Propagation of Waves in a Micropolar Elastic Layer Immersed in an Infinite Liquid

by

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Summary. The effect of the liquid on the propagation of two characteristic types of waves in a micropolar elastic layer are investigated in this paper. In both the cases the frequency equations for symmetric and antisymmetric vibrations are derived and then applied in investigating the particular cases of modified surface waves, viz (a) Rayleigh (b) Love (c) Stoneley waves, etc with the effect of the liquid.

1. Introduction. In recent years, a good deal of attention has been paid to the likely effects of couple stresses into the theory of elasticity. Most of the results obtained are based on the work of Eringen [2, 3, 4], Mindlin [5, 6, 7], Palmov [8] and Nowacki [9, 10, 11]. Of particular relevance to the present paper we cite the works [2, 3, 8, 11] in which equivalent theories for linear micropolar (Cosserat) continuum are developed. This particular formulation assumes that the microstructure may rotate (but not deform) relative to the macrostructure.

Within the context of the Cosserat theory the propagation of monochromatic waves on an infinite elastic layer, immersed in an infinite liquid is the object of studies in this paper. This note considers two characteristic types of propagation of monochromatic waves. We shall consider the layer as a two-dimensional waveguide. The basic equations are established, using the notations of Nowacki [11] (Sec. 2). The characteristic equations for symmetric and antisymmetric vibrations are found (Sec. 3) The expression for modified longitudinal and flexural vibrations with the effect of the liquid and the expression for modified Rayleigh waves and Stoneley waves on the floor (Cosserat continuum) of the ocean have been derived (Sec. 4). In the latter section the present authors arrived at an interesting result that the liquid has no effect on the propagation of modified Love waves in an elastic micropolar medium (Sec. 4). Furthermore the expression for Love waves in a micropolar elastic medium in the presence of an isotropic homogeneous elastic half-space is obtained (Sec 5). It has been shown there that in the derived expression for twisted Love waves in the presence of the elastic half-space, if we make the modulus of rigidity of the elastic half-space to be zero, then it also helps us to make the conclusion that
the liquid has no effect on the propagation of twisted Love waves in a micropolar medium. It has also been shown in the subsequent sections that among the material constants, $\alpha$, $\beta$, $\gamma$, $\varepsilon$ if and only if $\alpha \to 0$, then we get the corresponding classical results.

2. Basic equations. We shall consider an isotropic homogeneous and centro-symmetric layer. The basic equations of motion in a micropolar elastic solid [9] are:

$$
(\mu + \alpha) \nabla^2 u + (\lambda + \mu - \alpha) \text{grad } \text{div } u + 2\alpha \text{rot } \omega = \rho \frac{\partial^2 u}{\partial t^2},
$$

(2.1)

$$
(\gamma + \varepsilon) \nabla^2 \omega + (\gamma + \beta - \varepsilon) \text{grad } \text{div } \omega - 4\alpha + 2\alpha \text{rot } u = J \frac{\partial^2 \omega}{\partial t^2}.
$$

The asymmetric force stress tensor $\sigma_{ij}$ and the couple stress tensor $\mu_{ij}$ are connected with the strain tensor $\gamma_{ij}$ and the curvature twist tensor $\kappa_{ij}$ by the following constitutive relations:

$$
\sigma_{ij} = (\mu + \alpha) \gamma_{ij} + (\mu - \alpha) \gamma_{ij} + \lambda_{ijkl} \delta_{ij},
$$

(2.2a)

$$
\mu_{ij} = (\gamma + \varepsilon) \kappa_{ij} + (\gamma - \varepsilon) \kappa_{ij} + \beta_{ijkl} \delta_{ij},
$$

(2.2b)

in which

$$
\gamma_{ij} = u_{i,j} - u_{j,i}, \quad \kappa_{ij} = \omega_{i,j},
$$

where $u(x, t)$, $\omega(x, t)$ denote respectively the displacement and rotation fields; $\lambda$, $\mu$, $\alpha$, $\beta$, $\gamma$, $\varepsilon$ are the material constants, $\rho$ is the density of the material and $J$ is the rotational inertia.

From the nature of the problems let us now consider that the vectors $u$ and $\omega$ depend only on the variables $x_1$, $x_3$ and time, $t$. Then Eqs. (2.1) can be grouped into two independent systems:

$$
(\mu + \alpha) \nabla^2 u_1 + (\lambda + \mu - \alpha) \frac{\partial e}{\partial x_1} - 2\alpha \frac{\partial \omega_2}{\partial x_3} = \rho \frac{\partial^2 u_1}{\partial t^2},
$$

(2.3a)

$$
(\mu + \alpha) \nabla^2 u_3 + (\lambda + \mu - \alpha) \frac{\partial e}{\partial x_3} + 2\alpha \frac{\partial \omega_2}{\partial x_1} = \rho \frac{\partial^2 u_3}{\partial t^2},
$$

and

$$
(\gamma + \varepsilon) \nabla^2 \omega_2 - 4\alpha \omega_2 + 2\alpha \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) = J \frac{\partial^2 \omega_2}{\partial t^2},
$$

(2.3b)

$$
(\gamma + \varepsilon) \nabla^2 \omega_1 + (\gamma + \beta - \varepsilon) \frac{\partial \kappa}{\partial x_1} - 4\alpha \omega_1 - 2\alpha \frac{\partial u_2}{\partial x_3} = J \frac{\partial^2 \omega_1}{\partial t^2},
$$

$$
(\gamma + \varepsilon) \nabla^2 \omega_3 + (\gamma + \beta - \varepsilon) \frac{\partial \kappa}{\partial x_3} - 4\alpha \omega_3 + 2\alpha \frac{\partial u_2}{\partial x_1} = J \frac{\partial^2 \omega_3}{\partial t^2},
$$

where

$$
e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}, \quad \kappa = \frac{\partial \omega_1}{\partial x_1} + \frac{\partial \omega_3}{\partial x_3} \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}.$$
It is easy to note that the displacement and rotation fields in the system of Eqs (2.3a) are described by the displacement vector \( \mathbf{u} (u_1, 0, u_3) \) and the rotation vector \( \omega (0, \omega_3, 0) \). These two vectors induce the following stress \( \sigma \) and couple stress \( \mu \) state:

\[
\sigma = \begin{bmatrix}
\sigma_{11} & 0 & \sigma_{13} \\
0 & \sigma_{22} & 0 \\
\sigma_{31} & 0 & \sigma_{33}
\end{bmatrix}, \quad \mu = \begin{bmatrix}
0 & \mu_{12} & 0 \\
\mu_{21} & 0 & \mu_{23} \\
0 & \mu_{32} & 0
\end{bmatrix}
\]  

(2.4)

Analogously, the displacement and the rotation fields in the system (2.3b) are described by the displacement vector \( \mathbf{u} (0, u_2, 0) \) and the rotation vector \( \omega (\omega_1, 0, \omega_3) \). These two vectors induce the following stress \( \sigma \) and couple stress \( \mu \) state:

\[
\sigma = \begin{bmatrix}
0 & \sigma_{12} & 0 \\
\sigma_{21} & 0 & \sigma_{23} \\
0 & \sigma_{32} & 0
\end{bmatrix}, \quad \mu = \begin{bmatrix}
\mu_{11} & 0 & \mu_{13} \\
0 & \mu_{22} & 0 \\
\mu_{31} & 0 & \mu_{33}
\end{bmatrix}
\]  

(2.5)

3. Micropolar elastic solid layer immersed in an infinite liquid. In this case the authors make inquiries about propagation of waves in a micropolar elastic solid layer being placed in between two semi-infinite liquid of the same type. In investigating the stated problem we are going to find out the solutions of Eqs (2.3a) and (2.3b) separately with the appropriate boundary conditions relevant to the problem.

(A) Solution of Eqs. (2.3a) Let us consider that a monochromatic wave propagates along the \( x_1 \)-axis of a micropolar elastic layer of thickness \( 2H \) and immersed in an infinite liquid. Let \( x_3 = 0 \) be the median plane of the layer and \( x_3 \)-axis be directed downwards the layer.

Boundary conditions. On the plane surface \( x_3 = \pm H \) the following boundary conditions are satisfied:

\[
\begin{align*}
\sigma_{31} &= \mu_{32} \text{ for the layer} = \mu_{32} \text{ for the liquid}, \\
\sigma_{33} &= \sigma_{33} \text{ for the layer} = \sigma_{33} \text{ for the liquid}, \\
\mathbf{u}_3 &= \mathbf{u}_3 \text{ for the layer} = \mathbf{u}_3 \text{ for the liquid}.
\end{align*}
\]  

(3.1)

Let us express the displacement components \( u_1, u_3 \) of the layer in terms of two potential functions \( \Phi \) and \( \Psi \) by the following relations:

\[
\begin{align*}
u_1 &= \frac{\partial \Phi}{\partial x_1} - \frac{\partial \Psi}{\partial x_3}, \\
u_3 &= \frac{\partial \Phi}{\partial x_3} + \frac{\partial \Psi}{\partial x_1}.
\end{align*}
\]  

(3.2)

Similarly, for the liquid

\[
\begin{align*}
u_1^{(0)} &= \frac{\partial \Phi_0}{\partial x_1}, \\
\nu_3^{(0)} &= \frac{\partial \Phi_0}{\partial x_3}, \\
\nu_1^{(2)} &= \frac{\partial \Phi_2}{\partial x_1}, \\
\nu_3^{(2)} &= \frac{\partial \Phi_2}{\partial x_3}.
\end{align*}
\]  

(3.3)
where the superscripts (0) and (2) and the displacement potentials \( \phi_0, \phi_2 \) are, respectively, for the liquids above and below the layer.

Expressing the set of Eqs. (2.3a) in terms of \( \phi \) and \( \psi \) we have

\[
\begin{align*}
\left( \nabla_i^2 - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right) \phi &= 0, \\
\left( \nabla_i^2 - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right) \psi + \frac{2\alpha}{\mu + \alpha} \psi &= 0, \\
\left( \nabla_i^2 - \nabla_j^2 - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right) \omega_2 &= \frac{2\alpha}{\gamma + \epsilon} \nabla_i^2 \psi = 0, \\
\end{align*}
\]

and similarly for the liquid

\[
\begin{align*}
\left( \nabla_i^2 - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right) \phi_0 &= 0, \\
\left( \nabla_i^2 - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right) \phi_2 &= 0.
\end{align*}
\]

From the last two Eqs. of (3.4), eliminating \( \psi \) or \( \omega_2 \) we have

\[
\begin{align*}
\left[ \left( \nabla_i^2 - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right) \nabla_j^2 - \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} \right] + \eta^2 \nabla_i^2 \left( \Psi, \omega_2 \right) &= 0.
\end{align*}
\]

To obtain the solution of (3.4) and (3.6) we put

\[
\begin{align*}
\phi &= f(x_3) e^{(i k x_1 - \omega t)}, \\
\psi &= g(x_3) e^{(i k x_1 - \omega t)}, \\
\omega_2 &= h(x_3) e^{(i k x_1 - \omega t)}.
\end{align*}
\]

Now putting (3.7) in (3.4) and (3.6), and after a simple calculation we get

\[
\begin{align*}
\phi &= (A \sinh (\delta x_3) + B \cosh (\delta x_3)) e^{(i k x_1 - \omega t)} + C \sinh (\lambda_1 x_3) + D \cosh (\lambda_1 x_3) + E \sinh (\lambda_2 x_3) + F \cosh (\lambda_2 x_3), \\
\psi &= (C' \sinh (\lambda_1 x_3) + D' \cosh (\lambda_1 x_3) + E' \sinh (\lambda_2 x_3) + F' \cosh (\lambda_2 x_3)) e^{(i k x_1 - \omega t)}, \\
\end{align*}
\]

Again, from (3.5) we obtain

\[
\begin{align*}
\phi_0 &= A_0 e^{(i k x_1 - \omega t)} e^{(i k x_1 - \omega t) + i \theta x_1),} \\
\phi_2 &= A_2 e^{(i k x_1 - \omega t + i \theta x_1),}
\end{align*}
\]

where

\[
\begin{align*}
\omega_0^2 &= \frac{\lambda + 2\mu}{c_0^2} = \frac{\mu + \alpha}{c_0^2} = \frac{\gamma + \epsilon}{c_0^2}, \\
\omega_1^2 &= \frac{\sigma_1^2}{c_1^2}, \\
\omega_2^2 &= \frac{\sigma_2^2}{c_2^2}, \\
\eta^2 &= \frac{4\alpha}{\gamma + \epsilon}, \\
\delta^2 &= k^2 - \sigma_1^2, \\
\nu_0^2 &= k^2 - \omega_0^2 - \sigma_1^2, \\
\lambda_1^2, \lambda_2^2 &= k^2 + \frac{1}{2} \left[ (\nu_0^2 - \omega_1^2 - \omega_2^2) \pm \frac{1}{2} \right] \left[ (\sigma_1^2 + \sigma_2^2 + \eta^2 - \nu_0^2) \pm 4\sigma_2^2 (\sigma_2^2 - \nu_0^2)^{1/2} \right],
\end{align*}
\]

and \( c_\omega \) is the velocity of dilatational (sound) wave in the liquid.
Now we shall consider the general problem of wave propagation as the solution of two problems, i.e. by considering the symmetric and antisymmetric vibrations.

**Symmetric vibrations** Considering the symmetry of the displacement \( u_1 \) and the force stresses \( \sigma_{33} \) and the couple stresses \( \mu_{32} \) with respect to the plane \( x_3=0 \) we have from (3.8), (3.4)\(_2\) and (3.4)\(_3\)

\[
A=D=F=D'=F'=0,
\]

\[
(C'=\kappa_1 C \quad \text{and} \quad E'=\kappa_2 E),
\]

where

\[
\kappa_r = \frac{1}{p} \left( k^2 - \lambda_r^2 - \sigma_0^2 \right), \quad p = \frac{2\alpha}{\mu + \alpha}, \quad r = 1, 2.
\]

By using the relations (2.2a), (2.2b) and (3.2), the values of \( \sigma_{33}, \sigma_{31} \) and \( \mu_{32} \) for the layer now take the form

\[
\sigma_{31} = \mu \left( \frac{\partial^2 \Phi}{\partial x_1 \partial x_3} + \frac{\partial^2 \Psi}{\partial x_1 \partial x_3} - \frac{\partial^2 \Psi}{\partial x_3^2} \right) - 2\nu \partial^2 \Psi \partial x_3^2 - 2\omega_2,
\]

\[
\sigma_{33} = \lambda \nu^2 \Phi + 2\mu \left( \frac{\partial^2 \Phi}{\partial x_3^2} + \frac{\partial^2 \Psi}{\partial x_1 \partial x_3} \right),
\]

\[
\mu_{32} = (\gamma + \delta) \frac{\partial \omega_2}{\partial x_3},
\]

and for the liquid

\[
\sigma_{33} = -\rho_0 \omega^2 \Phi_0 \quad \text{or} \quad -\rho_0 \omega^2 \Phi_2,
\]

accordingly, as the liquid is above or below the layer, and \( \rho_0 \) is the density of the liquid.

Further using the boundary conditions (3.1), the values of the stresses from (3.12) and (3.13), the values of the normal displacements from (3.2) and (3.3), the expression for displacement potential from (3.8) and (3.9) with (3.11), and taking \( A_0 = A_2 \), we have a system of four homogeneous equations with four unknown constants. Eliminating the constants from this system of equations we get the following period equation in the form of a transcendental equation

\[
\left[ b \left( \frac{a_1 \kappa_2}{\kappa_2 - \kappa_1} - \frac{a_2 \kappa_1}{\kappa_2 - \kappa_1} \cdot \frac{\lambda_1}{\lambda_2} \cdot \tan h (\lambda_2 H) \tan h (\lambda_1 H) \right) - 4\mu^2 k^2 \lambda_2 \delta \tan h (\delta H) \tan h (\lambda_1 H) \right] + \frac{\rho_0 \omega^2 \delta}{\nu_0} \left[ \frac{(2\mu k^2 - a_1) \kappa_2}{\kappa_2 - \kappa_1} \right] = \left[ \frac{(2\mu k^2 - a_2) \kappa_1}{\kappa_2 - \kappa_1} \right] \tan h (\lambda_2 H) \tan h (\lambda_1 H) \tan h (\delta H) = 0.
\]

where

\[
b = (\lambda + 2k\delta)^2 - k^2 \lambda, \quad a_r = \mu (k^2 + \lambda_r^2) + \alpha (\lambda_r^2 - k^2) + 2\omega \kappa_r, \quad r = 1, 2.
\]

\[
(3.15) \quad b = (\lambda + 2k\delta)^2 - k^2 \lambda, \quad a_r = \mu (k^2 + \lambda_r^2) + \alpha (\lambda_r^2 - k^2) + 2\omega \kappa_r, \quad r = 1, 2.
\]
Now, if \( \alpha \to 0 \) then (3.14) reduces to

\[
(2 - c^2/\xi^2)^2 \tan h \left( KH \sqrt{1 - c^2/\xi^2} \right) - 4 \left( 1 - c^2/\xi^2 \right)^{1/2} \left( 1 - c^2/\xi^2 \right)^{1/2} x \times \tan h \left( KH \sqrt{1 - c^2/\xi^2} \right) + \frac{\rho_0}{v_0} \frac{\delta}{\epsilon_2^2} \tan h \left( KH \sqrt{1 - c^2/\xi^2} \right) x \times \tan h \left( KH \sqrt{1 - c^2/\xi^2} \right) = 0,
\]

where

\[ \xi^2 = \frac{\mu}{\rho} \]

This is in agreement with the corresponding classical result [1] with some change of notations. The first part of (3.14) corresponds to the result obtained by Nowacki [11] for the free symmetrical vibration of a micropolar elastic plate. Here the second part of (3.14) is the effect produced by the presence of the liquid.

**Antisymmetric vibrations.** Let us assume

\[
(3.17) \quad B = C = E = C' = E' = 0 \quad \text{and} \quad D' = \kappa_1 D, \quad F' = \kappa_2 F,
\]

and as such it is easy to see that the displacement \( u_1 \) and the stresses \( \sigma_{11}, \sigma_{33} \) and \( \mu_{32} \) are antisymmetric with respect to the plane \( x_3 = 0 \).

Now, using the boundary conditions (3.1), the relations (3.12), (3.13), (3.2), (3.3), (3.5) and (3.9) with (3.17) and \( A_0 = -A_2 \), we obtain a system of four homogeneous equations. If we make the determinant of this system equal to zero, it will give the following wave velocity equation for the antisymmetric vibrations of a micropolar elastic layer immersed in an infinite liquid

\[
(3.18) \quad \left[ b \left( \frac{a_1 \kappa_2 \lambda_2}{\tan h (\lambda_1 H)} - \frac{a_2 \kappa_1 \lambda_1}{\tan h (\lambda_2 H)} \right) \tan h(\delta H) - 4 \mu_2 \kappa_2^2 \delta \lambda_1 \lambda_2 (\kappa_2 - \kappa_1) \right] + \frac{\rho_0}{v_0} \omega^2 \delta \left[ \frac{(2\mu k^2 - a_1) \kappa_2 \lambda_2}{\tan h (\lambda_1 H)} - \frac{(2\mu k^2 - a_2) \kappa_1 \lambda_1}{\tan h (\lambda_2 H)} \right] = 0.
\]

For, \( \alpha \to 0 \) we get form (3.18)

\[
(3.19) \quad (2 - c^2/\xi^2)^2 \tan h \left( KH \sqrt{1 - c^2/\xi^2} \right) - 4 \left( 1 - c^2/\xi^2 \right)^{1/2} \left( 1 - c^2/\xi^2 \right)^{1/2} x \times \tan h \left( KH \sqrt{1 - c^2/\xi^2} \right) + \frac{\rho_0}{v_0} \frac{\delta}{\epsilon_2^2} \cdot c^2 = 0,
\]

which, with some change of notations is the classical result [1].

The first part of (3.18) represent the wave velocity equation for free antisymmetric vibration of a micropolar plate [11]. The second part represents a modification due to the presence of the liquid.

(B) **Solution of Eqs (2.3b).** As in (A) let us consider a micropolar elastic layer of thickness \( 2H \), immersed in an infinite liquid wherein the plane waves are propagated with constant velocity \( c \) along the \( x_1 \)-axis. Here the boundary conditions are

\[
(3.20) \quad \mu_{33} = \mu_{31} = \sigma_{32} = 0 \quad \text{for} \quad x_3 = \pm H,
\]
and the displacement continuity will not arise as the common surface of separation between the layer, and the liquid cannot be taken to be welded in contact. It is clear from the boundary conditions that these are the same conditions for the free vibrations of a micropolar plate in vacuum.

Now, proceeding in a similar manner as in [11] we arrive at the following wave velocity equation for the symmetric and antisymmetric vibrations respectively:

\[
\frac{\tan \vartheta (\delta H)}{\tan h (\lambda_1 H)} = \frac{4\gamma k^2 \alpha \left( c_1 \lambda_2 - c_2 \lambda_1 \right) \tan h (\lambda_2 H) + \left[ (2\gamma + \beta) \sigma^2 - \beta k^2 \right] \left( c_1 d_2 - c_2 d_1 \right) \tan h (\lambda_1 H)}{4\gamma^2 k^2 \sigma \left( \lambda_1 d_2 - \lambda_2 d_1 \right)}
\]

and

\[
\frac{\tan h (\sigma H)}{\tan h (\lambda_1 H)} = \frac{4\gamma k^2 \sigma \left( \lambda_1 d_2 - \lambda_2 d_1 \right)}{4\gamma^2 k^2 \sigma \left( \lambda_1 d_2 - \lambda_2 d_1 \right)}
\]

where

\[
\sigma = (k^2 - \gamma^2 - \alpha^2)^{1/2}, \quad \sigma_2^2 = \frac{\omega^2}{c_2^2}, \quad c_2^2 = \frac{\beta + 2\gamma}{\lambda}
\]

\[
\rho_r = \frac{p (\lambda_r^2 - k^2)}{\lambda_r^2 - k^2 + \sigma_r^2}, \quad c_r = \gamma (k^2 + \lambda_r^2) + \epsilon (\lambda_r^2 - k^2),
\]

\[
d_r = (\mu + \alpha) \lambda_r \rho_r - 2\alpha \lambda r, \quad r = 1, 2
\]

From the above results (3.21) and (3.22) we conclude that the liquid exerts no effect on the propagation of modified twisted shear waves in a micropolar elastic layer.

4. Discussions of the characteristic equations, obtained in Sec. 3. It is interesting to discuss the result of wave velocity Eqs (3.14), (3.18), (3.21) and (3.22) in the following cases.

Case 1. For very small values of $KH$, i.e., for a wavelength large as compared with the thickness of layer, the hyperbolic tangents can be replaced by their arguments, we have thus from Eq. (3.14)

\[
\beta (\kappa_2 - \kappa_1) - 4\mu k^2 \delta \left( \kappa_2 - \kappa_1 \right) + \frac{\rho \omega^2 \gamma^2 H}{\nu_0} \left[ (2\mu k^2 - \alpha_1) \kappa_2 - (2\mu k^2 - \alpha_2) \kappa_1 \right] = 0.
\]

Nowacki [11] gives, under the same conditions for a plate in vacuum, just the above expression with $\rho_\infty = \sigma_0 = 0$. Therefore, (4.1) represents the wave velocity equation for the longitudinal vibration of micropolar elastic layer immersed in an infinite
Now, if $\alpha \to 0$, then (4.1) reduces to an expression [1] which gives the longitudinal vibrations of an elastic plate immersed in an infinite liquid with the unchanged real part of the phase velocity but with an additive small attenuation factor

$$(4.2) \quad c = 2\varepsilon_z (1 - \varepsilon_z^2/c_i^2)^{1/2} \left[ 1 + \frac{i}{2} \frac{\rho_0 \omega \cnum{1} H}{pc_i^2} \left( \frac{1}{4(1 - \varepsilon_z^2/c_i^2)} - \frac{\varepsilon_z^2}{c_i^2} \right) \right].$$

For a long wavelength the expression (3.18) for the antisymmetric modes gives the expression for the phase velocity of long flexural waves.

$$(4.3) \quad b \left( 1 - \frac{\delta^2 H^2}{3} \right) \left[ \frac{a_1 \kappa_1}{\lambda_1^2 \left( \frac{\lambda_1^2 H^2}{3} \right)} - \frac{a_2 \kappa_2}{\lambda_2^2 \left( 1 - \frac{\lambda_2^2 H^2}{3} \right)} \right] - 4\mu^2 k^2 (\kappa_2 - \kappa_1) +$$

$$\quad + \frac{\rho_0 \omega^2}{v_0 H} \left[ \frac{(2\mu k^2 - a_1) \kappa_2}{\lambda_1^2 \left( 1 - \frac{\lambda_1^2 H^2}{3} \right)} - \frac{(2\mu k^2 - a_2) \kappa_1}{\lambda_2^2 \left( 1 - \frac{\lambda_2^2 H^2}{3} \right)} \right] = 0.$$

This is the same expression as that obtained by Nowacki [11] for a plate in vacuum with an additive term, being correction due to the presence of the liquid.

If $\alpha \to 0$, then (4.3) reduces to the classical result [1]

$$(4.4) \quad c^2 = \frac{4}{3} (KH)^3 \delta_z^2 (1 - \varepsilon_z^2/c_i^2) \frac{\rho}{\rho_0} \frac{1}{(1 + KH/\rho_0)}.$$

**Case 2.** For a short wavelength or large $KH$ both the wave velocity equations (3.14) and (3.18), symmetric and antisymmetric motions, respectively, approach the same limiting form

$$(4.5) \quad b \left[ \frac{a_1 \kappa_2}{\kappa_2 - \kappa_1} - \frac{a_2 \lambda_1 \kappa_1}{\lambda_2 (\kappa_2 - \kappa_1)} \right] - 4\mu^2 k^2 \lambda_1 \delta +$$

$$\quad + \frac{\rho_0 \omega^2 \delta}{v_0} \frac{1}{\lambda_2 (\kappa_2 - \kappa_1)} \left[ \frac{(2\mu k^2 - a_1) \kappa_2}{\kappa_2 - \kappa_1} - \frac{(2\mu k^2 - a_2) \lambda_1 \kappa_1}{\lambda_2 (\kappa_2 - \kappa_1)} \right] = 0.$$

We note that the first part of (4.5) represents the expression for the modified Rayleigh waves in a micropolar medium [12], and the second part represents the effect of the liquid on the propagation of modified Rayleigh waves. Therefore, the above expression corresponds to the modified Rayleigh waves on the floor (if we suppose the floor to be a micropolar medium) of the ocean.

For $\alpha \to 0$, (4.5) reduces to the classical result [1], as presented below

$$(4.6) \quad (2 - c^2/c_i^2)^2 - 4(1 - c^2/c_i^2)^{1/2} (1 - c^2/c_i^2)^{1/2} + \frac{\rho_0 \delta e^2}{\rho v_0} = 0.$$

The wave velocity equation (3.21) and (3.22) for symmetric and antisymmetric motions approaches the same limiting form for short wavelengths, or, $KH$ (large)

$$(4.7) \quad 4\gamma^2 k^2 \sigma (\lambda_1 d_2 - \lambda_2 d_1) = 4\gamma k^2 \sigma (e_1 \lambda_2 - c_2 \lambda_1) + [(2\gamma + \beta) \sigma^2 - \beta k^2] (e_1 d_2 - c_2 d_1),$$
which corresponds to the expression for modified Love waves in a micropolar medium [11]. From this we conclude that the liquid has no effect upon the propagation of twisted Love waves in a micropolar elastic medium. Again if \( c < c_a \) and the length of the wave is very small as compared with the thickness of the layer, then both Eqs (3.14) and (3.18) for symmetric and antisymmetric motions, respectively, reduce to

\[
(4.8) \quad (1 - \frac{c^2}{c_a^2})^{1/2} \left[ b \left( \frac{a_1 \kappa_2}{\kappa_2 - \kappa_1} - \frac{a_2 \kappa_1}{\kappa_2 - \kappa_1} \right) \frac{\lambda_1}{\lambda_3} - 4 \mu^2 k^2 \lambda_1 \delta \right] + \rho_a \omega^2 (1 - \frac{c^2}{c_a^2})^{1/2} \left[ \frac{(2 \mu k^2 - a_1) \kappa_3}{\kappa_2 - \kappa_1} - \frac{(2 \mu k^2 - a_2) \kappa_1}{\lambda_2 (\kappa_2 - \kappa_1)} \right] = 0.
\]

From (4.8) we get approximately

\[
(4.9) \quad \frac{c^2}{c_a^2} = 1 - \frac{\rho_a c_a^4 \left( 1 - \frac{c^2}{c_a^2} \right)^{1/2}}{\rho D'' c_4^2} \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} + \sqrt{\left( 1 - \frac{c^2}{c_4^2} \right) \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right)} \right)^2,
\]

where

\[
D'' = \left[ 2 - \frac{c^2}{c_a^2} \right] \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} + \sqrt{\left( 1 - \frac{c^2}{c_a^2} \right) \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right)} \right)^{1/2} \left( 1 + \frac{1}{2} \left( \frac{v^2 - \eta^2}{k^2} \right) \right) - \frac{c^2}{c_4} \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right) - 4 \sqrt{\left( 1 - \frac{c^2}{c_a^2} \right) \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right)} \right) + \left( 1 + \frac{1}{2} \left( \frac{v^2 - \eta^2}{k^2} \right) \right) - \frac{c^2}{c_4} \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right) - 4 \sqrt{\left( 1 - \frac{c^2}{c_a^2} \right) \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right)} \right)^{1/2} - \frac{c^2}{c_4} \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right) - 4 \sqrt{\left( 1 - \frac{c^2}{c_a^2} \right) \left( 1 - \frac{c^2}{c_4^2} + \frac{v^2 - \eta^2}{k^2} \right)} \right)^{1/2}.
\]

This equation will determine the phase velocity of Stoneley waves on the liquid micropolar solid interface. If the bed of the ocean is made of micropolar elastic solid and if the waves are propagated over the liquid-solid interface, then this phase velocity equation will play an important role in ascertaining the velocity of the waves and connected information.

If we make \( \alpha \to 0 \) then (4.9) reduced to the classical result [1]

\[
(4.11) \quad \frac{c^2}{c_a^2} = \frac{\rho_a c_a^4 \left( 1 - \frac{c^2}{c_a^2} \right)^{1/2}}{4 \rho D'' c_4^2 \left( 1 - \frac{c^2}{c_a^2} \right)^{1/2} \left( 1 - \frac{c^2}{c_4^2} \right)^{1/2} - \left( 1 - \frac{1}{2} \frac{c^2}{c_4^2} \right)^{1/2}}.
\]
5. Micropolar elastic layer in between two isotropic homogeneous solid half-spaces. Let us consider in this section that between two elastic half-spaces of the same isotropic homogeneous material a micropolar layer of thickness $2H$ has been placed, so that they are welded in contact (i.e., sufficiently rough to prevent any sliding on the common surface) at their common surface of separation. Let us take the origin on the middle plane of the micropolar elastic layer and the $x_3$-axis to be directed downwards. We assume that a monochromatic wave propagates along the $x_1$-axis.

Expressing the rotational components $\omega_1$, $\omega_3$ at any point of the micropolar elastic layer in terms of the potential functions $\theta$ and $\chi$ we have

\[
\omega_1 = \frac{\partial \theta}{\partial x_1} - \frac{\partial \chi}{\partial x_3}, \quad \omega_3 = \frac{\partial \theta}{\partial x_3} + \frac{\partial \chi}{\partial x_1}.
\]

With the help of Eqs (5.1), the equations of motion (2.3b) reduce to

\[
\begin{align*}
\left( \nabla_1^2 - \nu_1^2 - \frac{1}{c_3^2} \frac{\partial^2}{\partial t^2} \right) \theta &= 0, \\
\left( \nabla_1^2 - \nu_1^2 - \frac{1}{c_4^2} \frac{\partial^2}{\partial t^2} \right) \chi + \frac{2\eta}{\gamma + \eta} u_2 &= 0, \\
\left( \nabla_1^2 - \nu_1^2 - \frac{1}{c_5^2} \frac{\partial^2}{\partial t^2} \right) u_2 - \frac{2\eta}{\mu + \eta} \nabla_1^2 \chi &= 0.
\end{align*}
\]

From the final two equations of (5.2), eliminating $\chi$ or $u_2$, we get

\[
\left\{ \left[ \nabla_1^2 - \nu_1^2 - \frac{1}{c_4^2} \frac{\partial^2}{\partial t^2} \right] \left[ \nabla_1^2 - \frac{1}{c_5^2} \frac{\partial^2}{\partial t^2} \right] + \eta^2 \nabla_1^2 \right\} (\chi, u_2) = 0.
\]

To solve Eq. (5.2), we assume:

\[
\begin{align*}
\theta &= f(x_3) \, e^{i(\nu_1 t - \omega t)}, \\
\chi &= g(x_3) \, e^{i(\nu_1 t - \omega t)}, \\
u_2 &= h(x_3) \, e^{i(\nu_1 t - \omega t)}
\end{align*}
\]

and therefore,

\[
\begin{align*}
\theta &= [A \sin h(\sigma x_3) + B \cos h(\sigma x_3)] \, e^{i(\nu_1 t - \omega t)}, \\
\chi &= [C \sin h(\lambda_1 x_3) + D \cos h(\lambda_1 x_3) + E \sin h(\lambda_2 x_3) + F \cos h(\lambda_2 x_3)] \, e^{i(\nu_1 t - \omega t)}, \\
u_2 &= [C' \sin h(\lambda_1 x_3) + D' \cos h(\lambda_1 x_3) + E' \sin h(\lambda_2 x_3) + F' \cos h(\lambda_2 x_3)] \, e^{i(\nu_1 t - \omega t)}.
\end{align*}
\]

where the new notation is

\[
\nu_1^2 = \frac{4\eta}{\beta + 2\eta}.
\]
For the isotropic homogeneous solid half-space the equation of motion is

\[(5.7) \quad \left( \frac{\nu^2}{c_0^2} - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2} \right) u'_2 = 0. \]

Therefore,

\[
(5.8) \quad u'_2 = A'_0 e^{(kx_1 - \nu'_0 x_2 - \omega t)}, \quad \text{for the upper half-space,}
\]

\[= A'_2 e^{(kx_1 + \nu'_0 x_2 - \omega t)}, \quad \text{for the lower half-space.} \]

where \(\nu'_0 = (k - \omega^2/c'_2)^{1/2}\), \(c'_2\) is the distortional wave velocity, and \(u'_2\) is the transverse displacement component, while \(\mu', \rho'\) are the material constants for the elastic half-spaces.

**Boundary conditions.** On the plane surface of separation \(x_3 = \pm H\), the following boundary conditions are satisfied:

\[
\mu_{32} = \mu_{31} = 0,
\]

\[
s_{32} = s_{31}
\]

\[
u_3 = \nu_1
\]

where the stresses for the micropolar layer in terms of the potential functions \(\theta\) and \(\chi\) are

\[
s_{32} = (\mu + \alpha) \frac{\partial u_2}{\partial x_3} + 2\alpha \left( \frac{\partial \theta}{\partial x_1} - \frac{\partial \chi}{\partial x_2} \right),
\]

\[
(5.10) \quad s_{33} = (\beta + 2\gamma) \frac{\partial^2 \theta}{\partial x_3 \partial x_3} + \beta \left( \frac{\partial^2 \theta}{\partial x_1 \partial x_1} - \frac{\partial^2 \chi}{\partial x_1 \partial x_2} \right),
\]

\[
\mu_{31} = 2\gamma \frac{\partial^2 \theta}{\partial x_1 \partial x_3} + \gamma \left( \frac{\partial^2 \chi}{\partial x_1^2} - \frac{\partial^2 \chi}{\partial x_3^2} \right) - \sigma^2 \frac{\partial \chi}{\partial x_1}.
\]

Symmetric vibrations. Let the rotation \(\omega_1\), the stresses \(\mu_{11}, \mu_{22}, \sigma_{32}\) of the elastic micropolar layer are symmetric with respect to plane \(x_3 = 0\), and therefore

\[
A = D = F = D^* = F^* = 0
\]

\[
C' = \rho_1 C, \quad E' = \rho_2 E
\]

where \(\rho_r = \frac{p}{\lambda_r^2 - \kappa^2 + \sigma_r^2}, \quad p = \frac{2\alpha}{\mu + \alpha}; \quad r = 1, 2.\)

Using the boundary conditions (5.9), the relations (5.10), (5.8), (5.5) and (5.11), and taking \(A_0 = -A'_2\), we get a system of four homogeneous equations. Making the determinant of this system to be zero, we get the following characteristic equation of the problem considered

\[
(5.12) \quad 4k^2 \gamma^2 \sigma (\lambda_1 \lambda_2 - d_2 \lambda_1) \tan h (\sigma H) \tan h (\lambda_1 H) + a \left( c_1 d_2 - c_2 d_1 \frac{\tan h (\lambda_1 H)}{\tan h (\lambda_1 H)} \right) +
\]

\[
+ 4k^2 \gamma^2 \nu \chi (c_1 \lambda_2 - c_2 \lambda_1) \tan h (\lambda_2 H) + \mu' \nu'_0 \left[ 4k^2 \gamma^2 \sigma \times (\lambda_2 \rho_1 - \lambda_1 \rho_2) \frac{\tan h (\lambda_1 H)}{\tan h (\lambda_1 H)} \right] = 0,
\]

\[\times \left( \lambda_2 \rho_1 - \lambda_1 \rho_2 \frac{\tan h (\lambda_1 H)}{\tan h (\lambda_1 H)} \right) \frac{\tan h (\sigma H) - \tan h (\lambda_1 H)}{\tan h (\lambda_1 H) - \tan h (\lambda_2 H)} = 0\].
Antisymmetric vibrations. If we set

$$b = C = C' = E = E' = 0 \quad \text{and} \quad D' = D = D' = F = F',$$

then it is readily observed that the rotation component $a_1$ the stresses $\mu_{11}, \mu_{33}, \sigma_{32}$ are antisymmetric with respect to the plane $x_3 = 0$.

Now from the boundary conditions (5.9), and the expressions (5.10), (5.8), (5.5), (5.14), and also assuming $A'_0 = A'_2$ as before we get a system of four homogeneous equations with four unknown constants. Eliminating the unknown constants, we get the following period equation

$$4k^2 \gamma^2 \sigma (\lambda_2 - \lambda_1) \tan \frac{h (\lambda_1 H)}{h (\lambda_2 H)} + \left[ c_1 d_2 - c_2 d_1 \tan \frac{h (\lambda_1 H)}{h (\lambda_2 H)} \right] \tan \frac{h (\sigma H)}{h (\lambda_1 H)} +$$

$$4k^2 \gamma^2 \sigma \left[ c_1 \lambda_2 - c_2 \lambda_1 \right] \left[ \tan \frac{h (\lambda_1 H)}{h (\lambda_2 H)} \right] - \lambda_1 \rho_2 \tan \frac{h (\lambda_1 H)}{h (\lambda_2 H)}}{h (\lambda_1 H)} = 0.$$
[10] — , ibid., 16 (1968), 555 [899]

D K. Banerji, A P. Sengupta, Распространение волн в микрополярном упругом слое, уменьшенным в бесконечную жидкость.

Содержание. В работе исследуется эффект влияния жидкости на распространение двух характерных типов волн в микрополярном упругом слое. Для случаев симметричных и антисимметричных колебаний выводятся частотные уравнения и затем применяются для исследования конкретных случаев модифицированных поверхностных волн, а именно (а) волн Релея, (б) волн Лове, (в) волн Стоунли с учетом жидкости.