CHAPTER - III

UPPER AND LOWER $L_{S_k}$ INTEGRALS

1. Introduction and preliminaries

In this chapter we define upper and lower integrals of a function defined on a closed interval following the method of Perron of introducing major and minor functions. We show that when the upper and the lower integrals coincide we obtain an integral completely equivalent to the $L_{S_k}$-integral in Chapter I. Thus the upper and the lower integrals can be treated as the upper and the lower $L_{S_k}$-integrals. Some properties of such upper and lower integrals are established and in particular it is shown that these are $AC_{gk}$ on the interval. Finally we show that if the upper and lower integrals of a function are different from each other then the function is not $gk$-measurable.

We present the following results which will be needed in developing some results of the subsequent sections.
Theorem 1.1. If \( f(x) \) is \( \text{AC}_{g_k} \) on \( [a, b] \), then for \( x \in (a, b] \) and \( S \) \nexists, \( S \), \( S \)
\[
f(x) = f(a^+) + (\text{LS}_k) \int_a^x f'_g(t) \frac{d^k g(t)}{dt^{k-1}}.
\]

Proof. Since \( f(x) \) is \( \text{AC}_{g_k} \) on \( [a, b] \), by Theorem 1.3 of Chapter II, it is \( \text{BV}_{g_k} \) on \( [a, b] \). So, by Theorems 3.3 and 3.4 of Chapter II, \( f'_g(x) \) exists and is finite in \( [a, b] \) except a set of \( g_k \)-measure zero and \( f'_g(x) \) is summable \( (\text{LS}_k) \) on \( [a, b] \). We define the function \( g(x) \) by
\[
g(x) = f(a^+) + (\text{LS}_k) \int_a^x f'_g(t) \frac{d^k g(t)}{dt^{k-1}} \quad \text{for} \quad a \leq x \leq b
\]
\[
= f(a^+) \quad \text{for} \quad x < a
\]
\[
= f(b^-) \quad \text{for} \quad x > b.
\]

By Theorem 1.1 of Chapter I, \( g(x) \) is in class \( \cup \) and is \( \text{AC}_{g_k} \) on \( [a, b] \). By Theorem 1.3 of Chapter I, \( f(x) - g(x) = C \) (constant) on \( S \). Letting \( x \to a^+ \) over the points of \( S \) we see that \( f(a^+) - g(a^+) = C \). Now for \( x \in (a, b] \) and
\[
A = \{ a \}
\]
\[
g(x) = f(a^+) + \int_a f'_g(t) \frac{d^k g(t)}{dt^{k-1}} + \int_{(a, x]} f'_g(t) \frac{d^k g(t)}{dt^{k-1}}.
\]
If \( k^{-1} g(a) - b^{-1} g(a) \neq 0 \), then \( f'_{g_k}(a) = 0 \) and so in any case the first integral on the right of the above equality is zero. So, letting \( x \to a^+ \) over the points of \( S \) in the above equality we get \( g(a^+) = f(a^+) \), and hence \( C = 0 \). Thus

\[
f(x) = f(a^+) + (L_S) \int_a^x f'_{g_k}(t) \frac{dkg(t)}{dt} \, dt
\]

for all \( x \in [a, b] \cap S \) and the theorem is proved.

**Theorem 1.2.** If \( f(t) \in \mathcal{C}_g \) on \( [a, b] \) and if \( f(a^+) = 0 \), then

\[
f(x) = f_1(x) - f_2(x)
\]

for all \( x \in [a, b] \cap S \), where \( f_1(x) \) and \( f_2(x) \) belong to class \( \mathcal{C}_g \), are \( AC_g \) and non-decreasing on \( [a, b] \), and \( f_1(a^+) = f_2(a^+) = 0 \).

**Proof.** Since \( f(x) \) is \( AC_g \) on \( [a, b] \), then, by Theorem 1.3 of Chapter II, it is \( BV_g \) on \( [a, b] \). Again by Theorem 3.3 of Chapter II, \( f'_{g_k}(x) \) exists and is finite except a set of \( g_k \)-measure zero in \( [a, b] \). Let

\[
A = \{ x: a \leq x \leq b, f'_{g_k}(x) \text{ exists and finite} \}
\]
and write \( E_x = \left[a, x\right) \cap A \) so that \( \left| E_x \right|_{g_k} = \left| \left[a, x\right) \right|_{g_k} \).

Let the two functions \( p(x) \) and \( q(x) \) be defined on \( \left[a, b\right] \) as follows:

\[
p(x) = f'_{g_k}(x) \quad \text{if} \quad f'_{g_k}(x) > 0 \\
= 0 \quad \text{if} \quad f'_{g_k}(x) < 0,
\]

and

\[
q(x) = -f'_{g_k}(x) \quad \text{if} \quad f'_{g_k}(x) < 0 \\
= 0 \quad \text{if} \quad f'_{g_k}(x) > 0,
\]

for all \( x \in A \) and \( p(x) = q(x) = 0 \) for all \( x \in \left[a, b\right] - A \).

Then

\[
f'_{g_k}(x) = p(x) - q(x),
\]

for \( x \in A \) and \( p(x) \) and \( q(x) \) are two non-negative functions on \( \left[a, b\right] \). Since \( f(x) \) is BV on \( \left[a, b\right] \), then, by Theorem 3.4 of Chapter II, \( f'_{g_k}(x) \) is summable (LS) on \( \left[a, b\right] \). We define \( f_1(x) \) and \( f_2(x) \) on \( \left[a, b\right] \) by

\[
f_1(x) = \begin{cases} 
  f_1(a+) & \text{for } x < a, \\
  \int_a^x p(t) \frac{d}{dt} g(t) dt & \text{for } a \leq x \leq b, \\
  f_1(b-) & \text{for } x > b.
\end{cases}
\]
and

\[
f_2(x) = \begin{cases} 
  f_2(a^+) & \text{for } x < a, \\
  \int_a^x q(t) \frac{d^k g(t)}{dt^{k-1}} & \text{for } a \leq x \leq b, \\
  f_2(b^-) & \text{for } x > b.
\end{cases}
\]

Then, by Theorem 1.1 of Chapter I, \( f_1(x) \) and \( f_2(x) \) belong to class \( \mathcal{M} \) and are \( AC_{g_k} \) on \([a, b]\). Since \( p(x) \) and \( q(x) \) are non-negative then clearly \( f_1(x) \) and \( f_2(x) \) are non-decreasing on \([a, b]\) and so \( f_1(x), f_2(x) \) both are in class \( \mathcal{M} \). If \( x \in [a, b] \cap S \), then by Theorem 1.1 above

\[
f(x) = f(a^+) + \int_a^x f'_g(t) \frac{d^k g(t)}{dt^{k-1}}
\]

\[
= f(a^+) + \int_{E_x} \left( \int_{E_x} p(t) - q(t) \right) \frac{d^k g(t)}{dt^{k-1}},
\]

\[
= f(a^+) + \int_{E_x} p(t) \frac{d^k g(t)}{dt^{k-1}} - \int_{E_x} q(t) \frac{d^k g(t)}{dt^{k-1}},
\]

\[
= f(a^+) + f_1(x) - f_2(x).
\]

Since \( f(a^+) = 0 \) the theorem follows.
Theorem 1.3. If \( f(x) \in \mathcal{U} \) is \( AC_{gk} \)-below (or \( AC_{gk} \)-above) on \( [a, b] \), then on \( [a, b] \cap S \), \( f(x) \) can uniquely be represented in the form \( f(x) = \phi(x) + r(x) \), where \( \phi(x) \) is \( AC_{gk} \) on \( [a, b] \), \( \phi(a+) = f(a+) \), \( r(x) \) is continuous and non-decreasing on \( [a, b] \cap S \), and \( r'_{gk}(x) = 0 \) except a set of \( gk \)-measure zero in \( [a, b] \).

Proof. We prove the theorem when \( f(x) \) is \( AC_{gk} \)-below on \( [a, b] \). The other case is similar.

Since \( f(x) \) is \( AC_{gk} \)-below on \( [a, b] \), then, by Theorem 1.8 of Chapter II, \( f(x) \) is \( BV_{gk} \) on \( [a, b] \) and by Theorems 3.3 and 3.4 of Chapter II, \( f'_{gk}(x) \) is finite except a set of \( gk \)-measure zero and is summable (\( IS_k \)) on \( [a, b] \). Let

\[
\phi(x) = \begin{cases} 
  f(a+) & \text{for } x < a, \\
  f(a+) + \int_a^x f'_{gk}(t) \frac{d_{gk}(t)}{d^{k-1}} dt & \text{for } a < x \leq b, \\
  \phi(b-) & \text{for } x > b,
\end{cases}
\]

and \( r(x) = f(x) - \phi(x) \). Then \( f(x) = \phi(x) + r(x) \). By Theorems 1.1 and 1.2 of Chapter I, \( \phi(x) \) belongs to \( \mathcal{U} \) and is \( AC_{gk} \) on \( [a, b] \) and \( \phi'_{gk}(x) = f'_{gk}(x) \) except a
set of gk-measure zero in \([a, b]\). So \(r'_{gk}(x) = 0\) except a set of gk-measure zero in \([a, b]\). Also \(r(x)\) is \(AC_{gk}\)-below on \([a, b]\) and \(r(x)\) is continuous and non-decreasing on \([a, b] \cap S\). Now if \(A = \{a\}\), then

\[
\phi(x) = f(a+) + \int_a^x f'_{gk}(t) \frac{d^k g(t)}{dt^{k-1}} \, dt + \int_{(a, x]} f'_{gk}(t) \frac{d^k g(t)}{dt^{k-1}} \, dt
\]

If \(|A|_{gk} = D^{k-1}_+ g(a) - D^{k-1}_- g(a) \neq 0\), then \(f'_{gk}(a) = 0\) and so in any case

\[
\int_a^{a+} f'_{gk}(t) \frac{d^k g(t)}{dt^{k-1}} = 0.
\]

Letting \(x \to a^+\) over the points of \(S\) we have

\[
\phi(a+) = f(a+).
\]

Now we prove the uniqueness of this theorem. If possible, let \(f(x) = \phi_1(x) + r_1(x)\). This gives

\[
\phi(x) - \phi_1(x) = r_1(x) - r(x).
\]

Clearly \((\phi - \phi_1)'(x) = 0\) except a set of gk-measure zero in \([a, b]\) and \(\phi(x) - \phi_1(x)\) is \(AC_{gk}\) on \([a, b]\). Then, by Theorem 1.5 of Chapter I, \(\phi(x) - \phi_1(x) = C\), a constant, on \([a, b] \cap S\). Letting \(x \to a^+\) over the points of \(S\) we have \(C = 0\). Hence
\[ \phi(x) = \phi(x) \text{ on } [a, b] \cap S \]

and so \[ r(x) = r_1(x) \text{ on } [a, b] \cap S. \]

This proves the theorem.

Combining Theorems 1.2 and 1.3 we obtain

**Theorem 1.4.** If \( f(x) \in \mathcal{U}_0 \) is either AC\(_{gk}\) below (or above) on \([a, b]\) and if \( f(a^+) = 0 \), then

\[ f(x) = f_1(x) - f_2(x) \]

for all \( x \in [a, b] \cap S \) where \( f_1(x) \) and \( f_2(x) \) belong to class \( \mathcal{U}_0 \), are AC\(_{gk}\) and non-decreasing on \([a, b]\), and \( f_1(a^+) = f_2(a^+) = 0 \).

**Lemma 1.1.** If a function \( f(x) \) of class \( \mathcal{U}_0 \) is AC\(_{gk}\) below on \([a, b]\) and if \( f'_{gk}(x) > 0 \) except a set of \( gk \)-measure zero in \([a, b]\), then \( f(x) \) is non-decreasing on \([a, b] \cap S\).

**Proof.** Let \( \alpha, \beta \) be any two points of \([a, b] \cap S\) and

\[ E = \left\{ x : x \in (\alpha, \beta) \cap S \text{ and } 0 \leq f'_{gk}(x) < \infty \right\}. \]

In view of Theorems 1.8 and 3.3 of Chapter II, we have

\[ |E|_{gk} = \left| \frac{[\alpha, \beta]}{gk} \right|. \]

Let \( \varepsilon > 0 \) be arbitrary. If \( x \in E \) there is
a sequence \( \{ h_1 \} \), \( (h_1 > 0) \), such that \( h_1 \to 0 \), \( x + h_1 \in S \) and

\[
(1.1) \quad \frac{f(x+h_1) - f(x)}{B^{k-1}(x+h_1) - B^{k-1}g(x)} > f'_k(x) - \epsilon.
\]

Let \( F \) denote the family of open intervals \( (x, x+h_1) \) thus associated to the points of \( E \). Then, by Lemma 1.3 of Chapter I, there exists a finite number of non-overlapping open intervals \( (x_1, x'_1), (x_2, x'_2), \ldots, (x_n, x'_n) \) from \( F \) with \( x_i, x'_i \) belonging to \( (\alpha, \beta) \cap S \) for each \( i \) such that

\[
\sum_{i=1}^{n} |E \cap (x_i, x'_i)|_{g_k} > \left| [\alpha, \beta] \right|_{g_k} - \delta
\]

where \( \delta \) is any preassigned positive number. If \( \alpha < x_1 < x'_1 < x_2 < x'_2 < \cdots < x_n < x'_n < \beta \) and if we write \( \alpha = x'_0, \beta = x'_{n+1} \), then

\[
\sum_{i=1}^{n+1} \left| \int_{x_{i-1}}^{x'_i} \right|_{g_k} < \delta.
\]

Since \( f(x) \) is \( AC_{g_k} \)-below then the number \( \delta \) can be chosen such that

\[
(1.2) \quad \sum_{i=1}^{n+1} \left| \int f(x_i) - f(x'_{i-1}) \right| > -\epsilon.
\]
Combining (1.1) and (1.2) we obtain

\[ f(\beta) - f(\alpha) > -\varepsilon + \sum_{i=1}^{n} \left[ f'_{gk}(x) - \varepsilon \right] \left| J_{x_i, x'_i} \right|_{gk} \]

> - \varepsilon - \varepsilon \left[ [\alpha \ beta] \right]_{gk}.

Arbitrariness of \( \varepsilon > 0 \) gives \( f(\beta) > f(\alpha) \). Since this is true for arbitrary \( \alpha, \beta(>\alpha) \) in \( [a,b] \cap S \), the lemma is proved.

2. The \( \mathcal{L}_{gk} \)-integral

Definition 2.1. Let \( f(x) \) be defined on \( [a,b] \). A function \( M(x) \in \mathcal{M} \) is said to be a \( \mathcal{L}_{gk} \)-major function of \( f(x) \) on \( [a,b] \) if

(i) \( M(a-) = 0 \),

(ii) \( M(x) \) is AC\( gk \)-below on \( [a,b] \),

(iii) \( D_{gk}M(x) > f(x) \) except a set of \( gk \)-measure zero in \( S_3 \cup D \).

A \( \mathcal{L}_{gk} \)-minor function \( m(x) \) is defined similarly. Throughout the chapter we shall call a \( \mathcal{L}_{gk} \)-major (respectively \( \mathcal{L}_{gk} \)-minor) function simply a major (respectively minor) function of \( f \) on \( [a,b] \).
Lemma 2.1. If \( M(x) \) is a major function and \( m(x) \) is a minor function of \( f(x) \) on \( [a, b] \), then \( M(x) - m(x) \) is non-decreasing on \( [a, b] \setminus S \), and \( M(b+) \geq m(b+) \).

Proof. In view of Theorem 1.8 of Chapter II, \( M(x) \), \( m(x) \) are \( BV_{gk} \) on \( [a, b] \). Hence, by Theorem 3.3 of Chapter II, the \( gk \)-derivatives of \( M(x) \) and \( m(x) \) exist finitely in \( [a, b] \) except a set of \( gk \)-measure zero. Now

\[
R'_{gk}(x) = M'_{gk}(x) - m'_{gk}(x) = \frac{dM_{gk}(x)}{dx} - \frac{dm_{gk}(x)}{dx} > 0
\]

except a set of \( gk \)-measure zero. Since \( R(x) \) is \( AC_{gk} \)-below on \( [a, b] \) and \( R'_{gk}(x) > 0 \) except a set of \( gk \)-measure zero, then by Lemma 1.1 above, \( R(x) \) is non-decreasing on \( [a, b] \setminus S \). So if \( \xi \) and \( \eta \) are any two points of \( [a, b] \setminus S \) with \( \xi < \eta \) we have \( R(\xi) \leq R(\eta) \). Proceeding to the limit as \( \xi \to a^+ \) and \( \eta \to b^- \) over \( S \), we have \( M(b+) \geq m(b+) \) and this proves the theorem.

Definition 2.2. Let the function \( f(x) \) possess major functions \( M(x) \) on \( [a, b] \). We define the function \( U(x) \) by

\[
U(x) = \inf \{ M(x) \} \quad \text{for } x \in [a, b] \setminus S.
\]

Clearly \( U(b^-) \) exists. For \( x < a \), we take \( U(x) = 0 \) and for
we take $U(x) = U(b-)$. This function $U(x)$ is said to be the upper integral function of $f(x)$ on $[a, b]$.

Similarly if the function $f(x)$ possesses minor functions $m(x)$ on $[a, b]$ and if we define the function $L(x)$ by

$$L(x) = \begin{cases} 0 & \text{for } x < a, \\ \sup \{m(x)\} & \text{for } x \in [a, b] \cap S, \\ L(b-) & \text{for } x > b. \end{cases}$$

Then $L(x)$ is said to be the lower integral function of $f(x)$ on $[a, b]$.

If $f(x)$ has a major function $M(x)$ and a minor function $m(x)$, it has both the upper and the lower integral functions and

$$m(x) \leq U(x) \leq M(x) \quad \text{and} \quad m(x) \leq L(x) \leq M(x).$$

Throughout the section we assume that both major and minor functions of $f(x)$ exists.

**Theorem 2.1.** If $M(x)$ and $m(x)$ are respectively a major function and a minor function of $f(x)$ on $[a, b]$ and if $U(x)$ and $L(x)$ are the upper and the lower integral functions of $f(x)$, then each of the differences $M(x) - U(x)$ and $L(x) - m(x)$ is non-decreasing on $[a, b] \cap S$. 
Proof. We show that $M(x) - U(x)$ is non-decreasing on $[a, b] \cap S$. The other case is similar. Let $x_1$ and $x_2$ be any two points in $[a, b] \cap S$ and $a \leq x_1 < x_2 \leq b$. To each $\varepsilon > 0$ there exists a major function $M_1(x)$ of $f(x)$ on $[a, b]$ such that

$$M_1(x_1) < U(x_1) + \varepsilon.$$  

The function $M_2(x)$ defined by

$$M_2(x) = \begin{cases} 
0 & \text{for } x < a, \\
M_1(x) & \text{for } x \in [a, x_1] \cap S, \\
M_1(x_1) + M(x) - M(x_1) & \text{for } x \in [x_1, b] \cap S, \\
M_2(b-)) & \text{for } x > b 
\end{cases}$$

is also a major function function of $f(x)$. Hence

$$M_2(x_2) \geq U(x_2)$$

i.e., $U(x_2) \leq M_1(x_1) + M(x_2) - M(x_1)$

i.e., $M(x_2) - U(x_2) \geq M(x_1) - U(x_1) - \varepsilon$ .

As $\varepsilon > 0$ is arbitrary, it follows that $M(x) - U(x)$ is non-decreasing on $[a, b] \cap S$, and so the theorem is proved.
Theorem 2.2. If $U(x)$ and $L(x)$ are the upper and the lower integral functions of $f(x)$ on $[a, b]$, then $U(x) - L(x)$ is non-decreasing on $[a, b] \cap S$.

Proof. Let $x_1$ and $x_2$ be any two points in $[a, b] \cap S$ with $x_1 < x_2$. To each $\varepsilon > 0$ arbitrary, there is a major function $M(x)$ and a minor function $m(x)$ such that

$$M(x_2) < U(x_2) + \varepsilon/2 \quad \text{and} \quad m(x_2) > L(x_2) - \varepsilon/2.$$

Now

$$U(x_1) - L(x_1) \leq M(x_1) - m(x_1) \leq M(x_2) - m(x_2), \quad \text{by Lemma 2.1}$$

$$\leq U(x_2) - L(x_2) + \varepsilon.$$

Arbitrariness of $\varepsilon > 0$ proves the theorem.

Theorem 2.3. The upper integral function $U(x)$ associated to a function $f(x)$ on $[a, b]$ is in class $\mathcal{U}_0$ and is $AC_{gk}$-below on $[a, b]$ and the lower integral function $L(x)$ associated to $f(x)$ belongs to class $\mathcal{U}_0$ and is $AC_{gk}$-above on $[a, b]$.

Proof. Let $\{\varepsilon_n\}$ be a sequence of positive constant terms such that $\varepsilon_n \to 0$ as $n \to \infty$ and let for each $\varepsilon_n$, $M_n(x)$ be a major function of $f(x)$ on $[a, b]$ such that for $x \in [a, b] \cap S$...
Hence for \( x \in [a, b] \cap S \), \( U(x) \) is the limit of a uniformly convergent sequence of functions \( \{ M_n(x) \} \). Further every member of \( \{ M_n(x) \} \) belongs to class \( \cap \cup_o \) and so each \( M_n(x) \) is continuous on \([a, b] \cap S\). Consequently \( U(x) \) is continuous on \([a, b] \cap S\) and \( U(x) \in \cap \cup_o \). Let \( M(x) \) be a major function of \( f(x) \) on \([a, b] \) such that

\[
M(b^-) - U(b^-) < \varepsilon/2,
\]

where \( \varepsilon > 0 \) is arbitrary. Since \( M(x) \) is \( AC_{g_k} \)-below on \([a, b]\) there exists a \( \delta > 0 \) such that for any elementary system

\[
\left\{ (x_i, x'_i) \right\} \text{ in } [a, b],
\]

with

\[
\sum_{i} \int_{x_i}^{x'_i} \left( \frac{d^{k-1} g(x'_i)}{d^{k-1} g(x_i)} - \frac{d^{k-1} g(x_i)}{d^{k-1} g(x'_i)} \right) < \delta
\]

the relation

\[
\sum_{i} \int_{M(x'_i)}^{M(x_i)} - M(x_i^-) < \varepsilon/2
\]

holds. Then

\[
\sum_{i} \left[ \int_{x_i}^{x'_i} U(x_i^-) - U(x_i^-) \right] = \sum_{i} \left[ \int_{M(x'_i)}^{M(x_i)} - M(x_i^-) \right]
\]

\[
= \sum_{i} \int_{M(x'_i)}^{M(x_i)} - M(x_i^-) - \sum_{i} \left( \int_{M(x'_i)}^{M(x_i)} - U(x'_i) \right) - \int_{M(x'_i)}^{M(x_i)} - U(x_i^-)
\]
and so \( U(x) \) is \( AC_{gk} \)-below on \( [a, b] \). Similar is the case for \( L(x) \). This proves the theorem.

**Corollary 2.1.** If \( f(x) \) is bounded on \( [a, b] \), then its upper and lower integral functions are \( AC_{gk} \) on \( [a, b] \).

**Proof.** If \( A < f(x) < B, x \in [a, b] \), then \( A \sum_{n=1}^{\infty} [D_{+}^{k-1} g(x) - D_{-}^{k-1} g(a)] \) and \( B \sum_{n=1}^{\infty} [D_{+}^{k-1} g(x) - D_{-}^{k-1} g(a)] \) are respectively a minor and a major function of \( f(x) \) on \( [a, b] \). Let \( U(x) \) be the upper integral function of \( f(x) \). Then by Theorem 2.1, \( U(x) - B \sum_{n=1}^{\infty} [D_{+}^{k-1} g(x) - D_{-}^{k-1} g(a)] \) is non-increasing on \( [a, b] \). Let

\[
(2.1) \quad \eta(x) = U(x) - B \sum_{n=1}^{\infty} [D_{+}^{k-1} g(x) - D_{-}^{k-1} g(a)]
\]

Then clearly \( \eta(x) \in \mathcal{U}_{c} \) and \( \eta(x) \) is non-increasing on \( [a, b] \). From (2.1), it is not difficult to show that \( U(x) \) is \( AC_{gk} \)-above on \( [a, b] \) and so by Theorem 2.3, \( U(x) \) is \( AC_{gk} \) on \( [a, b] \).

Similarly, it can be shown that the lower integral function is also \( AC_{gk} \) on \( [a, b] \), and thus the corollary is proved.
Theorem 2.4. The upper integral function is a major function and the lower integral function is a minor function.

Proof. Let \( U(x) \) be the upper integral function of \( f(x) \) on \([a, b] \). We first show that

\[
\mathcal{D} U_{gk}(x) \geq f(x)
\]

except a set of \( gk \)-measure zero in \([a, b] \). Assume on the contrary that

\[
\mathcal{D} U_{gk}(x) < f(x),
\]

on a set of positive outer \( gk \)-measure. In view of Theorem 2.3 above and Theorem 1.8 of Chapter II, \( U(x) \) is \( BV_{gk} \) on \([a, b] \) and so by Theorem 3.3 of Chapter II, the \( gk \)-derivative of \( U(x) \), \( U'_{gk}(x) \), exists finitely except a set of \( gk \)-measure zero in \([a, b] \). Then there exists a positive number \( \varepsilon \) such that

\[
f(x) - U'_{gk}(x) > \varepsilon
\]

at the points of a set \( E \subset S_3 \cup D \), where \( |E|_{gk} > 0 \). We first assume that

\[
|E \cap S_3|_{gk} = p > 0.
\]
Let $M(x)$ be a major function of $f(x)$ such that

$$(2.2) \quad M(h-) - U(h-) < (1/4)p \epsilon.$$ 

Put

$$R(x) = M(x) - U(x).$$

Then clearly $R(x) \in \mathcal{U}_0$ and by Theorem 2.1, $R(x)$ is non-decreasing on $[a, b] \cap S$. Also by Theorems 1.8 and 3.3 of Chapter II, $M'_{gk}(x)$ exists finitely except a set of $gk$-measure zero. Then from the relation

$$R'_{gk}(x) = M'_{gk}(x) - U'_{gk}(x),$$

$R'_{gk}(x)$ also exists finitely except a set of $gk$-measure zero in $[a, b]$. The set $E_1 \subset S_2$ where

$$M'_{gk}(x) - U'_{gk}(x) > \epsilon$$

is $gk$-measurable and $E_1$ contains $E \cap S^2$ with the possible exception of a set of $gk$-measure zero. Hence

$$|E_1|_{gk}^0 > |E \cap S^2|_{gk}^0$$

i.e.

$$|E_1|_{gk} > p.$$
Let $H \subset E_1$ be a closed set with

\[(2.3) \quad |H|_{gk} > p/2.\]

Let $[a, b]$ be the smallest closed interval containing $H$. If $x \in H$ there exists a null sequence $\{h_i\}$, $h_i > 0$, $x + h_i \in S$, such that

\[R_{gk}(x) > \varepsilon.\]

This gives

\[(2.4) \quad R(x+h_i) - R(x) > \varepsilon \leq g(x+h_i) - g(x).\]

Let $F$ denote the family of closed intervals $[x, x+h_i]$ thus associated with the set $H$. Then there exists a finite family of pairwise disjoint closed intervals $\Delta_i = [x_i, x_i + l_i]$,

\[i = 1, 2, \ldots, n\] of $F$ for which

\[(2.5) \quad \sum_{i=1}^{n} |\Delta_i|_{gk} > |H|_{gk} - p/4.\]

We may suppose that $x_1 < x_2 < \ldots < x_n$, $x_i = x_i$, $x_{n+1} = \beta$. Then $x_i + l_i < x_{i+1}$, $i = 1, 2, \ldots, n-1$. We have

\[R(\beta) - R(\alpha) \geq \sum_{i} \{R(x_i + l_i) - R(x_i)\} \]
\[ \varepsilon(\vert H \vert_{gk} - (1/4)p), \text{ using (2.4) and (2.5)} \]

\[ (1/4)p \varepsilon, \text{ by (2.3).} \]

Since \( R(x) \) is non-negative and non-decreasing on \( \overline{[a,b]} \) it follows that \( M(b-) - U(b-) > (1/4)p \), which contradicts (2.2).

If we assume that \( E \cap D \) is different from a gk-null set we arrive at a similar contradiction. Hence

\[ D U_{gk}(x) \succ f(x) \]

except a set of gk-measure zero. Also \( U(a-) = 0 \) and by Theorem 2.3, \( U(x) \) is \( AC_{gk} \)-below on \( \overline{[a,b]} \). Hence \( U(x) \) is a major function of \( f(x) \) on \( \overline{[a,b]} \).

Similarly we can show that the lower integral function is a minor function of \( f(x) \) on \( \overline{[a,b]} \) and the theorem is proved.

**Definition 2.3**: If \( f(x) \) has major and minor functions and if \( L(x) \) and \( U(x) \) are its lower and upper integral functions on \( \overline{[a,b]} \), then \( L(b+) \) and \( U(b+) \) are finite and \( U(b+) \succ L(b+) \).

If \( U(b+) = L(b+) \), we say that \( f(x) \) is \( \xi_{gk} \)-integrable on \( \overline{[a,b]} \). The common value is called the \( \xi_{gk} \)-integral and is denoted by
More generally, we say that $L(b^+)$ and $U(b^+)$ are respectively the lower and the upper $\mathcal{S}_k$-integrals of $f(x)$ on $[a, b]$. From the definition of $\mathcal{S}_k$-integral the following two lemmas are immediate.

**Lemma 2.2.** A necessary and sufficient condition that the function $f(x)$ be $\mathcal{S}_k$-integrable on $[a, b]$ is that to every $\varepsilon > 0$ there correspond a major function $M(x)$ and a minor function $m(x)$ such that

$$M(b^+) - m(b^+) < \varepsilon.$$ 

**Lemma 2.3.** If the $\mathcal{S}_k$-integral of $f(x)$ exists on $[a, b]$, then it exists on every closed subinterval of $[a, b]$.

**Definition 2.4.** If $f(x)$ is $\mathcal{S}_k$-integrable on $[a, b]$, then the function $G(x)$ defined by

$$G(x) = \begin{cases} 
0 & \text{for } x < a , \\
(\mathcal{S}_k)^k \left[ f(t) \frac{d^k g(t)}{dt^{k-1}} \right]_a^x & \text{for } x \in [a, b], \\
G(b^-) & \text{for } x > b 
\end{cases}$$
is called the indefinite $\mathcal{L}_k$-integral of $f(x)$. We next show that the $\mathcal{L}_k$-integral is identical with the $\mathcal{L}_k$-integral in Chapter I.

**Theorem 2.5.** If $f(x)$ is summable ($\mathcal{L}_k$) on $[a, b]$, then $f(x)$ is $\mathcal{L}_k$-integrable on $[a, b]$, and

\[
(\mathcal{L}_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = (\mathcal{L}_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

**Proof.** We write

\[
F(x) = \begin{cases} 
0 & \text{for } x < a, \\
(\mathcal{L}_k) \int_a^x f(t) \frac{d^k g(t)}{dt^{k-1}} & \text{for } x \in [a, b], \\
F(b-) & \text{for } x > b.
\end{cases}
\]

Then, by Theorems 1.1 and 1.2 of Chapter I, $F(x)$ is in class $\mathcal{L}_k$ and is $AC_{g_k}$ on $[a, b]$ and $F'_g(x) = f(x)$ except a set of $g_k$-measure zero in $[a, b]$. Thus $F(x)$ is both a $\mathcal{L}_k$-major function and a $\mathcal{L}_k$-minor function of $f(x)$ on $[a, b]$ and so $F(x)$ is also the upper and lower $\mathcal{L}_k$-integral functions of $f(x)$. Hence $f(x)$ is $\mathcal{L}_k$-integrable on $[a, b]$ and

\[
(\mathcal{L}_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = F(b+) = (\mathcal{L}_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

This proves the theorem.
Theorem 2.6. If $f(x)$ is $\Delta_{k}^{k}$-integrable on $[a, b]$, then $f(x)$ is summable $(\text{LS}_{k})$ on $[a, b]$, and

\[
\left(\text{LS}_{k}\right) \int_{a}^{b} f(x) \frac{d^{k}g(x)}{dx^{k-1}} = \left(\Delta_{k}^{k}\right) \int_{a}^{b} f(x) \frac{d^{k}g(x)}{dx^{k-1}}.
\]

Proof. Since $f(x)$ is $\Delta_{k}^{k}$-integrable on $[a, b]$, the upper and lower integral functions associated to $f(x)$ coincide at points where $B^{k-1}g$ exists. Hence if we write

\[
G(x) = \begin{cases} 
0 & \text{for } x < a, \\
\left(\Delta_{k}^{k}\right) \int_{a}^{x} f(t) \frac{d^{k}g(t)}{dt^{k-1}} & \text{for } x \in [a, b], \\
G(b-) & \text{for } x > b,
\end{cases}
\]

then $G(x) = L(x) = U(x)$ for $x \in [a, b] \cap S$. So by Theorem 2.3, $G(x)$ is in class $\mathcal{N}_{0k}$ and is $AC_{gk}$ on $[a, b]$. By Theorem 1.3 of Chapter II, it is therefore $BV_{gk}$ on $[a, b]$ and hence by Theorem 3.4 of Chapter II, $G'_{gk}(x)$ is summable $(\text{LS}_{k})$ on $[a, b]$. Since, by Theorem 1.4 of Chapter I, $G'_{gk}(x) = f(x)$ in $[a, b]$ except a set of $gk$-measure zero, $f(x)$ is also summable $(\text{LS}_{k})$ on $[a, b]$. Write
\[ F(x) = \begin{cases} 0 & \text{for } x < a, \\ (L_{S_k}) \int_a^x f(t) \frac{d^k g(t)}{dt^k-1} & \text{for } x \in [a, b], \\ F(b-) & \text{for } x > b. \end{cases} \]

Then, by Theorem 1.4 of Chapter I, \( F'_{gk}(x) = f(x) \) except a set of \( gk \)-measure zero in \([a, b] \). Hence, by Theorem 1.3 of Chapter I,

\[ F(x) - G(x) = c, \text{ constant, on } S. \]

Letting \( x \rightarrow a^+ \) over the points of \( S \) we get \( c = 0 \).

So

\[ F(x) = G(x) \text{ for } x \in [a, b] \cap S. \]

Therefore

\[ (L_{S_k}) \int_a^b f(x) \frac{d^k g(x)}{dx^k-1} = F(b+) = G(b+) = (L_{S_k}) \int_a^b f(x) \frac{d^k g(x)}{dx^k-1}, \]

and the theorem is proved.

Combining Theorems 2.5 and 2.6, we obtain

Theorem 2.7. \( f(x) \) is summable \((L_{S_k})\) on \([a, b] \) if, and only if, \( f(x) \) is \( \Delta S_k \)-integrable.
Theorem 2.7 helps in establishing many properties of \( \text{LS}_k \)-integral very simply. We submit a little account of them in the next section.

3. Upper and lower \( \text{LS}_k \)-integrals

We denote \( U(b^+) \) and \( L(b^+) \) as the upper and lower \( \text{LS}_k \)-integrals of \( f(x) \) on \( [a, b] \) with the notations

\[
U(b^+) = \int_a^b f(x) \frac{d^{k-1}g(x)}{dx^{k-1}} \quad \text{and} \quad L(b^+) = \int_a^b f(x) \frac{d^{k-1}g(x)}{dx^{k-1}}.
\]

Theorem 3.1. If the function \( f(x) \) has upper and lower \( \text{LS}_k \)-integrals on \( [a, b] \), then the function \( |f(x)| \) has also upper and lower \( \text{LS}_k \)-integrals on \( [a, b] \).

Proof. To prove this we will show that \( |f(x)| \) has major and minor functions on \( [a, b] \). Let \( L(x) \) and \( U(x) \) be the lower and the upper integral functions of \( f(x) \) on \( [a, b] \). Then, by Theorem 2.3, \( L(x) \) is \( \text{AC}_{gk} \)-above on \( [a, b] \) and \( U(x) \) is \( \text{AC}_{gk} \)-below on \( [a, b] \) and \( L(x) \) and \( U(x) \) belong to the class \( \mathcal{L}_p \). Then, by Theorem 1.4 above \( L(x) \) and \( U(x) \) can be expressed in the forms

\[
L(x) = \alpha(x) - \beta(x), \quad U(x) = \theta(x) - \phi(x),
\]
for all $x \in [a, b] \cap S$, where $\alpha(x), \beta(x), \theta(x), \varphi(x)$ are in class $\mathcal{L}$ and are non-decreasing on $[a, b]$ and $\alpha(a^+) = \beta(a^+) = \theta(a^+) = \varphi(a^+) = 0$. Then except a set of $g$-measure zero in $[a, b]$ we have

\[(3.1) \quad \beta'_{gk}(x) > -L'_{gk}(x) > -f(x),\]

and

\[(3.2) \quad \theta'_{gk}(x) > U'_{gk}(x) > f(x).\]

If we write $M(x) = \beta(x) + \theta(x)$, then, from (3.1) and (3.2), we have

$$D_{gk}(x) > |f(x)|$$

except a set of $g$-measure zero in $[a, b]$. Since $M(x)$ is $AC_{gk}$ below on $[a, b]$ then clearly we say that $M(x)$ is a major function for $|f(x)|$ on $[a, b]$.

The function $m(x) = 0$ is a minor function for $|f(x)|$ on $[a, b]$. Hence $|f(x)|$ has upper and lower $L_{gk}^*$-integrals, and the theorem is proved.

**Theorem 3.2.** If $f_1(x)$ and $f_2(x)$ has upper and lower $L_{gk}^*$-integrals on $[a, b]$ and
\[ f(x) = f_1(x) + f_2(x), \]

then \( f(x) \) has upper and lower \( \text{LS}_k \)-integrals on \( [a, b] \), and

\[
(3.3) \quad \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} \leq \int_a^b f_1(x) \frac{d^k g(x)}{dx^{k-1}} + \int_a^b f_2(x) \frac{d^k g(x)}{dx^{k-1}},
\]

and

\[
(3.4) \quad \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} \geq \int_a^b f_1(x) \frac{d^k g(x)}{dx^{k-1}} + \int_a^b f_2(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

If one of the functions \( f_1(x) \) and \( f_2(x) \) is \( \text{LS}_k \)-integrable on \( [a, b] \), then each of the inequalities (3.3) and (3.4) becomes an equality.

Proof: Let \( L_1(x) \) and \( L_2(x) \) be the lower integral functions of \( f_1(x) \) and \( f_2(x) \) on \( [a, b] \) respectively. Then, by Theorem 2.3, \( L_1(x) \) and \( L_2(x) \) belong to class \( \Lambda_0 \) and are \( \text{AC}_{gk}^- \) above on \( [a, b] \). Also by Theorem 2.4, \( L_1(x) \) and \( L_2(x) \) are minor functions of \( f_1(x) \) and \( f_2(x) \) respectively. If we write

\[ m(x) = L_1(x) + L_2(x), \]

then \( m(x) \) belongs to class \( \Lambda_0 \) and \( m(x) \) is \( \text{AC}_{gk}^- \) above on \( [a, b] \) and \( m(a-) = 0 \). Also
\[ \bar{D}_{m_{gk}}(x) \leq \bar{D}_{L_{gk}}(x) + \bar{D}_{L_{gk}^2}(x) \leq f_1(x) + f_2(x) \]

except a set of \( g_k \)-measure zero in \( [a, b] \). Hence \( m(x) \) serves as a minor function of \( f(x) \) on \( [a, b] \). Similarly we can show that the function

\[ M(x) = U_1(x) + U_2(x), \]

where \( U_1(x) \) and \( U_2(x) \) are the upper integral functions of \( f_1(x) \) and \( f_2(x) \) respectively, is a major function of \( f(x) \) on \( [a, b] \). Therefore the function \( f(x) \) has upper and lower \( L_{S_k} \)-integrals on \( [a, b] \), and

\[ \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} \leq \int_a^b f_1(x) \frac{d^k g(x)}{dx^{k-1}} + \int_a^b f_2(x) \frac{d^k g(x)}{dx^{k-1}}, \]

\[ \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} \geq \int_a^b f_1(x) \frac{d^k g(x)}{dx^{k-1}} + \int_a^b f_2(x) \frac{d^k g(x)}{dx^{k-1}}. \]

Now suppose that \( f_2(x) \) is \( L_{S_k} \)-integrable on \( [a, b] \) so that

\[ U_2(x) = L_2(x), \quad x \in [a, b] \cap S. \]

Hence \( U_2(x) \) is \( \text{AC}_{g_k} \) on \( [a, b] \) and by Theorem 1.2 of Chapter I,

\[ U_{2_{gk}}(x) = f_2(x). \]
except a set of \(g_k\)-measure zero in \(\overline{a, b}\). Let \(L(x)\) be the lower integral function of \(f(x)\) and

\[
p(x) = L(x) - L_2(x).
\]

Then \(p(x)\) is \(AC_{g_k}\)-above on \(\overline{a, b}\) and

\[
p'_k(x) = L'_{g_k}(x) - L'_{g_k}(x) = L'_{g_k}(x) - U'_{g_k}(x)
\]

\[\leq f_1(x) + f_2(x) - f_2(x)
\]

\[= f_1(x)
\]

except a set of \(g_k\)-measure zero in \(\overline{a, b}\). Hence \(p(x)\) serves as a minor function of \(f_1(x)\) on \(\overline{a, b}\) and so

\[
(3.5) \quad \int_{a}^{b} f_1(x) \frac{d^k g(x)}{dx^{k-1}} \geq \int_{a}^{b} f(x) \frac{d^k g(x)}{dx^{k-1}} - \int_{a}^{b} f_2(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

From the relations (3.4) and (3.5) we have

\[
\int_{a}^{b} f(x) \frac{d^k g(x)}{dx^{k-1}} = \int_{a}^{b} f_1(x) \frac{d^k g(x)}{dx^{k-1}} + \int_{a}^{b} f_2(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

Similar is the case for the upper integral functions. This proves the theorem.
For an unbounded function \( f(x) \) defined on \([-a,b]\) we introduce the functions \( f_n(x) \) and \( f_{-n}(x) \) for every natural number \( n \) by the rules

\[
f_n(x) = \min \{ f(x), n \} \quad \text{and} \quad f_{-n}(x) = \max \{ f(x), -n \}.
\]

**Theorem 3.3.** If the unbounded function \( f(x) \) has upper and lower \( L^k \)-integrals on \([-a,b]\), then its upper and lower integral functions are \( AC^k \) on \([-a,b]\).\)

\[
\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_{-n}(x) \frac{d^k g(x)}{dx^{k-1}},
\]

and

\[
\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_{-n}(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

**Proof.** Let \( L(x) \) and \( U(x) \) be the lower and the upper integral functions of \( f(x) \) on \([-a,b]\).

**Case (i).** \( f(x) \) is bounded below on \([-a,b]\).

In this case the function \( f_n(x) \) is bounded on \([-a,b]\) and by Corollary 2.1, its upper integral function \( U_n(x) \) is \( AC^k \) on \([-a,b]\). Now consider the sequence of functions \( \{ U_n(x) \} \), \( x \in [-a,b] \) \( \cap S \). By Theorem 2.4, \( U_{n+1}(x) \) is a major function of \( f_{n+1}(x) \) and since \( f_{n+1}(x) \geq f_n(x) \) it follows that \( U_{n+1}(x) \) is a major function of \( f_n(x) \) for every \( n \). Consequently,
\[ U_{n+1}(x) - U_n(x) \geq 0 \]

for every \( x \in [a, b] \cap S \) and for every value of \( n \). Therefore the sequence \( \{ U_n(x) \} \) is non-decreasing. Again, by Theorem 2.4, the function \( U(x) \) is a major function of \( f(x) \). Since \( f(x) \geq f_n(x) \) for each natural number \( n \), \( U(x) \) serves as a major function for \( f_n(x) \) for every positive integral value of \( n \), and so

\[ U(x) \geq U_n(x) \]

for every \( x \in [a, b] \cap S \) and for every value of \( n \). Let

\[
\lim_{n \to \infty} U_n(x) = X(x).
\]

Then

(3.6) \[ U(x) \geq X(x), \quad x \in [a, b] \cap S. \]

Define \( V_n(x) = X(x) - U_n(x) \). Let \( x_1 \) and \( x_2 \) be any two points of \([a, b] \cap S\), and let \( \epsilon > 0 \) be arbitrary. There exists positive integer \( p \) such that

\[ X(x_1) < U_{n+p}(x_1) + \epsilon. \]

Then since \( U_{n+p}(x) - U_n(x) \) is non-decreasing for each positive integral value of \( n \) and \( p \), it follows that

\[ V_n(x_1) = X(x_1) - U_n(x_1) < U_{n+p}(x_1) + \epsilon - U_n(x_1) \]

\[ \leq U_{n+p}(x_2) - U_n(x_2) + \epsilon < X(x_2) - U_n(x_2) + \epsilon. \]
As ε > 0 is arbitrary we have \( V_n(x_1) \leq V_n(x_2) \) and so the function \( V_n(x) \) is non-decreasing on \( [-a, b] \cap S \). This shows that the sequence \( \{ U_n(x) \} \) converges uniformly to \( X(x) \). Hence \( X(x) \in \sim U_0 \) and so \( V_n(x) \) is in class \( \sim U_0 \) and is \( AC_{gk} \) below on \( [-a, b] \). Consequently \( X(x) \) is \( AC_{gk} \) below on \( [-a, b] \).

Next let \( n \) be such that

\[
X(b-) - U_n(b-) < \frac{\varepsilon}{2}.
\]

Since by Corollary 2.1, \( U_n(x) \) is \( AC_{gk} \) above on \( [-a, b] \), there exists \( \delta > 0 \) such that for any elementary system \( I = \{ (x_i, x_i') \} \) on \( [-a, b] \) with

\[
\sum_i \int_{x_i'}^{x_i} g(x_i') - \sum_i \int_{x_i}^{x_i'} g(x_i') < \delta,
\]

Then

\[
\sum_i \int_{x_i'}^{x_i} x(x_i') - x(x_i') \, \delta < \varepsilon / 2.
\]

Then

\[
\sum_i \int_{x_i'}^{x_i} x(x_i') - x(x_i') \, \delta = \sum_i \int_{x_i'}^{x_i} \{ x(x_i') - U_n(x_i') \} - \{ x(x_i) - U_n(x_i) \} \, \delta + \sum_i \{ U_n(x_i') - U_n(x_i) \} \leq x(b-) - U_n(b-) + \varepsilon / 2 < \varepsilon.
\]
So, $X(x)$ is also $AC_{g_k}$ above on $[a, b]$ and hence $X(x)$ is $AC_{g_k}$ on $[a, b]$.

Now

$$X'_g(x) > U'_n(x) > f_n(x)$$

except a set of $g_k$-measure zero in $[a, b]$. Since the above relation is true for all $n$, it follows that

$$X'_g(x) > f(x)$$

except a set of $g_k$-measure zero. Hence $X(x)$ is a major function of $f(x)$ and so $X(x) > U(x)$. Thus $U(x)$ is $AC_{g_k}$ on $[a, b]$, and utilising (3.6)

$$U(x) = \lim_{n \to \infty} U_n(x), \quad x \in [a, b] \cap S.$$  

This gives

$$\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}}.$$  

Similarly we can show that the lower integral function $L(x)$ of $f(x)$ is $AC_{g_k}$ on $[a, b]$, and

$$\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}}.$$
Case (ii). \( f(x) \) is bounded above on \([-a, b]\).

Since the upper integral function of the function \(-f(x)\) is the negative of the lower integral function of \(f(x)\) and the lower integral function of the function \(-f(x)\) is the negative of the upper integral function of \(f(x)\), if we proceed with \(-f(x)\) as in case (i), we will get that both \(U(x)\) and \(L(x)\) are \(ACg_k\) on \([-a, b]\), and

\[
\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_{-n}(x) \frac{d^k g(x)}{dx^{k-1}}
\]

and

\[
\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \int_a^b f_{-n}(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

Case (iii). \( f(x) \) is neither bounded above nor bounded below on \([-a, b]\).

In this case \( f_n(x) \) is bounded above on \([-a, b]\). Since the upper integral function \(U(x)\) serves as a major function for \(f_n(x)\) and the upper integral function of \(|f(x)|\) is a major function for \(-f_n(x)\), it follows that the function \(f_n(x)\) has upper and lower integral functions, and by case (ii), they are \(ACg_k\) on \([-a, b]\). If we now proceed as in case (i) we will similarly get that the functions \(U(x)\) and \(L(x)\) are \(ACg_k\) on \([-a, b]\), and
\[
\frac{1}{b} \int_{a}^{b} f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \frac{1}{b} \int_{a}^{b} f_n(x) \frac{d^k g(x)}{dx^{k-1}} ;
\]

\[
\frac{1}{b} \int_{a}^{b} f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \frac{1}{b} \int_{a}^{b} f_n(x) \frac{d^k g(x)}{dx^{k-1}} .
\]

If \(-f(x)\) is considered in place of \(f(x)\) we again obtain the same property for \(U(x)\) and \(L(x)\), and

\[
\frac{1}{b} \int_{a}^{b} f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \frac{1}{b} \int_{a}^{b} f_{-n}(x) \frac{d^k g(x)}{dx^{k-1}} ;
\]

\[
\frac{1}{b} \int_{a}^{b} f(x) \frac{d^k g(x)}{dx^{k-1}} = \lim_{n \to \infty} \frac{1}{b} \int_{a}^{b} f_{-n}(x) \frac{d^k g(x)}{dx^{k-1}} .
\]

This completes the proof of the theorem.

A function \(f(x)\) can be represented by

\[
f(x) = f_+(x) - f_-(x)
\]

where \(f_+(x)\) and \(f_-(x)\) are two non-negative functions defined by the rules

\[
f_+(x) = \max \{ f(x), 0 \} \quad \text{and} \quad f_-(x) = \max \{ -f(x), 0 \} .
\]

**Theorem 3.4.** If the function \(f(x)\) has upper and lower \(L^k\)-integrals on \([a, b]\), then the positive part \(f_+(x)\) and negative part \(f_-(x)\) of \(f(x)\) have upper and lower \(L^k\)-integrals on \([a, b]\). Further
Proof. By Theorem 3.1, \(|f(x)|\) has upper and lower \(L^k\)-integrals on \([a, b]\) and since

\[|f(x)| = f_+(x) + f_-(x),\]

the upper integral function of \(|f(x)|\) is a major function of both \(f_+(x)\) and \(f_-(x)\). Further \(m(x) = 0\) is a minor function of both \(f_+(x)\) and \(f_-(x)\). Hence both the functions \(f_+(x)\) and \(f_-(x)\) have upper and lower \(L^k\)-integrals on \([a, b]\). This proves the first part of this theorem.

Let \(L(x)\) and \(U(x)\) be the lower and upper integral functions of \(f(x)\) on \([a, b]\). Then, by Theorem 3.3, both \(L(x)\) and \(U(x)\) are AC on \([a, b]\) and so by Theorem 1.2, \(L(x)\) and \(U(x)\) can be represented on \([a, b]\) in the forms

\[
L(x) = \theta(x) - \psi(x),
\]
\[
U(x) = \alpha(x) - \beta(x),
\]

where each of the functions \(\alpha(x), \beta(x), \theta(x), \psi(x)\) is in class \(\mathcal{U}_c\) and is AC and non-decreasing on \([a, b]\) and \(\theta(a^+) = \psi(a^+) = \alpha(a^+) = \beta(a^+) = 0\). Denote \(U'_g(x)\) by \(u(x)\) and the positive and negative parts of \(U'_g(x)\) by \(u_+ (x)\) and \(u_-(x)\).
respectively. Then, by Theorem 1.4 of Chapter I, we have, for \( x \in \overline{a,b} \cap S \),

\[
U(x) = \int_a^x u(t) \frac{d^k g(t)}{dt^{k-1}} dt,
\]

\[
\alpha(x) = \int_a^x u_+(t) \frac{d^k g(t)}{dt^{k-1}} dt,
\]

\[
\beta(x) = \int_a^x u_-(t) \frac{d^k g(t)}{dt^{k-1}} dt.
\]

Since \( U(x) \) is \( AC_{gk} \) on \( \overline{a,b} \), then, by Theorem 1.3 of Chapter II, it is \( BV_{gk} \) on \( \overline{a,b} \) and so, by Theorem 3.3 of Chapter II, the \( gk \)-derivatives of \( U(x) \) exist finitely except a set of \( gk \)-measure zero in \( \overline{a,b} \). Hence, by Theorem 2.4,

\[
f(x) \leq U_{gk}'(x) = u(x)
\]

except a set of \( gk \)-measure zero in \( \overline{a,b} \), and so

\[
f_+(x) \leq u_+(x) \text{ and } f_-(x) \geq u_-(x)
\]

except a set of \( gk \)-measure zero in \( \overline{a,b} \). Hence

\[
(3.7) \quad f_+(x) \leq \alpha_{gk}'(x) \text{ and } f_-(x) \geq \beta_{gk}'(x)
\]

except a set of \( gk \)-measure zero in \( \overline{a,b} \). Evidently, \( \alpha(x) \)
is a major function of \( f_+(x) \) on \( [a, b] \) and we have

\[
(3.8) \quad \phi(b+) \geq \int_a^b f_+(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

Similarly

\[
\beta(b+) \leq \int_a^b f_-(x) \frac{d^k g(x)}{dx^{k-1}}.
\]

Suppose the sign of equality of (3.8) does not hold and let

\[
N(x) = \begin{cases} 
0 & \text{for } x < a, \\
\int_a^x f_+(t) \frac{d^k g(t)}{dt^{k-1}} & \text{for } a \leq x \leq b, \\
N(b-) & \text{for } x > b.
\end{cases}
\]

and

\[
M(x) = N(x) - \beta(x).
\]

Then letting \( x \to b^- \) over \( S \) we get

\[
M(b-) = N(b-) - \beta(b-) = N(b+) - \beta(b-)
\]

\[
= \int_a^b f_+(x) \frac{d^k g(x)}{dx^{k-1}} - \beta(b-) < \phi(b-) - \beta(b-).
\]
So

\[ (3.9) \quad M(b-) < U(b-) \]

Now \( M(x) \) belongs to the class \( \mathcal{U} \) and is \( AC_{gk} \) on \( \int_a^b \) and \( M(a-) = 0 \). Also

\[ M'(x) \geq f_+(x) - \beta'(x) \geq f_+(x) - f_-(x), \] by \( (3.7) \)

except a set of \( g_k \)-measure zero in \( \int_a^b \). Thus \( M(x) \) is a major function of \( f(x) \) on \( \int_a^b \) which contradicts \( (3.9) \).

Hence

\[ \alpha(b+) = \int_a^b f_+(x) \frac{d^k g(x)}{dx^{k-1}}. \]

Similarly we can show

\[ \beta(b+) = \int_a^b f_-(x) \frac{d^k g(x)}{dx^{k-1}}; \]

\[ \theta(b+) = \int_a^b f_+(x) \frac{d^k g(x)}{dx^{k-1}}; \]

\[ \phi(b+) = \int_a^b f_-(x) \frac{d^k g(x)}{dx^{k-1}}. \]

Therefore

\[ \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \int_a^b f_+(x) \frac{d^k g(x)}{dx^{k-1}} - \int_a^b f_-(x) \frac{d^k g(x)}{dx^{k-1}}. \]
This completes the proof of the theorem.

**Theorem 5.5.** If the upper and lower LS$_k$-integrals of a function \( f(x) \) on \( [a, b] \) are not equal, then \( f(x) \) is not gk-measurable on \( [a, b] \).

**Proof.** We first consider the case when \( f(x) \) is non-negative on \( [a, b] \). If possible, let \( f(x) \) be gk-measurable on \( [a, b] \). Then the functions \( f_n(x) \) defined by

\[
 f_n(x) = \min \{ f(x), n \}, \quad n = 1, 2, \ldots
\]

are bounded and gk-measurable on \( [a, b] \). So

\[
 \left( \text{LS}_k \right) \int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}}
\]

exists for all \( n \). Since \( f(x) \) is not summable (LS$_k$) on \( [a, b] \), we get

\[
 (3.10) \quad \lim_{n \to \infty} \int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}} = +\infty.
\]

For \( x \in [a, b] \), let

\[
 m_n(x) = \left( \text{LS}_k \right) \int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}}.
\]
Then, by Theorem 1.1 of Chapter I, $m_n(x)$ belongs to class $\mathcal{H}L_0$ for every fixed $n$ and $m_n(x)$ is $AC_{gk}$ on $[a, b]$. Again, by Theorem 1.2 of Chapter I,

$$m'_{ngk}(x) = f_n(x) \leq f(x)$$

except a set of $gk$-measure zero in $[a, b]$. Also $m_n(a^-) = 0$. Then $m_n(x)$ is a minor function of $f(x)$ on $[a, b]$ and so for all values of $n$

$$\int_a^b f_n(x) \frac{d^k g(x)}{dx^{k-1}} \leq \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}}.$$

Utilising (3.10) we get

$$\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \infty,$$

which contradicts the fact that the lower integral must be a finite number. Hence $f(x)$ is not $gk$-measurable on $[a, b]$.

Now let $f(x)$ be a function without any restriction on its sign and let $f_+(x)$ and $f_-(x)$ be the positive and negative parts of $f(x)$. By Theorem 3.4, at least one of the functions $f_+$ and $f_-$ has upper and lower $L_{k}$-integrals on $[a, b]$ which are not equal. Since both the functions $f_+$ and $f_-$ are non-negative, at least one of them is not $gk$-measurable. Therefore $f(x)$ is not $gk$-measurable on $[a, b]$, and thus the theorem is proved.