Chapter 4

Common solutions of split generalized equilibrium problem and fixed point problem for a nonexpansive semigroup

4.1 Introduction

In this chapter, we introduce the following \textit{split generalized equilibrium problem} (in short, S\textsubscript{PGEP}): Find \(x^* \in C\) such that

\[ F_1(x^*, x) + h_1(x^*, x) \geq 0, \quad \forall x \in C, \]  \hspace{1cm} (4.1.1)

and such that

\[ y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + h_2(y^*, y) \geq 0, \quad \forall y \in Q, \]  \hspace{1cm} (4.1.2)

where \(F_1, h_1 : C \times C \to \mathbb{R}\) and \(F_2, h_2 : Q \times Q \to \mathbb{R}\) are nonlinear bifunctions and \(A : H_1 \to H_2\) is a bounded linear operator.

If \(h_2 = 0, \ F_2 = 0\) then \(\text{S\textsubscript{PGEP}}(4.1.1)-(4.1.2)\) reduces to the following generalized equilibrium problem (in short, GEP) studied by Cianciaruso \textit{et al.} [48]: Find \(x^* \in C\) such that

\[ F_1(x^*, x) + h_1(x^*, x) \geq 0, \quad \forall x \in C. \] \hspace{1cm} (4.1.3)

If \(h_1 = 0, \ h_2 = 0\) then \(\text{S\textsubscript{PGEP}}(4.1.1)-(4.1.2)\) reduces to \(\text{S\textsubscript{EP}}(3.1.4)-(3.1.5)\).

We denote the solution set of \(\text{GEP}(4.1.1)\) and \(\text{GEP}(4.1.2)\) by \(\text{Sol(} \text{GEP}(4.1.1)\)) and
Sol(GEP(4.1.2)), respectively.

The solution set of $S_p\text{GEP}(4.1.1)-(4.1.2)$ is denoted by $\text{Sol}(S_p\text{GEP}(4.1.1)-(4.1.2)) = \{p \in \text{Sol}(\text{GEP}(4.1.1)) : Ap \in \text{Sol}(\text{GEP}(4.1.2))\}$.

Recently, Plubtieng and Punpaeng [147] introduced and studied the following iterative method to prove a strong convergence theorem for FPP for a nonexpansive semigroup $S = \{T(s) : 0 \leq s < \infty\}$ on $C$ in a real Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s) x_n ds, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$ and $\{s_n\}$ is a positive real divergent sequence.

Motivated by the work of Cianciaruso et al. [47,48] and Plubtieng and Punpaeng [147], we introduce and study an iterative method for approximating a common solution of $S_p\text{GEP}(4.1.1)-(4.1.2)$ and FPP for a nonexpansive semigroup $S$ in real Hilbert spaces. Further, based on this method, we prove a strong convergence theorem for approximating a common solution of these problems. Some consequences from the main result are also derived. The results and iterative method presented here extend and generalize the corresponding results and iterative methods given in [43,47,147,152].

### 4.2 Preliminaries

First, we have the following assumption:

**Assumption 4.2.1.** [113] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $F(x, x) \geq 0, \quad \forall x \in C$;

(ii) $F$ is monotone;

(iii) $F$ is upper-hemicontinuous;
(iv) For each $x \in C$ fixed, the function $y \to F(x, y)$ is convex and lower semicontinuous;

let $h : C \times C \to \mathbb{R}$ such that

(i) $h(x, x) \geq 0, \quad \forall x \in C$;

(ii) For each $y \in C$ fixed, the function $x \to h(x, y)$ is upper semicontinuous,

(iii) For each $x \in C$ fixed, the function $y \to h(x, y)$ is convex and lower semicontinuous,

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty, compact and convex subset $K$ of $H_1$ and $x \in C \cap K$ such that

$$F(y, x) + h(y, x) + \frac{1}{r} \langle y - x, x - z \rangle < 0, \quad \forall y \in C \setminus K.$$ 

The proof of following lemma is similar to the proof of Lemma 2.13 [113] and hence is omitted.

**Lemma 4.2.1.** Assume that $F_1, h_1 : C \times C \to \mathbb{R}$ satisfying Assumption 4.2.1. Let $r > 0$ and $x \in H_1$. Then there exists $z \in C$ such that

$$F_1(z, y) + h_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$ 

The proof of following lemma is similar to the proof of Lemma 3.5 due to Cianciaruso et al. [48] and hence is omitted.

**Lemma 4.2.2.** Assume that the bifunctions $F_1, h_1 : C \times C \to \mathbb{R}$ satisfy Assumption 4.2.1 and $h_1$ is monotone. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{(F_1, h_1)} : H_1 \to C$ as follows:

$$T_r^{(F_1, h_1)}(x) = \{ z \in C : F_1(z, y) + h_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \}.$$ 

Then the following hold:

(i) $T_r^{(F_1, h_1)}$ is single-valued;
(ii) \( T_r^{(F_1,h_1)} \) is firmly nonexpansive, i.e.,
\[
\|T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y\|^2 \leq \langle T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y, x - y \rangle, \quad \forall x, y \in H_1;
\]

(iii) \( \text{Fix}(T_r^{(F_1,h_1)}) = \text{Sol}(\text{GEP}(4.1.1)) \);
(iv) \( \text{Sol}(\text{GEP}(4.1.1)) \) is compact and convex.

Further, assume that \( F_2, h_2 : Q \times Q \rightarrow \mathbb{R} \) satisfying Assumption 4.2.1. For \( s > 0 \) and for all \( w \in H_2 \), define a mapping \( T_s^{(F_2,h_2)} : H_2 \rightarrow Q \) as follows:
\[
T_s^{(F_2,h_2)}(w) = \{ d \in Q : F_2(d,e) + h_2(d,e) + \frac{1}{s} \langle e - d, d - w \rangle \geq 0, \quad \forall e \in Q \}.
\]

Then, we easily observe that \( T_s^{(F_2,h_2)} \) is single-valued and firmly nonexpansive; \( T_s^{(F_2,h_2)}(w) \neq \emptyset \) for every \( w \in H_2 \); \( \text{Sol}(\text{GEP}(4.1.2)) \) is compact and convex, and \( \text{Fix}(T_s^{(F_2,h_2)}) = \text{Sol}(\text{GEP}(4.1.2)) \), where \( \text{Sol}(\text{GEP}(4.1.2)) \) is the solution set of the following generalized equilibrium problem:

Find \( y^* \in Q \) such that \( F_2(y^*,y) + h_2(y^*,y) \geq 0, \quad \forall y \in Q \).

**Remark 4.2.1.** Lemmas 4.2.1-4.2.2 are slight generalization of Lemma 3.5 in [48] where the equilibrium condition \( F_1(x,x) = h_1(x,x) = 0 \) has been relaxed to \( F_1(x,x) \geq 0 \) and \( h_1(x,x) \geq 0 \) for all \( x \in C \). Further the monotonicity of \( h_1 \) in Lemma 4.2.1 is not required.

### 4.3 Iterative method

In this section, we prove a strong convergence theorem based on the proposed iterative method for computing an approximate common solution of \( \text{SpGEP}(4.1.1)-(4.1.2) \) and \( \text{FPP}(1.2.10) \) for a nonexpansive semigroup \( S \) in real Hilbert spaces.

We assume that \( \text{SpGEP}(4.1.1)-(4.1.2) \neq \emptyset \).
**Theorem 4.3.1.** Let $H_1$ and $H_2$ be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1, h_1 : C \times C \to \mathbb{R}$ and $F_2, h_2 : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Assumption 4.2.1; $h_1, h_2$ are monotone and $F_2$ is upper semicontinuous in first argument. Let $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $C$ such that $\Omega := \text{Fix}(S) \cap \text{Sol}(\text{SpGEP}(4.1.1)-(4.1.2)) \neq \emptyset$, where $\text{Fix}(S)$ is defined by (1.2.10). Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0, 1)$ and $B$ be a strongly positive self-adjoint bounded linear operator on $H_1$ with constant $\gamma > 0$ such that $0 < \gamma < \frac{s}{\alpha} < \gamma + \frac{1}{\alpha}$. Let $\{s_n\}$ be a positive real sequence which diverges to $+\infty$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$
\begin{align*}
  u_n &= T^{(F_1, h_1)}_{r_n}(x_n + \delta A^*(T^{(F_2, h_2)}_{r_n} - I)Ax_n), \\
  x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B) \frac{1}{s_n} \int_{s_n}^{s_{n+1}} T(s)u_n ds,
\end{align*}
$$

(4.3.1)

where $r_n \in (0, \infty)$ and $\delta \in (0, 1/L)$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;

(iii) $\liminf_{n \to \infty} r_n > 0$ and $\lim_{n \to \infty} |r_{n+1} - r_n| = 0$;

(iv) $\lim_{n \to \infty} \frac{|s_{n+1} - s_n|}{s_{n+1}} = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega(I - B + \gamma f)z$.

**Proof.** On the similar lines of proof of Theorem 3.4.1, we can easily observe that there exists an element $z \in H_1$ such that $z = q(I - B + \gamma f)z = P_\Omega(I - B + \gamma f)(z)$.

Let $p \in \Omega$, i.e., $p \in \text{Sol}(\text{SpGEP}(4.1.1)-(4.1.2))$, we have $p = T^{(F_1, h_1)}_{r_n}p$ and $Ap = T^{(F_2, h_2)}_{r_n}(Ap)$.  

89
As estimated (3.3.8), we obtain
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 + \delta (L\delta - 1) \| (T_{r_n}^{(F_{2,h^2})} - I) Ax_n \|^2.
\] (4.3.2)

Since, $\delta \in (0, \frac{1}{L})$, we obtain
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2.
\] (4.3.3)

Set $t_n := \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds$. As estimated (3.3.10), we obtain
\[
\|t_n - p\| \leq \|x_n - p\|.
\] (4.3.4)

Further, we estimate
\[
\|x_{n+1} - p\| = \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B)t_n - p\|
\]
\[
= \|\alpha_n (\gamma f(x_n) - Bp) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n B)(t_n - p)\|
\]
\[
\leq \alpha_n \|\gamma f(x_n) - Bp\| + \beta_n \|x_n - p\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|t_n - p\|
\]
\[
\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Bp\| + \beta_n \|x_n - p\|
\]
\[
+ (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - p\|
\]
\[
\leq \alpha_n \gamma \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \bar{\gamma})\|x_n - p\|
\]
\[
= (1 - (\bar{\gamma} - \gamma \alpha)\alpha_n)\|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\|
\]
\[
\leq \max \left\{\|x_n - p\|, \frac{1}{\bar{\gamma} - \gamma \alpha} \|\gamma f(p) - Bp\| \right\}, \quad n \geq 0
\]
\[
\leq \max \left\{\|x_0 - p\|, \frac{1}{\bar{\gamma} - \gamma \alpha} \|\gamma f(p) - Bp\| \right\}.
\] (4.3.5)

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$, $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

As estimated (3.4.8), we obtain
\[
\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \delta \|A\|\sigma_n + \delta_n + 2 \frac{|s_{n+1} - s_n|}{s_{n+1}} \|u_n - p\|.
\] (4.3.6)
where
\[ \sigma_n = \left| 1 - \frac{r_n}{r_{n+1}} \right| \left\| T_{r_{n+1}}^{(F_2,h_2)} Ax_{n+1} - Ax_{n+1} \right\| \]
and
\[ \delta_n = \left| 1 - \frac{r_n}{r_{n+1}} \right| \left\| T_{r_{n+1}}^{(F_1,h_1)} \left( x_{n+1} + \delta A^* \left( T_{r_{n+1}}^{(F_2,h_2)} - I \right) Ax_{n+1} \right) - \left( x_{n+1} + \delta A^* \left( T_{r_{n+1}}^{(F_2,h_2)} - I \right) Ax_{n+1} \right) \right\|. \]

Setting \( x_{n+1} := \beta_n x_n + (1 - \beta_n)e_n \), which implies from (4.3.1) that
\[ e_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n g_f(x_n) + ((1 - \beta_n)I - \alpha_n B)t_n}{1 - \beta_n}. \]

Further, it follows that
\[
e_{n+1} - e_n = \frac{\alpha_{n+1} g_f(x_{n+1}) + ((1 - \beta_n)I - \alpha_{n+1}B)t_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n g_f(x_n) + ((1 - \beta_n)I - \alpha_n B)t_n}{1 - \beta_n}
\]
\[= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} g_f(x_{n+1}) + \frac{(1 - \beta_{n+1})t_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n B t_n}{1 - \beta_n} - \frac{\alpha_n g_f(x_n)}{1 - \beta_n} + \frac{\alpha_n B t_n}{1 - \beta_n}
\]
\[= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (g_f(x_{n+1}) + B t_{n+1}) + t_{n+1} - t_n + \frac{\alpha_n}{1 - \beta_n} (B t_n - g_f(x_n)) \]

Using (4.3.6), we have
\[
\left\| e_{n+1} - e_n \right\| = \left\| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (g_f(x_{n+1}) + B t_{n+1}) + t_{n+1} - t_n + \frac{\alpha_n}{1 - \beta_n} (B t_n - g_f(x_n)) \right\|
\]
\[\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\|g_f(x_{n+1})\| + \|B t_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n} (\|g_f(x_n)\| + \|B t_n\|)
\]
\[+ \|t_{n+1} - t_n\|
\]
\[\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\|g_f(x_{n+1})\| + \|B t_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n} (\|g_f(x_n)\| + \|B t_n\|)
\]
\[+ \|x_{n+1} - x_n\| + \gamma \|A\| \sigma_n + \delta_n + 2 \frac{|s_{n+1} - s_n|}{s_{n+1}} \|u_n - p\|
\]

which implies that
\begin{align*}
\|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| & \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\|f(x_{n+1})\| + \|Bt_{n+1}\|) + \gamma\|A\|\sigma_n + \delta_n \\
& \quad + \frac{\alpha_n}{1 - \beta_n} (\|\gamma f(x_n)\| + \|Bt_n\|) + 2\frac{|s_{n+1} - s_n|}{s_{n+1}}\|u_n - p\|.
\end{align*}

Hence, it follows by conditions (i), (iii)-(iv) that

\[
\limsup_{n \to \infty} [\|e_{n+1} - e_n\| - \|x_{n+1} - x_n\|] \leq 0. \tag{4.3.7}
\]

From Lemma 1.2.8 and (4.3.7), we get \(\lim_{n \to \infty} \|e_n - x_n\| = 0\) and

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|e_n - x_n\| = 0. \tag{4.3.8}
\]

Now,

\[
x_{n+1} - x_n = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B)t_n - x_n
\]
\[
= \alpha_n (\gamma f(x_n) - x_n) + ((1 - \beta_n)I - \alpha_n B)(t_n - x_n).
\]

Since \(\|x_{n+1} - x_n\| \to 0\) and \(\alpha_n \to 0\) as \(n \to \infty\), we obtain

\[
\lim_{n \to \infty} \|t_n - x_n\| = 0. \tag{4.3.9}
\]

As estimated (3.4.14), we obtain

\[
\lim_{n \to \infty} \|T(s)x_n - x_n\| = 0. \tag{4.3.10}
\]

It follows from (4.3.2) and Lemma 1.2.7 (i) that

\[
\|x_{n+1} - p\|^2 = \|\alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n B]t_n - p\|^2
\]
\[
= \|\alpha_n (\gamma f(x_n) - Bp) + \beta_n (x_n - t_n) + (I - \alpha_n B)(t_n - p)\|^2
\]
\[
\begin{align*}
\leq & \quad \|(I - \alpha_n B)(t_n - p) + \beta_n(x_n - t_n)\|^2 + 2\alpha_n(\gamma f(x_n) - Bp, x_{n+1} - p) \\
\leq & \quad [\|(I - \alpha_n B)(t_n - p)\| + \beta_n\|x_n - t_n\|]^2 + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\
\leq & \quad [\|I - \alpha_n B\|\|u_n - p\| + \beta_n\|x_n - t_n\|]^2 + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\
= & \quad (I - \alpha_n \bar{\gamma})^2\|u_n - p\|^2 + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n \bar{\gamma})\beta_n\|u_n - p\|\|x_n - t_n\| \\
& + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\
\leq & \quad (1 - \alpha_n \bar{\gamma})^2\|x_n - p\|^2 + \delta(L\delta - 1)\|T_{r_n}^{(F_2,h_2)} - I\|A\|x_n\|^2 + \beta_n^2\|x_n - t_n\|^2 \\
& + 2(1 - \alpha_n \bar{\gamma})\beta_n\|u_n - p\|\|x_n - t_n\| + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\
\leq & \quad [1 - 2\alpha_n \bar{\gamma} + (\alpha_n \bar{\gamma})^2]\|x_n - p\|^2 + (1 - \alpha_n \bar{\gamma})^2\delta(L\delta - 1)\|T_{r_n}^{(F_2,h_2)} - I\|A\|x_n\|^2 \\
& + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n \bar{\gamma})\beta_n\|u_n - p\|\|x_n - t_n\| \\
& + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\|. \\
\end{align*}
\]
As estimated (3.3.15), we obtain
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\delta \|A(u_n - x_n)\|\|T^{(F_2,h_2)}_{T_{T_n}} - I\|A\|x_n\|. \tag{4.3.13}
\]

It follows from (4.3.11) and (4.3.12) that
\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n\gamma)^2 \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\delta \|A(u_n - x_n)\|\|T^{(F_2,h_2)}_{T_{T_n}} - I\|A\|x_n\|
\]
\[
+ \beta_n^2 \|x_n - t_n\|^2 + 2(1 - \alpha_n\gamma)\beta_n \|u_n - p\| \|x_n - t_n\|
\]
\[
+ 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\|
\]
\[
\leq \|x_n - p\|^2 + \alpha_n\gamma^2 \|x_n - p\|^2 - (1 - \alpha_n\gamma)^2 \|u_n - x_n\|^2
\]
\[
+ 2(1 - \alpha_n\gamma)^2 \delta \|A(u_n - x_n)\|\|T^{(F_2,h_2)}_{T_{T_n}} - I\|A\|x_n\|
\]
\[
+ \beta_n^2 \|x_n - t_n\|^2 + 2(1 - \alpha_n\gamma)\beta_n \|u_n - p\| \|x_n - t_n\|
\]
\[
+ 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\|
\]

Therefore,
\[
(1 - \alpha_n\gamma)^2 \|u_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2 \|x_n - t_n\|^2
\]
\[
+ 2(1 - \alpha_n\gamma)^2 \delta \|A(u_n - x_n)\|\|T^{(F_2,h_2)}_{T_{T_n}} - I\|A\|x_n\|
\]
\[
+ \alpha_n\gamma^2 \|x_n - p\|^2 + 2(1 - \alpha_n\gamma)\beta_n \|u_n - p\| \|x_n - t_n\|
\]
\[
+ 2\alpha_n \|\gamma f(x_n) - Bp\| \|x_{n+1} - p\|
\]
\[
\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| + \beta_n^2 \|x_n - t_n\|^2
\]
\[
+ 2(1 - \alpha_n\gamma)^2 \delta \|A(u_n - x_n)\|\|T^{(F_2,h_2)}_{T_{T_n}} - I\|A\|x_n\|
\]
\[
+ \alpha_n\gamma^2 \|x_n - p\|^2 + 2(1 - \alpha_n\gamma)\beta_n \|u_n - p\| \|x_n - t_n\|
\]
\[ +2\alpha_n \| \gamma f(x_n) - Bp \| \| x_{n+1} - p \|. \]

Since \( \alpha_n \to 0, \| x_n - t_n \| \to 0, \| T_{r_n}^{(F_2,h_2)} - I \| A x_n \| \to 0 \) and \( \| x_{n+1} - x_n \| \to 0 \) as \( n \to \infty \), we obtain
\[
\lim_{n \to \infty} \| u_n - x_n \| = 0. \tag{4.3.14}
\]

Since \( u_n = T_{r_n}^{(F_1,h_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)A x_n) \), on setting \( d_n = x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)A x_n \), we have
\[
\| u_n - d_n \| = \| u_n - (x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)A x_n) \|
\leq \| u_n - x_n \| + \delta \| A^*(T_{r_n}^{(F_2,h_2)} - I)A x_n \|
\leq \| u_n - x_n \| + \delta \| A^* \| \| (T_{r_n}^{(F_2,h_2)} - I)A x_n \|. \tag{4.3.15}
\]

It follows from (4.3.12) and (4.3.14) that
\[
\lim_{n \to \infty} \| u_n - d_n \| = 0. \tag{4.3.16}
\]

Since, we can write
\[
\| T(s) t_n - x_n \| \leq \| T(s) t_n - T(s) x_n \| + \| T(s) x_n - x_n \|
\leq \| t_n - x_n \| + \| T(s) x_n - x_n \|
\to 0 \text{ as } n \to \infty.
\]

Also, we have
\[
\| T(s) t_n - t_n \| \leq \| T(s) t_n - T(s) x_n \| + \| T(s) x_n - x_n \| + \| x_n - t_n \|
\leq \| t_n - x_n \| + \| T(s) x_n - x_n \| + \| x_n - t_n \|
\to 0 \text{ as } n \to \infty.
\]

Since \( \{ t_n \} \) is bounded, there exists a subsequence \( \{ t_{n_i} \} \) of \( \{ t_n \} \) which converges weakly to some \( w \in C \).
Now, we prove that \( w \in \Omega \). It is proved in Theorem 3.4.1 that \( w \in \text{Fix}(S) \). Next, we show that \( w \in \text{Sol}(\text{GEP}(4.1.1)) \). Since \( u_n = T_{r_n}^{(F_1, h_1)}d_n \), we have

\[
F_1(u_n, y) + h_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - d_n \rangle \geq 0, \ \forall y \in C.
\]

It follows from monotonicity of \( F_1 \) that

\[
h_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - d_n \rangle \geq F_1(y, u_n)
\]

and hence

\[
h_1(u_n, y) + \left\langle y - u_n, \frac{u_n - d_n}{r_n} \right\rangle \geq F_1(y, u_n).
\]

Since \( \|u_n - x_n\| \to 0, \|t_n - x_n\| \to 0 \) and \( \|u_n - d_n\| \to 0 \) as \( n \to \infty \), we get \( u_n \rightharpoonup w \) and hence it follows from condition (iii) that \( \frac{u_n - d_n}{r_n} \to 0 \). It follows by Assumption 4.2.1 (iv) that \( 0 \geq F_1(y, w), \ \forall w \in C \). For \( t \) with \( 0 < t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1 - t)w \).

Since \( y \in C \), \( w \in C \), we get \( y_t \in C \) and hence \( F_1(y_t, w) \leq 0 \). So from Assumption 4.2.1 (i) and (iv), we have

\[
0 \leq F_1(y_t, y_t) + h_1(y_t, y_t) \leq t[F_1(y_t, y) + h_1(y_t, y)] + (1 - t)[F_1(y_t, w) + h_1(y_t, w)]
\]

\[
\leq t[F_1(y_t, y) + h_1(y_t, y)] + (1 - t)[F_1(y, y_t) + h_1(y, y_t)]
\]

\[
\leq [F_1(y_t, y) + h_1(y_t, y)]
\]

Therefore \( 0 \leq F_1(y_t, y) + h_1(y_t, y) \). From Assumption 4.2.1 (iii), we have \( 0 \leq F_1(w, y) + h_1(w, y) \). This implies that \( w \in \text{Sol}(\text{GEP}(4.1.1)) \).

Next, we show that \( Aw \in \text{Sol}(\text{GEP}(4.1.2)) \). Since \( \|u_n - x_n\| \to 0 \), \( u_n \rightharpoonup w \) as \( n \to \infty \) and \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \rightharpoonup w \) and since \( A \) is a bounded linear operator so that \( Ax_{n_k} \rightharpoonup Aw \). Now setting \( v_{n_k} = Ax_{n_k} - T_{r_{n_k}}^{F_2} Ax_{n_k} \). It follows that from (4.3.12) that \( \lim_{k \to \infty} v_{n_k} = 0 \) and \( Ax_{n_k} - v_{n_k} = T_{r_{n_k}}^{F_2} Ax_{n_k} \).

Therefore from Lemma 4.2.2, we have
\[ F_2(Ax_n - v_{n_k}, z) + h_2(Ax_n - v_{n_k}, z) + \frac{1}{r_{n_k}}(z - (Ax_n - v_{n_k}), (Ax_n - v_{n_k}) - Ax_{n_k}) \geq 0, \forall z \in Q. \]

Since \( F_2 \) and \( h_2 \) are upper semicontinuous in first argument, taking \( \lim \sup \) to above inequality as \( k \to \infty \) and using condition (iii), we obtain

\[ F_2(Aw, z) + h_2(Aw, z) \geq 0, \forall z \in Q, \]

which means that \( Aw \in \text{Sol}(\text{GEP}(4.1.2)) \) and hence \( w \in \Omega \).

It is proved in Theorem 3.4.1 that

\[ \lim \sup_{n \to \infty} \langle (B - \gamma f)z, x_n - z \rangle \leq 0, \tag{4.3.17} \]

Finally, we show that \( x_n \to z \).

\[
\|x_{n+1} - z\|^2 = \|\alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n B]t_n - z\|^2 \\
= \|\alpha_n (\gamma f(x_n) - Bz) + \beta_n (x_n - z) + [(1 - \beta_n)I - \alpha_n B](t_n - z)\|^2 \\
\leq \|\beta_n (x_n - z) + [(1 - \beta_n)I - \alpha_n B](t_n - z)\|^2 \\
+ 2\alpha_n \langle \gamma f(x_n) - Bz, x_{n+1} - z \rangle \\
\leq \left[ \|\beta_n (x_n - z)\|^2 + \|\beta_n (x_n - z)\| \right]^2 \\
+ 2\alpha_n \langle f(x_n) - f(z), x_{n+1} - z \rangle + 2\alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z \rangle \\
\leq \left[ (1 - \beta_n) - \alpha_n \gamma \|x_n - z\| + \beta_n \|x_n - z\| \right]^2 + 2\alpha_n \gamma \alpha \|x_n - z\| ||x_{n+1} - z|| \\
+ 2\alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z \rangle \\
\leq (1 - \alpha_n \gamma)^2 \|x_n - z\|^2 + \alpha_n \gamma \alpha \left\{ \|x_n - z\|^2 + ||x_{n+1} - z||^2 \right\} \\
+ 2\alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z \rangle \\
\leq (1 - \alpha_n \gamma)^2 \|x_n - z\|^2 + \alpha_n \gamma \alpha \|x_n - z\|^2 + \gamma \alpha \|x_{n+1} - z\|^2 \\
+ 2\alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z \rangle.
\]

This implies that
\[ \|x_{n+1} - z\|^2 \leq \frac{1 - 2\alpha_n \bar{\gamma} + (\alpha_n \bar{\gamma})^2 + \alpha_n \gamma \alpha}{1 - \gamma \alpha} \|x_n - z\|^2 \]

\[ + \frac{2\alpha_n}{1 - \gamma \alpha} \langle \gamma f(z) - Bz, x_{n+1} - z \rangle \]

\[ = \left[ 1 - \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha} \right] \|x_n - z\|^2 + \frac{(\alpha_n \bar{\gamma})^2}{1 - \gamma \alpha} \|x_n - z\|^2 \]

\[ + \frac{2\alpha_n}{1 - \gamma \alpha} \langle \gamma f(z) - Bz, x_{n+1} - z \rangle \]

\[ \leq \left[ 1 - \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha} \right] \|x_n - z\|^2 + \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha} \]

\[ \times \left\{ \frac{(\alpha_n \bar{\gamma}^2)M}{2(\bar{\gamma} - \gamma \alpha)} + \frac{1}{\bar{\gamma} - \gamma \alpha} \langle \gamma f(z) - Bz, x_{n+1} - z \rangle \right\} \]

\[ = (1 - \delta_n) \|x_n - z\|^2 + \delta_n \sigma_n, \quad (4.3.18) \]

where \( M := \sup\{\|x_n - z\|^2 : n \geq 1\} \), \( \delta_n = \frac{2(\bar{\gamma} - \gamma \alpha)\alpha_n}{1 - \gamma \alpha} \) and \( \sigma_n = \frac{(\alpha_n \bar{\gamma}^2)M}{2(\bar{\gamma} - \gamma \alpha)} + \frac{1}{\bar{\gamma} - \gamma \alpha} \langle \gamma f(z) - Bz, x_{n+1} - z \rangle \). Since \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \), it is easy to see that \( \lim \delta_n = 0 \), \( \sum_{n=0}^{\infty} \delta_n = \infty \) and \( \limsup_{n \to \infty} \sigma_n \leq 0 \). Hence from (4.3.17), (4.3.18) and Lemma 1.2.11, we deduce that \( x_n \to z \). This completes the proof. \( \square \)

### 4.4 Consequences

We have the following consequences of Theorem 4.3.1.

**Corollary 4.4.1.** Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces and let \( C \subseteq H_1 \) and \( Q \subseteq H_2 \) be nonempty, closed and convex sets. Let \( A : H_1 \to H_2 \) be a bounded linear operator. Assume that \( F_1 : C \times C \to \mathbb{R} \) and \( F_2 : Q \times Q \to \mathbb{R} \) are bifunctions satisfying Assumption 4.2.1 and \( F_2 \) is upper semicontinuous in first argument. Let \( S = \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( C \) such that \( \Omega_1 := \text{Fix}(S) \cap \text{Sol}(\text{SpEP}(3.1.4)-(3.1.5)) \neq \emptyset \).

Let \( f : H_1 \to H_1 \) be a contraction mapping with constant \( \alpha \in (0, 1) \) and \( B \) be a strongly positive self-adjoint bounded linear operator on \( H_1 \) with constant \( \bar{\gamma} > 0 \) such that \( 0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha} \). Let \( \{s_n\} \) is a positive real sequence which diverges to \( +\infty \). For a given \( x_0 \in C \) arbitrarily, let the iterative sequences \( \{u_n\} \) and \( \{x_n\} \) be generated by...
\[
\begin{align*}
{u_n} &= T^F_{r_n}(x_n + \delta A^*(T^F_{r_n} - I)Ax_n), \\
{x_n + 1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds,
\end{align*}
\]

where \( r_n \subset (0,\infty) \) and \( \delta \in (0,1/L) \), \( L \) is the spectral radius of the operator \( A^*A \) and \( A^* \) is the adjoint of \( A \) and \( \{\alpha_n\} \) and \( \{\beta_n\} \) are the sequences in \( (0,1) \) satisfying the conditions (i)-(iv) of Theorem 4.3.1. Then the sequence \( \{x_n\} \) converges strongly to \( z \in \Omega_1 \), where \( z = P_{\Omega_1}(I - B + \gamma f)z \).

Proof. Taking \( h_1 = h_2 = 0 \) in Theorem 4.3.1 then the conclusion of Corollary 4.4.1 is obtained.

Corollary 4.4.2. [47] Let \( H \) be a real Hilbert space and let \( C \subseteq H \) be a nonempty, closed and convex set. Assume that \( F_1 : C \times C \to \mathbb{R} \) be a bifunction satisfying Assumption 4.2.1 for \( F_1 \) only. Let \( S = \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( C \) such that \( \Omega_2 := \text{Fix}(S) \cap \text{Sol}(\text{EP}(3.1.4)) \neq \emptyset \). Let \( f : H_1 \to H_1 \) be a contraction mapping with constant \( \alpha \in (0,1) \) and \( B \) be a strongly positive self-adjoint bounded linear operator on \( H_1 \) with constant \( \gamma > 0 \) such that \( 0 < \gamma < \frac{\gamma}{\alpha} < \gamma + \frac{1}{\alpha} \). Let \( \{s_n\} \) is a positive real sequence which diverges to \( +\infty \). For a given \( x_0 \in C \) arbitrarily, let the iterative sequences \( \{u_n\} \) and \( \{x_n\} \) be generated by

\[
\begin{align*}
{u_n} &= T^F_{r_n}x_n, \\
{x_n + 1} &= \alpha_n \gamma f(x_n) + (1 - \alpha_n B) \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds,
\end{align*}
\]

where \( r_n \subset (0,\infty) \) and \( \{\alpha_n\} \) is a sequences in \( (0,1) \) satisfying the conditions (i),(ii),(iv) of Theorem 4.3.1. Then the sequence \( \{x_n\} \) converges strongly to \( z \in \Omega_2 \), where \( z = P_{\Omega_2}(I - B + \gamma f)z \).

Proof. Taking \( H_2 = H_1, h_1 = h_2 = 0, F_2 = 0, \{\beta_n\} = 0 \) and \( A = 0 \) in Theorem 4.3.1 then the conclusion of Corollary 4.4.2 is obtained.

Corollary 4.4.3. [147] Let \( S = \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( C \) such that \( \text{Fix}(S) \neq \emptyset \). Let \( f : C \to C \) be a contraction mapping with constant \( \alpha \in (0,1) \). Let \( \{s_n\} \) is a positive real sequence which diverges to \( +\infty \). For a given
Let \( x_0 \in C \) arbitrarily, let the iterative sequence \( \{x_n\} \) be generated by

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n ds,
\]

where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are the sequences in \((0,1)\) satisfying the conditions (i), (ii), (iv) of Theorem 4.3.1. Then the sequence \( \{x_n\} \) converges strongly to \( z \in \text{Fix}(S) \), where \( z = P_{\text{Fix}(S)} f(z) \).

**Proof.** Taking \( H_2 = H_1, u_n = x_n, F_1 = F_2 = h_1 = h_2 = 0 \) and \( B = I \) in Theorem 4.3.1 then the conclusion of Corollary 4.4.3 is obtained. \( \square \)