U). \]

By Lemma 4.1, there exists a \( V \in \text{PO}(X, y) \) such that
\[ V \cap U = \emptyset \]

\[ \Rightarrow \text{spcl}(V) \cap U = \emptyset \]

\[ \Rightarrow \text{spcl}(V) \cap \{x\} = \emptyset \]

\[ \Rightarrow y \not\in \text{spcl}_0(\{x\}) \]

\[ \Rightarrow \text{a contradiction}. \]

Thus \( y \in \text{spcl}_0(\{x\}) \Rightarrow x \in \text{spcl}_0(\{y\}). \)

The proof of the converse part follows by pursuing the same argument.

**Theorem 5.9.** A topological space \( X \) is sp-R\(_1\) iff

\[ \text{spcl}(\{x\}) = \text{spcl}_0(\{x\}) \text{ for every } x \in X. \]

**Proof.** Assume \( X \) be sp-R\(_1\). If possible suppose there exists a point \( x \in X \) such that

\[ \text{spcl}(\{x\}) \neq \text{spcl}_0(\{x\}). \]

By Lemma 5.4

\[ \text{spcl}(\{x\}) \subset \text{spcl}_0(\{x\}). \]

This guarantees the existence of a \( y \in X \) such that \( y \in \text{spcl}_0(\{x\}) \) but

\[ y \not\in \text{spcl}(\{x\}). \]

Hence

\[ \text{spcl}(\{x\}) \neq \text{spcl}(\{y\}). \]

Again sp-R\(_1\)-ness of \( X \) provides us

\[ U_1 \in \text{SPO}(X, x), U_2 \in \text{SPO}(X, y) \]

such that

\[ \text{spcl}(\{x\}) \subset U_1, \text{spcl}(\{y\}) \subset U_2 \text{ and } U_1 \cap U_2 = \emptyset. \]
Thus,

$$\{x\} \cap \operatorname{spcl} (U_2) = \emptyset$$

$$\Rightarrow y \notin \operatorname{spcl}_0(\{x\})$$

$$\Rightarrow \text{a contradiction.}$$

Therefore, the foregoing gives

$$\operatorname{spcl}_0(\{x\}) = \operatorname{spcl}(\{x\}).$$

Conversely, suppose that the given condition holds for every $$x \in X$$. We assert that $$X$$ is sp-Ro. To this end let $$x \in X$$ and $$U \in \operatorname{SPO}(X, x)$$. We take $$y \notin U$$. Obviously

$$\operatorname{spcl}_0(\{y\}) = \operatorname{spcl}(\{y\}) \subseteq X - U.$$  

$$\Rightarrow x \notin \operatorname{spcl}_0(\{y\}).$$

So, Lemma 5.5 induces that

$$y \notin \operatorname{spcl}_0(\{x\}).$$

Using Lemma 5.4 one infers that

$$y \notin \operatorname{spcl}(\{x\})$$

$$\Rightarrow \operatorname{spcl}(\{x\}) \subseteq U.$$  

Hence $$X$$ is sp-Ro. Therefore. By Theorem 5.2, $$X$$ is sp-T$_1$. Next let

$$\alpha, \beta \in X$$ with

$$\alpha \neq \beta.$$  

Now sp-T$_1$-ness of $$X$$ indicates that

$$\operatorname{spcl}(\{\alpha\}) = \{\alpha\} \text{ and } \operatorname{spcl}(\{\beta\}) = \{\beta\}.$$  

Clearly

$$\beta \notin \operatorname{spcl}(\{\alpha\}) = \operatorname{spcl}_0(\{\alpha\}).$$

Therefore there exists a $$V \in \operatorname{SPO}(X, \beta)$$ such that

$$\operatorname{spcl}(V) \cap \{\alpha\} = \emptyset$$

$$\Rightarrow \alpha \in X - \operatorname{spcl}(V) \in \operatorname{SPO}(X).$$

Thus for every $$\alpha, \beta \in X$$ with $$\alpha \neq \beta$$
there exist  
\[ X - \text{spcl} (V) \in \text{SPO} (X, \alpha), V \in \text{SPO} (X, \beta) \]
such that \((X - \text{spcl} (V)) \cap V = \emptyset.\]
This indicates that \(X\) is sp-T_2 and hence, by Theorem 5.8, \(X\) is sp-R_1.

Now we turn our attention to the third separation axiom of Part I which we mentioned in the beginning of this chapter.

**WEAKLY SEMI-PRE R_0 SPACES**

Our line of development parallels that of S.P. Arya et al. [4] where they studied R_0 space with the help of semi-open sets of Levine [40] and termed their spaces as "Weakly-semi-R_0" spaces.

We start with the following definition.

**Definition 5.4.** A topological space \(X\) is said to be weakly semi-pre-R_0 (briefly wsp-R_0) iff

\[ \bigcap_{x \in X} \text{spcl} (\{x\}) = \phi. \]

**Remark 5.2.** Obviously every sp-R_0 space is wsp-R_0 but the converse need not be true as the following shows.

**Example 5.2.** Let \(X = \{a, b, c, d\}\) be the set with the topology

\[ \tau = \{ \emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}. \]

Then \(X\) is wsp-R_0 but not sp-R_0.

**Remark 5.3.** Every weakly R^1_0 space is wsp-R_0 follows from the fact that

\[ \bigcap_{x \in X} \text{spcl} (\{x\}) \subseteq \bigcap_{x \in X} \text{Cl} (\{x\}). \]

But the reverse relation does not hold in general which is clear from the following example.

**Example 5.3.** Let \((X, \tau)\) be the space of Example 5.1.

---

1 See Definition 0.20 (x).
Then
\[ \bigcap_{x \in X} \text{Cl} \{\{x\}\} = \{b, c\} \neq \emptyset \]
but
\[ \bigcap_{x \in X} \text{spcl} \{\{x\}\} = \emptyset \]
which shows that \((X, \tau)\) is wsp-R_0 but not weakly R_0.

**Remark 5.4.** Maio [44] showed that a set equipped with the point exclusion topology cannot be weakly R_0. On the other hand, this space may be wsp-R_0 as shown below.

**Example 5.4.** Let \(X = \{a, b, c\}\) be the set with the topology
\[ \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}. \]
Then \((X, \tau)\) is wsp-R_0 but not weakly R_0.

The concept of sp-Kernel of a singleton is a useful tool to examine the nature of a weakly sp-R_0 space as is revealed in the next theorem.

**Theorem 5.10.** A topological space \(X\) is wsp-R_0 iff sp-Ker \{\{x\}\) $\neq X$ for any \(x \in X\).

**Proof. Necessity.** Suppose the theorem is false. Then there exists a \(x_0 \in X\) such that
\[ \text{sp-Ker} \{\{x_0\}\} = X. \]
This yields
\[ \text{sp-Ker} \{\{x_0\}\} = \bigcap \{G : G \in SPO (X, x_0)\} = X, \]
which indicates that \(X\) is the only s.p.o. set containing \(x_0\). This reveals that every semi-preclosed subset of \(X\) contains \(x_0\). Thus
\[ x_0 \in \text{spcl} \{\{x\}\} \text{ for any } x \in X. \]
Therefore
\[ \bigcap_{x \in X} \text{spcl} \{\{x\}\} \neq \emptyset \]
\[ \Rightarrow \text{a contradiction to the hypothesis that } X \text{ is wsp-R}_0. \text{ Hence} \]
Sufficiency. Suppose \( \text{sp-Ker}(\{x\}) \neq X \) for every \( x \in X \). If possible suppose \( X \) is not wsp-\( R_0 \) which means
\[
\bigcap_{x \in X} \text{spcl}(\{x\}) \neq \emptyset.
\]

Then there exists a \( x_0 \in X \) such that
\[
x_0 \in \bigcap_{x \in X} \text{spcl}(\{x\}).
\]

This implies that \( x_0 \in \text{spcl}(\{x\}) \) for every \( x \in X \).

Let \( U \in \text{SPO}(X, x_0) \). Then from above
\[
U \cap \{x\} \neq \emptyset \text{ for every } x \in X
\]
\[
\Rightarrow x \in U \text{ for every } x \in X
\]
\[
\Rightarrow X \subseteq U
\]
\[
\Rightarrow U = X.
\]

This then ensures
\[
\text{sp-Ker}(\{x_0\}) = X
\]
\[
\Rightarrow \text{a contradiction to the assumption}
\]
\[
\Rightarrow X \text{ is wsp-} R_0.
\]

We need the following definition and the lemma to establish the invariance of wsp-\( R_0 \)-ness.

**Definition 5.5.** A mapping \( f: X \to Y \) is called sp-closed iff
\[
f[A] \in \text{SPF}(Y) \text{ for all } A \in \text{SPF}(X).
\]

**Lemma 5.6.** If \( f: X \to Y \) is a sp-closed function then
\[
\text{spcl}_Y(\{f(x)\}) \subseteq f[\text{spcl}_Y(\{x\})] \text{ for every } x \in X.
\]

**Proof.** For any \( x \in X \)
\[
\{x\} \subseteq \text{spcl}_Y(\{x\})
\]
\[
f[\{x\}] \subseteq f[\text{spcl}_X(\{x\})]
\]
This gives

\[ \{ f(x) \} \subset f[\text{spcl}_X \{ \{x\} \}] \]

\[ \Rightarrow \text{spcl}_Y \{ \{ f(x) \} \} \subset \text{spcl}_Y (f[\text{spcl}_X \{ \{x\} \}]) . \]

Since \( f \) is sp-closed

\[ \text{spcl}_Y (f[\text{spcl}_X \{ \{x\} \}]) = f[\text{spcl}_X \{ \{x\} \}] . \]

From above

\[ \text{spcl}_Y \{ \{ f(x) \} \} \subset f[\text{spcl}_X \{ \{x\} \}] . \]

That wsp-R_0-ness remains invariant under a sp-closed injection will be proved now. In fact we have

**Theorem 5.11.** If \( f : X \rightarrow Y \) is an injective sp-closed mapping where \( X \) is wsp-R_0, then \( Y \) is so.

**Proof.** The injectivity of \( f \) produces the following containment relation

\[ \bigcap_{y \in Y} \text{spcl}_Y \{ \{ y \} \} \subset \bigcap_{x \in X} \text{spcl}_Y \{ \{ f(x) \} \} . \]

The sp-closedness of \( f \) gives, by Lemma 5.6,

\[ \text{spcl}_Y \{ \{ f(x) \} \} \subset f[\text{spcl}_X \{ \{x\} \}] . \]

So from above

\[ \bigcap_{y \in Y} \text{spcl}_Y \{ \{ y \} \} \subset \bigcap_{x \in X} f[\text{spcl}_X \{ \{x\} \}] . \]

Again the injectivity of \( f \) yields

\[ \bigcap_{x \in X} f[\text{spcl}_X \{ \{x\} \}] \subset f[\bigcap_{x \in X} \text{spcl}_X \{ \{x\} \}] . \]

Now wsp-R_0-ness of \( X \) gives

\[ \bigcap_{x \in X} \text{spcl}_X \{ \{x\} \} = \emptyset . \]

From the foregoing

\[ \bigcap_{y \in Y} \text{spcl}_Y \{ \{ y \} \} \subset f[\bigcap_{x \in X} \text{spcl}_X \{ \{x\} \}] = f[\emptyset] = \emptyset . \]

Hence \( Y \) is wsp-R_0.
PRODUCTIVITY OF wsp-R₀ SPACES

Our next endeavour is to examine the effect of the product of finite number of coordinate spaces if one of the coordinate space is a wsp-R₀ space.

To achieve our desired result the following two lemmas are needed. We first state and prove them.

Lemma 5.7. Let $X = \prod_{i=1}^{n} X_i$ be the product spaces of $X_i$’s, $i = 1, 2, \ldots, n$. Then for any point $<x_i> \in X$

$$\text{spcl}_X \left( \{<x_i>\} \right) \subseteq \prod_{i=1}^{n} \text{spcl}_{X_i} \left( \{x_i\} \right).$$

Proof. Let $<\alpha_i> \in \text{spcl}_X \left( \{<x_i>\} \right)$. Also let $U_i \in \text{SPO} (X_i, \alpha_i)$ and $U = \prod_{i=1}^{n} U_i$.

Lemma 0.9 gives

$$U \in \text{SPO} (X).$$

Obviously

$$<\alpha_i> \in U.$$

Now

$$<\alpha_i> \in \text{spcl}_X \left( \{<x_i>\} \right)$$

$$\Rightarrow \{<x_i>\} \cap U \neq \phi$$

$$\Rightarrow \{x_i\} \cap U_i \neq \phi, i = 1, 2, \ldots, n$$

$$\Rightarrow \alpha_i \in \text{spcl}_{X_i} \left( \{x_i\} \right), i = 1, 2, \ldots, n$$

$$\Rightarrow <\alpha_i> \in \prod_{i=1}^{n} \text{spcl}_{X_i} \left( \{x_i\} \right).$$

So,

$$\text{spcl}_X \left( \{<x_i>\} \right) \subseteq \prod_{i=1}^{n} \text{spcl}_{X_i} \left( \{x_i\} \right).$$
Lemma 5.8. Let $X = \prod_{i=1}^{n} X_i$ be the product space of $X_i$'s, $i = 1, 2, \ldots, n$.

Then for any point $<x_i> \in X$

$$\bigcap_{<x_i> \in X} \left[ \bigcap_{i=1}^{n} \text{spcl} \ X_i \left( \{x_i\} \right) \right] = \bigcap_{i=1}^{n} \left[ \bigcap_{<x_i> \in X} \text{spcl} \ X_i \left( \{x_i\} \right) \right].$$

Proof. Let $<\alpha_i> \in \bigcap_{<x_i> \in X} \left[ \bigcap_{i=1}^{n} \text{spcl} \ X_i \left( \{x_i\} \right) \right]$. Then $<\alpha_i> \in \prod_{i=1}^{n} \text{spcl} \ X_i \left( \{x_i\} \right) \forall <x_i> \in X$

$$\Rightarrow \alpha_i \in \text{spcl} \ X_i \left( \{x_i\} \right) \forall x_i \in X_i, i = 1, 2, \ldots, n$$

$$\Rightarrow \alpha_i \in \bigcap_{<x_i> \in X} \text{spcl} \ X_i \left( \{x_i\} \right)$$

$$\Rightarrow <\alpha_i> \in \prod_{i=1}^{n} \left[ \bigcap_{<x_i> \in X} \text{spcl} \ X_i \left( \{x_i\} \right) \right].$$

This gives

$$\bigcap_{<x_i> \in X} \left[ \bigcap_{i=1}^{n} \text{spcl} \ X_i \left( \{x_i\} \right) \right] \subset \bigcap_{i=1}^{n} \left[ \bigcap_{<x_i> \in X} \text{spcl} \ X_i \left( \{x_i\} \right) \right].$$

The reverse inclusion may be obtained using the same technique.

Theorem 5.12. A space $X = \prod_{i=1}^{n} X_i$ is wsp-R₀, if one of the $X_i$ is wsp-R₀.

Proof. Let $X_K$ be wsp-R₀, for some fixed index $k$, where $1 \leq k \leq n$. The Lemma 5.7 yields

$$\bigcap_{<x_i> \in X} \text{spcl} \left( \{<x_i>\} \right) \subset \bigcap_{i=1}^{n} \left[ \bigcap_{<x_i> \in X} \text{spcl} \left( \{x_i\} \right) \right].$$

An application of Lemma 5.8 gives

$$\bigcap_{<x_i> \in X} \text{spcl} \left( \{<x_i>\} \right) \subset \prod_{i=1}^{n} \left[ \bigcap_{<x_i> \in X} \text{spcl} \left( \{x_i\} \right) \right].$$

Now wsp-R₀-ness of $X_K$ ensures that

$$\bigcap_{<x_i> \in X} \text{spcl} \left( \{x_K\} \right) = \phi.$$

Therefore, from above
\[
\bigcap_{x \in X} \text{spcl} \left( \langle X_x \rangle \right) \subset X_1 \times \ldots \times X_{K-1} \times \phi \times X_{K+1} \times \ldots \times X_n = \phi.
\]

Hence \( X \) is wsp-Ro.

**PART - II**

**SEMI-PRE DOOR SPACES**

The concept of door\(^1\) space was studied in some details by J. L. Kelley [38]. Recently J. Dontchev [18] studied door space with great detail. In 1968, J.P.Thomas [83] defined semi-door space using semi open sets of Levine. We in this brief span introduce semi-pre door space (briefly sp-door space) utilising semi-preopen sets of Andrijević [1].

**Definition 5.6.** A topological space is called semi-pre door space (briefly sp-door space) if for every subset \( A \) of \( X \), either \( A \in \text{SPO}(X) \) or \( A \in \text{SPF}(X) \).

**Remark 5.5.** Clearly a door space is a sp-door space but not conversely as the following example shows.

**Example 5.5.** Let \((X, \tau)\) be the space of Example 5.1. Then \((X, \tau)\) is a sp-door space but not a door space.

**Remark 5.6.** Evidently every semi-door\(^2\) space is a sp-door space but the converse is not always true as is exhibited in the next example.

**Example 5.6.** Let \( X \) be the set of Example 5.1 endowed with the topology \( \tau = \{ \phi, X, \{a, b\} \} \). Then \((X, \tau)\) is a sp-door space but not a semi-door space.

To examine the various properties of a sp-door space we need the following definitions and lemmas.

**Definition 5.7.** Let \( x \in X \) and \( A \subset X \). Then \( x \) is said to be a semi-pre limit point (briefly sp-limit point) iff \( A \cap (U \setminus \{x\}) \neq \phi \) for all \( U \in \text{SPO}(X, x) \).

---

\(^1\) See Definition 0.20 (viii).
\(^2\) See Definition 0.20 (ix).
Remark 5.7. Every sp-limit point of a set $A$ is a limit point of $A$. But the converse does not always hold as is evident from the following example.

Example 5.7. Let $(X, \tau)$ be the space of Example 5.6. Take $A = \{a, c\}$. Then $a$ is a limit point of $A$ but not a sp-limit point of $A$.

Definition 5.8. A subset $A$ of $X$ is said to be semi-predense (briefly sp-dense) iff $\text{spcl} (A) = X$.

Remark 5.8. Obviously every sp-dense set is dense but a dense subset may not be sp-dense as is clear from the following example.

Example 5.8. Let $(X, \tau)$ be the space of Example 5.6. Take $A = \{a\}$. Then $A$ is dense but not sp-dense.

Definition 5.9. A space $X$ is said to be semi-presubmaximal (briefly sp-submaximal) iff every sp-dense subset of $X$ is s.p.o.

Remark 5.9. Clearly every submaximal$^1$ space is sp-submaximal but the converse need not be true. This is clear from Example 5.9.

Example 5.9. Let $(X, \tau)$ be the space of Example 5.1. Taking $A = \{a, b\}$ it is easy to see that $\text{Cl} (A) = X$ but $A \notin \tau$. Thus $(X, \tau)$ is not submaximal.

On the other hand the sp-dense subsets of $X$ are $\{a\}$, $\{a, b\}$, $\{a, c\}$ and $X$. All these sets are s.p.o. sets and hence $(X, \tau)$ is sp-submaximal.

Definition 5.10. A space $X$ is said to be semi-pre irreducible (briefly sp-irreducible) iff every semi-preopen subset of $X$ is sp-dense.

Remark 5.10. Evidently every sp-irreducible space is irreducible$^2$ but that the converse may not be true can be seen from the next example.

Example 5.10. Let $X = \{a, b\}$ be the set with the indiscrete topology $\tau$. Then $(X, \tau)$ is irreducible. That $(X, \tau)$ is not sp-irreducible follows from the fact that

$\{a\} \in \text{SPO} (X), \text{spcl} (\{a\}) = \{a\}$ but $\{a\}$ is not sp-dense.

$^1$ See Definition 0.20 (xi).
$^2$ See Definition 0.20 (xii).
**Lemma 5.9.** In a topological space $X$, if for a point $x \in X$, $\{x\} \in \text{SPO}(X)$, then $x$ is not a sp-limit point of $X$.

The straightforward proof is omitted.

**Theorem 5.13.** If $X$ is a Hausdorff sp-door space, then $X$ has at most one sp-limit point.

**Proof.** Let us assume that $x$, $y \in X$ ($x \neq y$) be two sp-limit points of $X$. Hausdorffness of $X$ provides two open sets $U \in \Sigma(x)$, $V \in \Sigma(y)$ such that $U \cap V = \emptyset$.

Set $A = (U - \{x\}) \cup \{y\}$.

Since $X$ is a sp-door space either $A \in \text{SPO}(X)$ or $A \in \text{SPF}(X)$.

**Case I.** Let $A \in \text{SPO}(X)$. Then by Lemma 0.3 and the fact that every open set is an $\alpha$-set it follows that $A \cap V \in \text{SPO}(X)$.

But $A \cap V = \{y\}$

$\Rightarrow \{y\} \in \text{SPO}(X)$.

**Case II.** Let $A \in \text{SPF}(X)$. This yields $X - A \in \text{SPO}(X)$

$\Rightarrow (X - A) \cap U = ((U - \{x\}) \cup \{y\})^C \cap U$.

An application of De Morgan's law produces

$$(X - A) \cap U = ((U^C \cup \{x\} \cap \{y\}^C) \cap U = \emptyset \cup \{x\} = \{x\}.$$  

So from above, we get

$\{x\} = (X - A) \cap U \in \text{SPO}(X)$.

Now from the above discussion one infers that one of $\{x\}$ and $\{y\}$ must be a s.p.o. set, which, in its turn, implies that at least one of $x$ and $y$ is not a sp-limit point. This contradicts our assumption. Hence the theorem.
**Theorem 5.14.** If $X$ is a sp-$T_2$ door space then $X$ has at most one sp-limit point.

**Proof.** Pursuing the same reasoning with minor modification in the proof of Theorem 5.13 in the result follows.

**Theorem 5.15.** Every $\alpha$-subspace of a sp-door space is a sp-door space.

**Proof.** Let $X$ be a sp-door space, $Y \in \alpha (X)$ and $A \subseteq Y$. Since $X$ is a sp-door space either $A \in SPO (X)$ or $A \in SPF (X)$.

**Case I.** Let $A \in SPO (X)$. Since every $\alpha$-set is a p.o. set, by Lemma 0.7 one obtains $A \in SPO (Y)$.

**Case II.** Let $A \in SPF (X)$. Then by Lemma 4.3 it follows that

$$
spcl_Y (A) = spcl_X (A) \cap Y = A \cap Y = A
$$

$$
\Rightarrow A \in SPF (Y).
$$

Combining these two cases one conclude that $Y$ is a sp-door space.

**Theorem 5.16.** Every sp-door space is sp-submaximal.

**Proof.** Let $A \subseteq X$ be sp-dense. sp-doorness of $X$ indicates that either $A \in SPO (X)$ or $A \in SPF (X)$.

**Case I.** If $A \in SPO (X)$, then there is nothing to prove.

**Case II.** If $A \in SPF (X)$, then

$$
spcl (A) = A.
$$

But sp-denseness of $X$ indicates that

$$
spcl (A) = X,
$$

whence from above, one obtains

$$
A = X
$$

$$
\Rightarrow A \in SPO (X).
$$

Hence $A$ is sp-door space.
**Theorem 5.17.** Every sp-irreducible and sp-submaximal space is a sp-door space.

**Proof.** Let $X$ be a sp-irreducible and sp-submaximal space. Let $A \subset X$. Two cases come up for consideration.

**Case I.** $A$ is sp-dense.

**Case II.** $A$ is not sp-dense.

In Case I, sp-submaximal ity of $X$ guarantees that

$$A \in \text{SPO}(X).$$

In the Case II we observe that

$$\text{spcl}(A) \neq X.$$

Let $x \in X$ be such that

$$x \notin \text{spcl}(A).$$

By Lemma 4.1, there exists a $U \in \text{SPO}(X, x)$ such that

$$U \cap A = \emptyset$$

$$\Rightarrow U \subset X - A.$$

The sp-irreducibility of $X$ yields that $U$ is sp-dense.

So,

$$X = \text{spcl}(U).$$

One now infers from above that

$$\text{spcl}(X - A) = X$$

$$\Rightarrow X - A$$

is also sp-dense.

Again sp-submaximality of $X$ implies that

$$X - A \in \text{SPO}(X)$$

$$\Rightarrow A \in \text{SPF}(X).$$

Therefore in any case

either $A \in \text{SPO}(X)$ or $A \in \text{SPF}(X)$.

Hence $X$ is a sp-door space.
TRANSFER TOPOLOGY OF \(sp\)-DOOR SPACE

**Theorem 5.18.** Let \( f : X \to Y \) be \( sp\)-open and surjective. If \( X \) is a \( sp\)-door space, then \( Y \) is a \( sp\)-door space.

**Proof.** Let \( A \subseteq Y \). Since \( X \) is a \( sp\)-door space either

\[
f^{-1} [A] \in SPO (X) \text{ or } f^{-1} [A] \in SPF (X).
\]

**Case I.** Let \( f^{-1} [A] \in SPO (X) \). The surjectivity and \( sp\)-openness of \( f \) together yield that

\[
A = f f^{-1} [A] \in SPO (Y).
\]

**Case II.** Let \( f^{-1} [A] \in SPF (X) \). Then

\[
X - f^{-1} [A] = f^{-1} [Y - A] \in SPO (X).
\]

Following the same technique we may see that

\[
Y - f^{-1} [A] = f^{-1} [Y - A] \in SPO (X).
\]

Pursuing the same technique we may see that

\[
Y - A \in SPO (Y)
\]

\[\Rightarrow A \in SPF (Y).
\]

Thus either \( A \in SPO (Y) \) or \( A \in SPF (Y) \).

In other words, \( Y \) is a \( sp\)-door space.