CHAPTER I

Some Set-theoretic Properties Of Linear Operators

1. Introduction

During the last two decades much investigations have been made for the existence of fixed point of operators $T$ which map a Banach space $E$ (or a metric space) into itself satisfying the condition

\[ ||Tx-Ty|| \leq a_1 ||x-y|| + a_2 ||x-Tx|| + a_3 ||y-Ty|| + a_4 ||x-Ty|| + a_5 ||y-Ty|| \]

... (1)

where $a_1 \notin 0$ and $\sum_{i=1}^{5} a_i < 1$, for all $x, y \notin E$.

The existence of fixed point of operators those satisfy the condition (1) has been investigated by many authors including that in $\sum_{22}$, where some further references may be found.
The condition (1) coincides with Banach's condition if \( a_2 = a_3 = a_4 = a_5 = 0 \), with Kannan's condition if \( a_1 = a_4 = a_5 = 0, a_2 = a_3 < \frac{1}{2} \); with Reich's condition if \( a_2 = a_3 = 0 \).

The operators with condition (1) have played a dominant role in many investigations but it appears that so far no investigation has been made to find out the set-theoretic position of these operators in the space \( L \) of all continuous linear operators mapping \( E \) into itself. The main purpose of this chapter is in this direction.

2. Definitions and Examples

It is well-known that \( L \) is a Banach space with norm
\[
||T|| = \sup_{||x|| \leq 1} ||Tx||.
\]

**Definition 1.1.** Let \( X \) be a bounded closed subset of \( E \) and \( S \) denote the set of all continuous linear operators \( T \in L \) mapping \( X \) into itself and satisfying the condition (1) with \( a_1 \gamma, 0, \sum_{i=1}^{5} a_i \leq 1, a_i > 0 \), for all \( x, y \in X \).
We present below two examples showing the existence of continuous linear operators those satisfy the condition (1) as well as those do not satisfy the condition (1).

**Example 1.1.** Let $E = \mathbb{R}$, the set of real numbers with usual norm. Let $T$ be such that $T x = \frac{1}{2^6} x$. Then $T$ is a continuous linear operator and $T \in S$ where $X = \bigcup_{0,1}$

and $a_1 = \frac{1}{2^9}$, $a_2 = \frac{1}{2^5}$, $a_3 = \frac{1}{2^4}$, $a_4 = \frac{1}{2^2}$, $a_5 = \frac{1}{2^3}$.

**Example 1.2.** Let $E = \mathbb{R}$ with its usual norm. Let $T$ be such that $T x = x/2$. Let $X = \bigcup_{0,1}$ and $a_i$'s be the same as in Ex. 1.1. Then, by taking the pair of points $x = 0$, $y = 1$ it may be verified that $T \in L$ but $T \notin S$.

**Definition 1.2.** Let $L_1 \subset L$ and $T \in L$. Then $T$ is said to be a limit element of $L_1$ if for arbitrary $\varepsilon > 0$ there exists $T_1 \in L_1$ different from $T$ such that $||T - T_1|| < \varepsilon$.

If all the limit elements of $L_1$ are members of $L_1$ then $L_1$ is said to be closed in $L$.

**Definition 1.3.** A non-empty set $L_1 \subset L$ is said to be dense-in-itself if every element of $L_1$ is a limit element of $L_1$. 

Definition 1.4. A non-empty set $L \subset L$ is said to be perfect if $L$ is closed and dense-in-itself.

Definition 1.5. Let $a_1 > 0$, $a_1 > 0$ be such that $\sum_{i=1}^{5} a_i \leq 1$. For $\varepsilon > 0$, $S_\varepsilon$ denotes the set of all continuous linear operators $T \in L$ mapping $X$ into itself such that

$$||Tx-Ty|| \leq a_1 ||x-y|| + a_2 ||x-Tx|| + a_3 ||y-Ty|| + a_4 ||x-Ty|| + a_5 ||y-Tx|| + \varepsilon$$

for all $x, y \in X$.

The operator $T$ of Example 1.2, although it does not belong to $S$, but belongs to $S_1$.

3. Theorems and Corollary

Theorem 1.1. The set $S$ is closed in $L$.

Proof. Since $X$ is bounded, there exists $M > 0$ such that

$$||x|| \leq M \text{ for all } x \in X.$$

Let $T \in L$ be a limit element of $S$ and let $T' \in S$. We show
that $T \in S$. There exists a sequence $\{T_n\}$, $T_n \in S$ such that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$. So, for any $x \in X$,

$$T_n = \lim_{n \rightarrow \infty} T_n x$$

where $T_n x \in X$ for all $n$. 

Since $X$ is closed and $T_n x \in X$ for all $n$, $T_n x \in X$, i.e., $T$ maps $X$ into itself.

Now for $x, y \in X$,

$$\|T_n x - T_n y\| \leq \|T_n x - T'_n x\| + \|T'_n x - T'_n y\| + \|T'_n y - T_n y\|$$

$$\leq \{ \|x\| + \|y\| \} \|T - T'_n\| + a_1 \|x - y\| + a_2 \|x - T'_n x\|$$

$$+ a_3 \|y - T'_n y\| + a_4 \|x - T'_n y\| + a_5 \|y - T'_n x\|$$

$$\leq 2M \|T - T'\| + a_1 \|x - y\| + a_2 \|x - T x\|$$

$$+ a_3 \|y - T y\| + a_4 \|x - T y\| + a_5 \|y - T x\|$$

$$+ \{ a_2 \|T x - T'_n x\| + a_3 \|T y - T'_n y\| + a_4 \|T y - T'_n y\| +$$

$$+ a_5 \|T x - T'_n x\| \}^2$$

i.e.,

$$\|T x - T y\| \leq 2M \|T - T'\| + a_1 \|x - y\| + a_2 \|x - T x\|$$

$$+ a_3 \|y - T y\| + a_4 \|x - T y\| + a_5 \|y - T x\|$$

$$+ \{ (a_2 + a_3^2) \|x\| + (a_3 + a_4) \|y\| \} \|T - T'\|$$
\[ \xi \leq 2M \|T - T'\| + a_1 \|x - y\| + a_2 \|x - Tx\| \\
+ a_3 \|y - Ty\| + a_4 \|x - Ty\| + a_5 \|y - Tx\| \\
+ (a_2 + a_3 + a_4 + a_5) M \|T - T'\|, \]

i.e., \[ \|T_x - Ty\| \leq (2 + \sum_{i=2}^{5} a_i) M \|T - T'\| + a_1 \|x - y\| + a_2 \|x - Tx\| \\
+ a_3 \|y - Ty\| + a_4 \|x - Ty\| + a_5 \|y - Tx\|. \]

Corresponding to the arbitrary \( \xi > 0 \), \( T' \) can be so selected that

\[ \|T - T'\| < \frac{\xi}{(2 + \sum_{i=2}^{5} a_i) M} \]

\[ \|T_x - Ty\| \leq \xi, \quad a_1 \|x - y\| + a_2 \|x - Tx\| + a_3 \|y - Ty\| \\
+ a_4 \|x - Ty\| + a_5 \|y - Tx\|. \]

Since \( \xi > 0 \) is arbitrary, it follows that \( T \in S \) and the theorem is proved.

**Corollary 1.1.** The set \( S_\xi \) is closed in \( L \).
From this stage onwards we assume that $X$ is a bounded closed convex subset of $E$ that contains the null element.

**Theorem 1.2.** The set $S_\xi$, $\xi \in M(\sum a_i)$, if it is not void, is dense-in-itself.

**Proof.** Let $T \in S_\xi$ and let $T_1$ be an operator defined on $E$ by

$$T_1x = T(x - \xi x) \text{ where } 0 \leq \xi \leq 1.$$ 

Since $X$ is convex containing the null element, $T_1$ is a continuous linear operator mapping $X$ into itself. Also for $x, y \in X$,

$$||T_1x - T_1y|| = ||T(x - \xi x) - T(y - \xi y)||$$

$$= ||Tx - Ty - \xi(Tx - Ty)||$$

$$= (1 - \xi) ||Tx - Ty||$$

$$\leq (1 - \xi) \left( a_1 ||x - y|| + a_2 ||x - Tx|| + a_3 ||y - Ty|| + a_4 ||x - Ty|| + a_5 ||y - Tx|| + \xi \right)$$

$$\leq (1 - \xi) \left( a_1 ||x - y|| + a_2 ||x - T_1x|| + a_3 ||y - T_1y|| + a_4 ||x - T_1y|| + a_5 ||y - T_1x|| + \xi \right)$$
\[ \xi (1 - \xi) \sum_{i=1}^{\xi} a_i \|x - y\| + a_2 \|x - T_1x\| + a_3 \|y - T_1y\| + a_4 \|x - T_1y\| + a_5 \|y - T_1x\| \]

\[ + (1 - \xi) \left( \sum_{i=\xi}^{\xi} a_i \right) M \xi + (1 - \xi) \varepsilon. \]

Since \( \xi > M \sum_{i=1}^{\xi} a_i \), we have
\[ (\sum_{i=1}^{\xi} a_i) M \xi (1 - \xi) + (1 - \xi) \varepsilon < \varepsilon. \]

\[ \|T_1x - T_1y\| \leq \sum_{i=1}^{\xi} a_i \|x - y\| + a_2 \|x - T_1x\| + a_3 \|y - T_1y\| + a_4 \|x - T_1y\| + a_5 \|y - T_1x\| + \varepsilon. \]

So, \( T_1 \in S_\varepsilon \). Also,
\[ \|T - T_1\| = \sup_{\|x\| \leq 1} \|Tx - T_1x\| = \sup_{\|x\| \leq 1} \|T_\xi x\| \]
\[ = \xi \|T\|. \]

Let \( \xi > 0 \) be arbitrary. Then \( 0 < \varepsilon < 1 \) can always be selected such that \( T_1 \in S_\varepsilon \) and \( \|T - T_1\| < \xi \). This, however, shows that \( T \) is a limit element of \( S_\varepsilon \), and therefore \( S_\varepsilon \) is dense-in-itself. The proof of the theorem is, therefore, complete.

Combining Theorem 1.2 with the corollary, we have the following theorem:
Theorem 1.2. The set $S_{\xi}$, $\xi > M\left(\sum_{i=1}^{\infty} a_i\right)$, $a_i > 0$ and

$$\sum_{i=1}^{\infty} a_i \leq 1$$

is perfect, if it is not void.

In Theorem 1.2 we see that if $\xi > M\left(\sum_{i=1}^{\infty} a_i\right)$ then $S_{\xi}$ is dense-in-itself and clearly $S \subset S_{\xi}$. This ensures that $S \subset S'_{\xi}$ provided $\xi > M\left(\sum_{i=1}^{\infty} a_i\right)$ where $S'_{\xi}$ is the derived set of $S_{\xi}$. In the following theorem we prove this result without the stated restriction on $\xi$.

Theorem 1.4. $S \subset S'_{\xi}$ where $S'_{\xi}$ is the set of all limit elements of $S_{\xi}$.

Proof. For $T \in S$ we define an operator $T_1$ on $E$ by $T_1x = (1 - \xi)Tx + \xi x$ where $0 < \xi < 1$. Since $X$ is convex containing the null-element, $T_1$ is a continuous linear operator mapping $X$ into itself. Also,

$$||T-T_1|| = \sup ||Tx-(1-\xi)Tx-\xi x||$$

$$||x|| \leq 1$$

$$= \sup \xi ||Tx-x||$$

$$||x|| \leq 1$$

$$\leq \xi \left(||T||+1\right)$$

$$= B\xi$$, say  

... (2)
where $B = ||T|| + 1$.

For $x, y \in X$,

$$||T_1 (x - T_1 y)|| = ||(1 - \xi)Tx + \xi x - (1 - \xi)Ty - \xi y||$$

$$\leq (1 - \xi) \|a_1 \|x - y\| + a_2 \|x - Tx\| + a_3 \|y - Ty\|$$

$$+ a_4 \|x - Ty\| + a_5 \|y - Tx\| \xi$$

$$+ 2 \xi M$$

$$\leq a_1 \|x - y\| + a_2 \|x - T_1 x\| + a_3 \|y - T_1 y\|$$

$$+ a_4 \|x - T_1 y\| + a_5 \|y - T_1 x\|$$

$$+ a_6 \|Ty - y\| + a_7 \|Tx - x\| \xi$$

$$+ 2 \xi M,$$

i.e., $||T_1 (x - T_1 y)|| \leq a_1 \|x - y\| + a_2 \|x - T_1 x\| + a_3 \|y - T_1 y\|$

$$+ a_4 \|x - T_1 y\| + a_5 \|y - T_1 x\| + 2M \xi (\sum_{i=2}^{7} a_i)$$

$$+ 2 \xi M.$$. 
\begin{align*}
&= a_1 ||x-y|| + a_2 ||x-T_1 x|| + a_3 ||y-T_1 y|| \\
&\quad + a_4 ||x-T_1 y|| + a_5 ||y-T_1 x|| + 2M \xi \left(1 + \sum_{i=2}^{5} a_i \right).
\end{align*}

\begin{equation}
\text{(3)}
\end{equation}

If $\xi < \frac{\varepsilon}{2M(1 + \sum_{i=2}^{5} a_i)}$, then from (3), it follows that $T_1 \in \mathcal{S}_{\varepsilon}$. Let $\varepsilon' (> 0)$ be arbitrary. Then if

\begin{equation}
0 < \xi < \min \left[ \frac{\varepsilon}{2M(1 + \sum_{i=2}^{5} a_i)}, 1, \frac{\varepsilon'}{B} \right],
\end{equation}

it follows from (2) that $||T-T_1|| < \varepsilon'$. This proves the theorem.

\textbf{Theorem 1.5.} Let $T_n \in L, n = 1, 2, 3, \ldots$ and $T_n$ converges to $T \in L$ in norm. If $x_0$ be a fixed point of $T_n$ for $n = 1, 2, 3, \ldots$ then $x_0$ is a fixed point of $T$. Conversely, if $x_0$ is a fixed point of $T$ then $T_n x_0 \to x_0$ as $n \to \infty$.

\textbf{Proof.} We have

\begin{equation}
||x_0 - Tx_0|| = ||x_0 - T_n x_0|| + ||T_n x_0 - Tx_0||
\quad = ||T_n x_0 - Tx_0|| \leq ||T_n - T|| ||x_0||
\quad \to 0 \text{ as } n \to \infty.
\end{equation}

So, $Tx_0 = x_0$. 
Conversely, we have

\[ ||x_0 - T_n x_0|| \leq ||T - T_n|| \cdot ||x_0|| \]
\[ \rightarrow 0 \text{ as } n \rightarrow \infty. \]

This proves the theorem.

**Theorem 4.6.** Let \( T_n \in L \) map \( X \) into itself and suppose that \( x_n \) is a fixed point of \( T_n \). Also let \( ||T_n - T|| \rightarrow 0 \) as \( n \rightarrow \infty \), \( T \in L \). Then the sequence \( \{ x_n \} \) is convergent and converges to a fixed point of \( T \) provided that \( X \) is compact and there exists \( \alpha \) such that \( ||T_n|| \leq \alpha < 1 \) for each \( n \).

**Proof.** Since \( X \) is compact, \( \{ x_n \} \supseteq \{ x_{n_i} \} \rightarrow x_0 \in X \), say as \( i \rightarrow \infty \).

Now, \[ ||x_0 - T_n x_0|| = ||x_0 - T_n x_{n_i} + T_n x_{n_i} - T_n x_0|| \]
\[ = ||x_0 - x_{n_i} + T_n(x_{n_i} - x_0)|| \]
\[ \leq ||x_{n_i} - x_0|| + ||T_n|| ||x_{n_i} - x_0|| \]
\[ \rightarrow 0 \text{ as } i \rightarrow \infty. \]

Also, \[ ||x_0 - T x_0|| = ||x_0 - T_n x_0 + T_n x_0 - T x_0|| \]
\[ \leq ||x_0 - T_n x_0|| + ||T_n - T|| ||x_0|| \]
\[ \rightarrow 0 \text{ as } i \rightarrow \infty. \]
So, \( T x_0 = x_0 \) i.e., \( x_0 \) is a fixed point of \( T \).

We now show that \( \{ x_n \} \) is a cauchy sequence.

We have, \( \| x_n - x_m \| = \| T_n x_n - T_m x_m \| \)

\[
\leq \| T_n \| \| x_n - x_m \| + \| T_n T_m \| \| x_m \|
\]

i.e., \( \| x_n - x_m \| \leq \frac{\| T_n T_m \|}{1 - \| T_n \|} \| x_m \| \rightarrow 0 \), as \( n, m \rightarrow \infty \).

This shows that \( \{ x_n \} \) is a cauchy sequence and the proof is complete.